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Research Article

Quaternionic Serret-Frenet Frames for Fuzzy Split Quaternion Numbers

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We build the concept of fuzzy split quaternion numbers of a natural extension of fuzzy real numbers in this study. Then, we give some differential geometric properties of this fuzzy quaternion. Moreover, we construct the Frenet frame for fuzzy split quaternions. We investigate Serret-Frenet derivation formulas by using fuzzy quaternion numbers.

1. Introduction

The Serret-Frenet formulas describe the kinematic properties of a particle moving along a continuous and differentiable curve in Euclidean space E^3 or Minkowski space E_1^3. These formulas are used in many areas such as mathematics, physics (especially in relative theory), medicine, and computer graphics.

Quaternions were discovered by Sir William R. Hamilton in 1843. The most widely used and most important feature of quaternions is that each unit quaternion represents a transformation. This representation has a special and important role on turns in 3-dimensional vector spaces. This situation is detailed in the study [1]. Nowadays, quaternions are used in many areas such as physics, computer graphics, and animation. For example, visualizing and translating with computer graphics are much easier with quaternions. It is known by especially mathematicians and physicists that any unit (split) quaternion corresponds to a rotation in Euclidean and Minkowski spaces.

The notion of a fuzzy subset was introduced by Zadeh [2] and later applied in various mathematical branches. According to the standard condition, a fuzzy number is a convex and a normalized fuzzy subset of real numbers. Basic operations on fuzzy quaternion numbers can be seen in study [3]. There are many applications of quaternions. In physics, we have highlighted applications in quantum mechanics [4] and theory of relativity [5]. In addition, there are applications in aviation projects and flight simulators [6]. On the other hand, the study [7] is a basic study for quaternionic fibonacci forms. All of references that we reviewed guided us to studying the geometry of quaternions.

In this paper, we have described the basic operations of fuzzy split quaternions. With this number of structures we aimed to achieve the frenet frame equation. Previously, frenet frame has been created by split quaternions in [8].

In these studies, we obtained Frenet frame by the fuzzy split quaternion.

2. Serret-Frenet Frame

The Serret-Frenet frame is defined as follows [8].

Let $\overrightarrow{\alpha}(t)$ be any second-order differentiable space curve with nonvanishing second derivative. We can choose this local coordinate system to be the Serret-Frenet frame consisting of the tangent vector $\overrightarrow{T}(t)$, the binormal vector $\overrightarrow{B}(t)$, and the normal vector $\overrightarrow{N}(t)$ vectors at any point on the curve given by

$$\overrightarrow{T}(t) = \frac{\overrightarrow{\alpha}(t)}{\left\| \overrightarrow{\alpha'}(t) \right\|}$$

$$\vec{B}(t) = \frac{\vec{\alpha}'(t) \times \vec{\alpha}''(t)}{\left\| \vec{\alpha}'(t) \times \vec{\alpha}''(t) \right\|}$$

$$\vec{N}(t) = \vec{B}(t) \times \vec{T}(t) \tag{1}$$

The Serret-Frenet frame for the curve $\vec{\alpha}(t)$ is given as the following differential equation. Writing this frame with matrices is easily for the mathematical calculations.

$$\begin{bmatrix} \vec{T}'(t) \\ \vec{B}'(t) \\ \vec{N}'(t) \end{bmatrix} = v(t) \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(t) \\ \vec{B}(t) \\ \vec{N}(t) \end{bmatrix} \tag{2}$$

The speed value of the curve $\vec{\alpha}(t)$ is denoted by $v(t) = \|\vec{\alpha}'(t)\|$. The scalar curvature of $\vec{\alpha}(t)$ is symbolized as $\kappa(t)$ and the torsion value of the curve $\vec{\alpha}(t)$ is symbolized as $\tau(t)$. The torsion of the curve $\vec{\alpha}(t)$ measures how sharply it is twisting out of the plane of curvature. The curvature of $\vec{\alpha}(t)$ is the magnitude of the acceleration of a particle moving along this curve. The torsion of curvature is related by the Serret-Frenet formulas and their generalization. These can be expressed with following formulas:

$$\kappa(t) = \frac{\left\| \vec{\alpha}'(t) \times \vec{\alpha}''(t) \right\|}{\left\| \vec{\alpha}'(t) \right\|^3}$$

$$\tau(t) = \frac{\vec{\alpha}'(t) \times \vec{\alpha}''(t) \times \vec{\alpha}'''(t)}{\left\| \vec{\alpha}'(t) \times \vec{\alpha}''(t) \right\|^2} \tag{3}$$

3. Split Quaternion Frames

In this section, firstly we will give the split quaternions definition and their characteristics properties.

Definition 1. The set $H' = \{q = q_0 1 + q_1 i + q_2 j + q_3 k,\ q_0, q_1, q_2, q_3 \in R\}$ is a vector space over R having basis $\{1, i, j, k\}$ with the following properties:

$$\begin{aligned} i^2 &= -1, \\ j^2 &= k^2 = 1 \\ ij &= -ji = k \\ kj &= -jk = -i \\ ki &= -ik = j \end{aligned} \tag{4}$$

Every element of the set H' is called a split quaternion. [9].

Definition 2. Let two split quaternions be $q = q_0 1 + q_1 i + q_2 j + q_3 k$ and $p = p_0 1 + p_1 i + p_2 j + p_3 k$. These two split quaternions multiplication is calculated as

$$\begin{aligned} q.p = {} & (q_0 p_0 - q_1 p_1 + q_2 p_2 + q_3 p_3) \\ & + (q_0 p_1 + q_1 p_0 - q_2 p_3 + q_3 p_2)\, i \\ & + (q_0 p_2 + q_2 p_0 + q_3 p_1 - q_1 p_3)\, j \\ & + (q_0 p_3 + q_3 p_0 + q_1 p_2 - q_2 p_1)\, k \end{aligned} \tag{5}$$

Definition 3. The conjugate of the split quaternion $q = q_0 1 + q_1 i + q_2 j + q_3 k$ is defined as

$$\bar{q} = q_0 1 - q_1 i - q_2 j - q_3 k \tag{6}$$

Definition 4. A unit-length split quaternion's norm is

$$N_q = q\bar{q} = \bar{q}q = (q_0)^2 + (q_1)^2 - (q_2)^2 - (q_3)^2 = 1 \tag{7}$$

Definition 5. Because of $H' \simeq E_2^4$, we can define the timelike, spacelike, and lightlike quaternions for $q = (q_0, q_1, q_2, q_3)$ as follows:

(i) Spacelike quaternion for $I_q < 0$

(ii) Timelike quaternion for $I_q > 0$

(iii) Lightlike quaternion for $I_q = 0$

Here, $I_q = q\bar{q} = \bar{q}q$. [1].

We can add to Definition 5 following descriptions. Timelike, spacelike, and lightlike vectors are important for the Minkowski space E_1^3. The Minkowski space E_1^3 is the accepted common space for the physical reality. We know that the general properties of the quaternions are similar to Minkowski space E_2^4. The Minkowski space E_2^4 is a vector space with real dimension $'4'$ and index $'2'$. Elements of Minkowski space E_2^4 are called events or four vectors. On Minkowski space E_2^4, there is an inner product of signature two "plus" and two "minus". Also, we prefer to define the vector structure of Minkowski space with quaternions.

Every possible rotation R (a 3×3 special split orthogonal matrix) can be constructed from either one of the two related split quaternions $q = q_0 1 + q_1 i + q_2 j + q_3 k$ or $-q = -q_0 1 - q_1 i - q_2 j - q_3 k$ using the transformation law [8]:

$$\begin{aligned} q w \bar{q} &= R.w \\ [q w \bar{q}]_i &= \sum_{j=1}^{3} R_{ij}.w_j \end{aligned} \tag{8}$$

where $w = v_1 i + v_2 j + v_3 k$ k is a pure split quaternion. We compute R_{ij} directly from (5)

$$R=\begin{bmatrix}(q_0)^2+(q_1)^2-(q_2)^2-(q_3)^2 & 2q_1q_2-2q_0q_3 & 2q_0q_2+2q_1q_3\\ 2q_0q_3+2q_1q_2 & -(q_0)^2+(q_1)^2+(q_2)^2-(q_3)^2 & 2q_2q_3+2q_0q_1\\ 2q_1q_3-2q_0q_2 & -2q_0q_1+2q_2q_3 & -(q_0)^2+(q_1)^2-(q_2)^2+(q_3)^2\end{bmatrix} \tag{9}$$

All columns of this matrix expressed in this form are orthogonal but not orthonormal. This matrix form is a special orthogonal group $SO(1,2)$. On the other hand, the matrix R can be obtained by the unit split quaternions q and $-q$. There are two unit timelike quaternions for every rotation in Minkowski 3-space. These timelike quaternions are q and $-q$. For this reason, a timelike quaternion R_q can be supposed as a 3×3 dimensional orthogonal rotation matrix.

The equations obtained as a result of this coincidence are quaternion valued linear equations. If we derive the column equation of (9), respectively, then we obtain the following results:

$$d\overrightarrow{T}=2\begin{bmatrix}q_0 & q_1 & q_2 & q_3\\ q_3 & q_2 & q_1 & q_0\\ -q_2 & q_3 & -q_0 & q_1\end{bmatrix}\begin{bmatrix}dq_0\\ dq_1\\ dq_2\\ dq_3\end{bmatrix}=2[A][q']$$

$$d\overrightarrow{N}=2\begin{bmatrix}-q_3 & q_2 & q_1 & -q_0\\ -q_0 & q_1 & q_2 & -q_3\\ -q_1 & -q_0 & q_3 & q_2\end{bmatrix}\begin{bmatrix}dq_0\\ dq_1\\ dq_2\\ dq_3\end{bmatrix}=2[B][q'] \tag{10}$$

$$d\overrightarrow{B}=2\begin{bmatrix}q_2 & q_3 & q_0 & q_1\\ q_1 & q_0 & q_3 & q_2\\ -q_0 & q_1 & -q_2 & q_3\end{bmatrix}\begin{bmatrix}dq_0\\ dq_1\\ dq_2\\ dq_3\end{bmatrix}=2[C][q']$$

4. Serret-Frenet Frames of Split Quaternions

In this section, we give the Serret-Frenet Frame equations for split quaternions. If we calculate the differential equations corresponding to Serret-Frenet Frames with split quaternions, we can obtain the following differential equations. These equations are the formulas Serret-Frenet frames with split quaternions.

$$2[A][q']=\overrightarrow{T'}=\nu\kappa\overrightarrow{N'} \tag{11}$$

$$2[B][q']=\overrightarrow{N'}=-\nu\kappa\overrightarrow{T'}+\nu\tau\overrightarrow{T'} \tag{12}$$

$$2[C][q']=\overrightarrow{B'}=-\nu\tau\overrightarrow{N'} \tag{13}$$

where

$$[q']=\begin{bmatrix}da_0\\ da_1\\ da_2\\ da_3\end{bmatrix}=\begin{bmatrix}b_0 & b_1 & b_2 & b_3\\ c_0 & c_1 & c_2 & c_3\\ d_0 & d_1 & d_2 & d_3\\ e_0 & e_1 & e_2 & e_3\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix} \tag{14}$$

Therefore, with using (11), (12), and (13) we obtain the H' split quaternion Frenet frame equations as [8]

$$[q']=\begin{bmatrix}da_0\\ da_1\\ da_2\\ da_3\end{bmatrix}=\frac{\nu}{2}\begin{bmatrix}0 & -\tau & 0 & -\kappa\\ \tau & 0 & \kappa & 0\\ 0 & \kappa & 0 & \tau\\ -\kappa & 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix} \tag{15}$$

5. Serret-Frenet Frames of Fuzzy Split Quaternions

In this section, we study obtaining the Frenet frame equations with split quaternions in the fuzzy space. For this, firstly we define a fuzzy real set and fuzzy real numbers.

Definition 6. The real number's set is denoted by R and let H be a set of quaternion numbers. A fuzzy real set is a function $\overline{A}:R\to[0,1]$.

A fuzzy real set $\overline{A}$ is a fuzzy real numbers set $\Leftrightarrow$.

(i) $\overline{A}$ is normal, i.e., there exists $x\in R$ whose $\overline{A}=1$.

(ii) For all $\alpha\in(0,1]$, the set $\overline{A}[\alpha]=\{x\in R:\overline{A}(x)\geq\alpha\}$ is a limited set.

The set of all fuzzy real numbers is denoted by R_F. We can see that $R\subset R_F$, since every $\alpha\in R$ can be written as $\alpha:R\to[0,1]$, where $\alpha(x)=1$ if $x=\alpha$ and $\alpha(x)=0$ if $x\neq\alpha$. [3]

Now, we define fuzzy numbers with quaternionic forms.

Definition 7. A fuzzy quaternion number is defined by a function $h:\mathrm{H}\to[0,1]$, where $h(a_01+a_1i+a_2j+a_3k)=\min\{\overline{A}_0(a_0),\overline{A}_1(a_1),\overline{A}_2(a_2),\overline{A}_3(a_3)\}$, for $\overline{A}_0,\overline{A}_1,\overline{A}_2,\overline{A}_3\in R_F$ [3].

Similarly, a fuzzy split quaternion number is given by $h':\mathrm{H}'\to[0,1]$ such that $h'(a_01+a_1i+a_2j+a_3k)=\min\{\overline{A}_0(a_0),\overline{A}_1(a_1),\overline{A}_2(a_2),\overline{A}_3(a_3)\}$, for $\overline{A}_0,\overline{A}_1,\overline{A}_2,\overline{A}_3\in R_F$.

The fuzzy quaternion number's set is denoted by H_F and the set of all fuzzy split quaternion numbers is denoted by H'_F and identified as R^4_F, where every element h' is associated with $(\overline{A},\overline{B},\overline{C},\overline{D})$.

We can define the fuzzy split quaternion numbers as follows:

$h' = (\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3) \in H'_F$, where $Re(h') = \overline{A}_0$ is called the real part and $Im1(h') = \overline{A}_1$, $Im2(h') = \overline{A}_2$, $Im3(h') = \overline{A}_3$ are called imaginary parts.

Let $h = a_0 1 + a_1 i + a_2 j + a_3 k \in H'$ and the function $h' : H' \to [0,1]$ is given by

$$h'(b_0 1 + b_1 i + b_2 j + b_3 k) = \begin{cases} 1, & \text{if } a_0 = b_0 \text{ and } a_1 = b_1 \text{ and } a_2 = b_2 \text{ and } a_3 = b_3 \\ 0, & \text{if } a_0 \neq b_0 \text{ or } a_1 \neq b_1 \text{ or } a_2 \neq b_2 \text{ or } a_3 \neq b_3 \end{cases} \tag{16}$$

Definition 8. In the fuzzy split quaternion numbers H'_F, we can define the addition and multiplication operations as follows [3].

Let $s', h' \in H'_F$, where $s' = (\overline{B}_0, \overline{B}_1, \overline{B}_2, \overline{B}_3)$ and $h' = (\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3)$; then,

$$\begin{aligned} s' + h' &= \left(\overline{B}_0 + \overline{A}_0, \overline{B}_1 + \overline{A}_1, \overline{B}_2 + \overline{A}_2, \overline{B}_3 + \overline{A}_3\right) \\ s'.h' &= \left(\overline{B}_0\overline{A}_0 - \overline{B}_1\overline{A}_1 + \overline{B}_2\overline{A}_2 + \overline{B}_3\overline{A}_3,\ \overline{B}_0\overline{A}_1 + \overline{B}_1\overline{A}_0 \right. \\ &\quad - \overline{B}_2\overline{A}_3 + \overline{B}_3\overline{A}_2,\ \overline{B}_0\overline{A}_2 + \overline{B}_1\overline{A}_3 + \overline{B}_2\overline{A}_0 \\ &\quad \left. - \overline{B}_3\overline{A}_1,\ \overline{B}_0\overline{A}_3 + \overline{B}_1\overline{A}_2 + \overline{B}_2\overline{A}_1 - \overline{B}_3\overline{A}_0\right) \end{aligned} \tag{17}$$

Definition 9. Let R be the field of real numbers and (R, τ) be a fuzzy topological vector space over the field R.

$f : R \to R, a \in R$; the function f is said to be fuzzy differentiable at the point a if there is a function ϕ that is fuzzy continuous at the point a and have

$$f(x) - f(a) = \phi(x)(x - a) \tag{18}$$

for all $x \in R$. $\phi(a)$ is said to be fuzzy derivative of f at and denote

$$f'(a) = \phi(a) \tag{19}$$

[10].

Definition 10. Let $h' = (\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3)$; the conjugate of h' is defined as

$$\overline{h'} = \left(\overline{A}_0, -\overline{A}_1, -\overline{A}_2, -\overline{A}_3\right) \tag{20}$$

The norm of h' is defined as

$$N_{h'} = \overline{h'}h' = h'\overline{h'} = \left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 - \left(\overline{A}_2\right)^2 - \left(\overline{A}_3\right)^2 \tag{21}$$

Because of $H' \subset H'_F$, the following equation can be written:

$$\left[h'.w'.\overline{h'}\right]_i = \sum_{j=1}^{3} R_{ij} w_j \tag{22}$$

where $w' = (V_1, V_2, V_3)$.

Here, R_{ij} is the component of the matrix R and the matrix is calculated from (17) as follows:

$$R = \begin{bmatrix} \left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 + \left(\overline{A}_2\right)^2 + \left(\overline{A}_3\right)^2 & 2\overline{A}_1\overline{A}_2 - 2\overline{A}_0\overline{A}_3 & 2\overline{A}_0\overline{A}_2 + 2\overline{A}_1\overline{A}_3 \\ 2\overline{A}_0\overline{A}_3 + 2\overline{A}_1\overline{A}_2 & -\left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 + \left(\overline{A}_2\right)^2 - \left(\overline{A}_3\right)^2 & 2\overline{A}_2\overline{A}_3 + 2\overline{A}_0\overline{A}_1 \\ 2\overline{A}_1\overline{A}_3 - 2\overline{A}_0\overline{A}_2 & -2\overline{A}_0\overline{A}_1 + 2\overline{A}_2\overline{A}_3 & -\left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 - \left(\overline{A}_2\right)^2 + \left(\overline{A}_3\right)^2 \end{bmatrix} \tag{23}$$

In this matrix (23), we calculate the derivative of the columns, respectively, to the elements $\overline{A}_0, \overline{A}_1, \overline{A}_2$, and $\overline{A}_3$. We will get the Fuzzy tangent vector $\overrightarrow{T'}$ to the derivation from the first column to the elements $\overline{A}_0, \overline{A}_1, \overline{A}_2$, and $\overline{A}_3$:

$$\begin{aligned} \overrightarrow{T'} = d\overrightarrow{T} &= 2\begin{bmatrix} \overline{A}_0 & \overline{A}_1 & \overline{A}_2 & \overline{A}_3 \\ \overline{A}_3 & \overline{A}_2 & \overline{A}_1 & \overline{A}_0 \\ -\overline{A}_2 & \overline{A}_3 & -\overline{A}_0 & \overline{A}_1 \end{bmatrix} \begin{bmatrix} d\overline{A}_0 \\ d\overline{A}_1 \\ d\overline{A}_2 \\ d\overline{A}_3 \end{bmatrix} \\ &= 2[X]\left[d\left(h'\right)\right] \end{aligned} \tag{24}$$

We will get the fuzzy normal vector $\overrightarrow{N'}$ to the derivation from the second column to the elements $\overline{A}_0, \overline{A}_1, \overline{A}_2$, and $\overline{A}_3$:

$$\begin{aligned} \overrightarrow{N'} = d\overrightarrow{N} &= 2\begin{bmatrix} -\overline{A}_3 & \overline{A}_2 & \overline{A}_1 & -\overline{A}_0 \\ -\overline{A}_0 & \overline{A}_1 & \overline{A}_2 & -\overline{A}_3 \\ -\overline{A}_1 & -\overline{A}_0 & \overline{A}_3 & \overline{A}_2 \end{bmatrix} \begin{bmatrix} d\overline{A}_0 \\ d\overline{A}_1 \\ d\overline{A}_2 \\ d\overline{A}_3 \end{bmatrix} \\ &= 2[Y]\left[d\left(h'\right)\right] \end{aligned} \tag{25}$$

We will get the fuzzy binormal vector $\overrightarrow{B'}$ to the derivation from the third column to the elements $\overline{A}_0, \overline{A}_1, \overline{A}_2$, and $\overline{A}_3$:

$$\overrightarrow{B'} = d\overrightarrow{B} = 2\begin{bmatrix} \overline{A}_2 & \overline{A}_3 & \overline{A}_0 & \overline{A}_1 \\ \overline{A}_1 & \overline{A}_0 & \overline{A}_3 & \overline{A}_2 \\ -\overline{A}_0 & \overline{A}_1 & -\overline{A}_2 & \overline{A}_3 \end{bmatrix} \begin{bmatrix} d\overline{A}_0 \\ d\overline{A}_1 \\ d\overline{A}_2 \\ d\overline{A}_3 \end{bmatrix}$$

$$= 2\,[Z]\left[d\left(h'\right)\right] \tag{26}$$

If we write, respectively, these founded matrices in (11), (12), and (13), we can obtain the following equalities for Serret-Frenet frame equations:

$$2\,[X]\left[d\left(h'\right)\right] = \overrightarrow{T'} = \nu\kappa\overrightarrow{N'} \tag{27}$$

$$2\,[Y]\left[d\left(h'\right)\right] = \overrightarrow{N'} = -\nu\kappa\overrightarrow{T'} + \nu\tau\overrightarrow{T'} \tag{28}$$

$$2\,[Z]\left[d\left(h'\right)\right] = \overrightarrow{B'} = -\nu\tau\overrightarrow{N'} \tag{29}$$

The differential of fuzzy split quaternion h' is expressed with matrix form as follows:

$$\left[d\left(h'\right)\right] = \begin{bmatrix} d\overline{A}_0 \\ d\overline{A}_1 \\ d\overline{A}_2 \\ d\overline{A}_3 \end{bmatrix} = \begin{bmatrix} \overline{B}_0 & \overline{B}_1 & \overline{B}_2 & \overline{B}_3 \\ \overline{C}_0 & \overline{C}_1 & \overline{C}_2 & \overline{C}_3 \\ \overline{D}_0 & \overline{D}_1 & \overline{D}_2 & \overline{D}_3 \\ \overline{E}_0 & \overline{E}_1 & \overline{E}_2 & \overline{E}_3 \end{bmatrix} \begin{bmatrix} \overline{A}_0 \\ \overline{A}_1 \\ \overline{A}_2 \\ \overline{A}_3 \end{bmatrix} \tag{30}$$

Here, $(\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3)$ is the real and imaginary elements of the fuzzy split quaternionic vector. Now, we must need to calculate the elements $\overline{B}_i, \overline{C}_i, \overline{D}_i, \overline{E}_i$, $(0 \le i \le 3)$ of the coefficient matrix. We need solutions of (27), (28), and (29) to obtain the elements $\overline{B}_i, \overline{C}_i, \overline{D}_i, \overline{E}_i$, $(0 \le i \le 3)$. For this reason, we put the differential of fuzzy split quaternion h', fuzzy tangent vector $\overrightarrow{T'}$, fuzzy normal vector $\overrightarrow{N'}$, and fuzzy binormal vector $\overrightarrow{B'}$ in (27), (28), and (29) in its places. When we make the needed calculations, we can obtain the following results:

$$\begin{aligned} &\overline{B}_0\overline{A}_0\overline{A}_3 + \overline{B}_1\overline{A}_1\overline{A}_3 + \overline{B}_2\overline{A}_2\overline{A}_3 + \overline{B}_3\left(\overline{A}_3\right)^2 + \overline{C}_0\overline{A}_0\overline{A}_2 \\ &\quad + \overline{C}_1\overline{A}_1\overline{A}_2 + \overline{C}_2\left(\overline{A}_2\right)^2 + \overline{C}_3\overline{A}_2\overline{A}_3 + \overline{D}_0\overline{A}_0\overline{A}_1 \\ &\quad + \overline{D}_1\left(\overline{A}_1\right)^2 + \overline{D}_2\overline{A}_1\overline{A}_2 + \overline{D}_3\overline{A}_1\overline{A}_3 + \overline{E}_0\left(\overline{A}_0\right)^2 \\ &\quad + \overline{E}_1\overline{A}_0\overline{A}_1 + \overline{E}_2\overline{A}_0\overline{A}_2 + \overline{E}_3\overline{A}_0\overline{A}_3 \\ &\quad = \frac{\nu}{2}\kappa\left(\left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 + \left(\overline{A}_2\right)^2 - \left(\overline{A}_3\right)^2\right) \end{aligned} \tag{31}$$

$$\begin{aligned} &-\overline{B}_0\overline{A}_0\overline{A}_3 - \overline{B}_1\overline{A}_1\overline{A}_3 - \overline{B}_2\overline{A}_2\overline{A}_3 - \overline{B}_3\left(\overline{A}_3\right)^2 \\ &\quad + \overline{C}_0\overline{A}_0\overline{A}_2 + \overline{C}_1\overline{A}_1\overline{A}_2 + \overline{C}_2\left(\overline{A}_2\right)^2 + \overline{C}_3\overline{A}_2\overline{A}_3 \\ &\quad + \overline{D}_0\overline{A}_0\overline{A}_1 + \overline{D}_1\left(\overline{A}_1\right)^2 + \overline{D}_2\overline{A}_1\overline{A}_2 + \overline{D}_3\overline{A}_1\overline{A}_3 \\ &\quad - \overline{E}_0\left(\overline{A}_0\right)^2 - \overline{E}_1\overline{A}_0\overline{A}_1 - \overline{E}_2\overline{A}_0\overline{A}_2 - \overline{E}_3\overline{A}_0\overline{A}_3 \\ &\quad = -\frac{\nu}{2}\kappa\left(\left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 + \left(\overline{A}_2\right)^2 + \left(\overline{A}_3\right)^2\right) \\ &\quad + \frac{\nu}{2}\tau\left(2\overline{A}_0\overline{A}_2 + 2\overline{A}_1\overline{A}_3\right) \end{aligned} \tag{32}$$

$$\begin{aligned} &-\overline{B}_0\overline{A}_0\overline{A}_1 - \overline{B}_1\left(\overline{A}_1\right)^2 - \overline{B}_2\overline{A}_1\overline{A}_2 - \overline{B}_3\overline{A}_1\overline{A}_3 \\ &\quad - \overline{C}_0\left(\overline{A}_2\right)^2 - \overline{C}_1\overline{A}_0\overline{A}_1 - \overline{C}_2\overline{A}_0\overline{A}_2 - \overline{C}_3\overline{A}_1\overline{A}_3 \\ &\quad + \overline{D}_0\overline{A}_0\overline{A}_3 + \overline{D}_1\overline{A}_1\overline{A}_3 + \overline{D}_2\overline{A}_1\overline{A}_3 + \overline{D}_3\left(\overline{A}_3\right)^2 \\ &\quad + \overline{E}_0\overline{A}_0\overline{A}_2 + \overline{E}_1\overline{A}_1\overline{A}_2 + \overline{E}_2\left(\overline{A}_2\right)^2 + \overline{E}_3\overline{A}_2\overline{A}_3 \\ &\quad = -\frac{\nu}{2}\kappa\left(2\overline{A}_1\overline{A}_3 - 2\overline{A}_0\overline{A}_2\right) \\ &\quad + \frac{\nu}{2}\tau\left(-\left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 - \left(\overline{A}_2\right)^2 + \left(\overline{A}_3\right)^2\right) \end{aligned} \tag{33}$$

$$\begin{aligned} &\overline{B}_0\overline{A}_0\overline{A}_1 + \overline{B}_1\left(\overline{A}_1\right)^2 + \overline{B}_2\overline{A}_1\overline{A}_2 + \overline{B}_3\overline{A}_1\overline{A}_3 + \overline{C}_0\left(\overline{A}_2\right)^2 \\ &\quad + \overline{C}_1\overline{A}_0\overline{A}_1 + \overline{C}_2\overline{A}_0\overline{A}_2 + \overline{C}_3\overline{A}_1\overline{A}_3 + \overline{D}_0\overline{A}_0\overline{A}_3 \\ &\quad + \overline{D}_1\overline{A}_1\overline{A}_3 + \overline{D}_2\overline{A}_1\overline{A}_3 + \overline{D}_3\left(\overline{A}_3\right)^2 + \overline{E}_0\overline{A}_0\overline{A}_2 \\ &\quad + \overline{E}_1\overline{A}_1\overline{A}_2 + \overline{E}_2\left(\overline{A}_2\right)^2 + \overline{E}_3\overline{A}_2\overline{A}_3 \\ &\quad = -\frac{\nu}{2}\tau\left(-\left(\overline{A}_0\right)^2 + \left(\overline{A}_1\right)^2 + \left(\overline{A}_2\right)^2 - \left(\overline{A}_3\right)^2\right) \end{aligned} \tag{34}$$

Finally, we get results for the elements $\overline{B}_i, \overline{C}_i, \overline{D}_i, \overline{E}_i$, $(0 \le i \le 3)$ as follows:

$$\begin{aligned} \overline{B}_0 &= 0, \\ \overline{B}_1 &= -\frac{\nu\tau}{2}, \\ \overline{B}_2 &= 0, \\ \overline{B}_3 &= -\frac{\nu\kappa}{2} \\ \overline{C}_0 &= \frac{\nu\tau}{2}, \\ \overline{C}_1 &= 0, \\ \overline{C}_2 &= \frac{\nu\kappa}{2}, \\ \overline{C}_3 &= 0 \\ \overline{D}_0 &= 0, \\ \overline{D}_1 &= \frac{\nu\tau}{2}, \\ \overline{D}_2 &= 0, \\ \overline{D}_3 &= \frac{\nu\kappa}{2} \end{aligned}$$

$$\begin{aligned}\overline{E}_0 &= -\frac{\nu\tau}{2},\\ \overline{E}_1 &= 0,\\ \overline{E}_2 &= -\frac{\nu\kappa}{2},\\ \overline{E}_3 &= 0\end{aligned} \tag{35}$$

Therefore, by using these values (35) we obtain the fuzzy split quaternionic Serret-Frenet frame equation as

$$\left[d\left(h'\right)\right] = \begin{bmatrix} d\overline{A}_0 \\ d\overline{A}_1 \\ d\overline{A}_2 \\ d\overline{A}_3 \end{bmatrix} = \frac{\nu}{2}\begin{bmatrix} 0 & \tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & \kappa & 0 & \tau \\ -\kappa & 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \overline{A}_0 \\ \overline{A}_1 \\ \overline{A}_2 \\ \overline{A}_3 \end{bmatrix} \tag{36}$$

6. Conclusion and Discussion

In this study, we redefined the algebraic operations for split quaternions on fuzzy split quaternions. The set of split quaternions is a subset of fuzzy split quaternions ($H' \subset H'_F$). This condition is important because the given definitions for fuzzy split quaternions are provided with it. As a result of this, given definitions are similar to definitions for split quaternions. We have seen that these definitions are similar to the split quaternion structures. We have obtained in this study fuzzy tangent vector $\overrightarrow{T'}$, fuzzy normal vector $\overrightarrow{N'}$, and fuzzy binormal vector $\overrightarrow{B'}$. These vector forms are a new description and calculation. Also, we have redefined these Serret-Frenet frames for fuzzy split quaternions on familiar Serret-Frenet frames. For fuzzy quaternionic forms the torsion and curvature functions are defined as

$$\begin{aligned}\tau &: I \subset R \longrightarrow [0,1]\\ \kappa &: I \subset R \longrightarrow [0,1]\end{aligned} \tag{37}$$

For this reason, Serret-Frenet frame elements in (36) for fuzzy split quaternions get values in the range $[-1,1]$. In Definition 7, we can see that if we take equal fuzzy split quaternion to the split quaternion, the function $h' \in H'$ can take the value $'1'$ and if we take not equal fuzzy split quaternion to the split quaternion, the function h' can take the value $'0'$. Hence, for calculating (27), (28), and (29), the necessary rule is

$$h'\left(b_0 1 + b_1 i + b_2 j + b_3 k\right) = 1 \tag{38}$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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