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Approximation of functions by de la Vallée-Poussin sums in weighted Orlicz spaces

Abstract We investigate problems of estimating the deviation of functions from their de la Vallée-Poussin sums in weighted Orlicz spaces $L_M(T, \omega)$ in terms of the best approximation $E_n(f)_{M,\omega}$.

Mathematics Subject Classification   30E10 · 41A10 · 41A25 · 46E30

الملخص

نبحث في مسائل تقريب انحراف الدوال عن مجاميع فالي – بوسن الخاصة بها في فضاءات أورلتز الموزونة (ω, T)L_M بدلالة أفضل تقريب $E_n(f)_{M,\omega}$.

1 Introduction, some auxiliary results and main results

Let $M(u)$ be a continuous increasing convex function on $[0, \infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary of M in Young's sense*, i.e., $N(u) = \max\{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$ with some constant c, independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi, \pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\, \mathrm{d}x < \infty.$$

Let N be the complementary Young function of M. It is well-known [28, p.69], [47, pp.52–68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{ \int_{\mathbb{T}} |f(x)g(x)|\, \mathrm{d}x : g \in \widetilde{L}_N(\mathbb{T}), \ \int_{\mathbb{T}} N(|g(x)|)\, \mathrm{d}x \leq 1 \right\},$$

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or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf \left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \le 1 \right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [28, p.26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$. If $M(x) = M(x, p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coincide with the usual Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics. Therefore, the investigation into the approximation of the functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm as

$$\|f\|^*_{L_M(\mathbb{T})} \le \|f\|_{L_M(\mathbb{T})} \le 2 \|f\|^*_{L_M(\mathbb{T})}, \quad f \in L_M(\mathbb{T})$$

holds true [28, p.80].

If we choose $M(u) = u^p/p$, $1 < p < \infty$, then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p} \|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \le \|u\|_{L_M(\mathbb{T})} \le q^{1/q} \|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int_{\mathbb{T}} |u(x)|^p dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$ space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$, then the so-called strong Hölder inequalities [28, p.80]

$$\int_{\mathbb{T}} |f(x)g(x)| dx \le \|f\|_{L_M(\mathbb{T})} \|g\|^*_{L_N(\mathbb{T})},$$

$$\int_{\mathbb{T}} |f(x)g(x)| dx \le \|f\|^*_{L_M(\mathbb{T})} \|g\|_{L_N(\mathbb{T})}$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [47, p.113].

Let $M^{-1} : [0, \infty) \to [0, \infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* [4, p.350]

$$\alpha_M := \lim_{t \to +\infty} -\frac{\log h(t)}{\log t}, \quad \beta_M := \lim_{t \to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h : (0, \infty) \to (0, \infty], \quad h(t) := \lim_{y \to \infty} \sup \frac{M^{-1}(y)}{M^{-1}(ty)}, \quad t > 0$$

first considered by Matuszewska and Orlicz [38], are called the *Boyd indices* of the Orlicz spaces $L_M(T)$.

It is known that the indices α_M and β_M satisfy $0 \le \alpha_M \le \beta_M \le 1$, $\alpha_N + \beta_M = 1$, $\alpha_M + \beta_N = 1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0 < \alpha_M \le \beta_M < 1$. The detailed information about the Boyd indices can be found in [3,5–7,39].

A measurable function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function* if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero. With any given weight ω we associate the *ω-weighted Orlicz space* $L_M(\mathbb{T}, \omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T}, \omega)} := \|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's A_p-condition* on $\mathbb{T}$ if

$$\sup_J \left(\frac{1}{|J|} \int_J \omega^p(t)\,\mathrm{d}t \right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-p'}(t)\,\mathrm{d}t \right)^{1/p'} < \infty,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us indicate by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

According to [33,34, Lemma 3.3], and [34, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and ω weight function satisfying the condition $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T}, \omega)$ is also reflexive.

Let $L_M(\mathbb{T}, \omega)$ be a weighted Orlicz space, let $0 < \alpha_M \leq \beta_M < 1$ and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f \in L_M(\mathbb{T}, \omega)$ we set

$$(\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\,\mathrm{d}t, \ 0 < h < \pi, \ x \in T.$$

By reference [18], Lemma 1.4, the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T}, \omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\omega)} \leq c \|f\|_{L_M(\mathbb{T},\omega)}.$$

The function

$$\Omega_{M,\omega}^l(\delta, \ f) := \sup_{\substack{0 < h_i \leq \delta \\ 1 \leq i \leq l}} \left\| \prod_{i=1}^{l} (I - \nu_{h_i}) f \right\|_{L_M(\mathbb{T},\omega)}, \quad \delta > 0, \quad l = 1, 2, \ldots$$

is called *k-th modulus of smoothness* of $f \in L_M(\mathbb{T}, \omega)$, where I is the identity operator.

It can easily be shown that $\Omega_{M,\omega}^k(\cdot, \ f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta \to 0} \Omega_{M,\omega}^k(\delta, \ f) = 0, \quad \Omega_{M,\omega}^k(\delta, \ f+g) \leq \Omega_{M,\omega}^k(\delta, \ f) + \Omega_{M,\omega}^k(\delta, g)$$

for $f, \ g \in L_M(\mathbb{T}, \omega)$.

The *function conjugate* to a 2π−periodic summable function on $[-\pi, \pi]$ given by

$$\widetilde{f}(x) = \lim_{\varepsilon \to +0} \left\{ -\frac{1}{\pi} \int_\varepsilon^\pi \frac{f(x+t) - f(x-t)}{2 \tan \frac{t}{2}}\,\mathrm{d}t \right\} = -\frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan \frac{t}{2}}\,\mathrm{d}t$$

exists almost everywhere.

Let

$$\frac{a_0}{2} + \sum_{k=1}^\infty A_k(x, \ f) \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $A_k(x, \ f) := (a_k(f) \cos kx + b_k(f) \sin kx)$, $k = 1, 2, \ldots$, $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$. For given $f \in L_1(\mathbb{T})$, let

$$\widetilde{f} \sim \sum_{k=1}^\infty (a_k(f) \sin kx - b_k(f) \cos kx) = \sum_{k=-\infty}^\infty \{-i\,signk\} c_k(f) e^{ikx}$$

be the *conjugate Fourier series* of f with $c_k(f) = (1/2)(a_k(f) - ib_k(f))$. It is known that the conjugate series to Fourier series $f \in L_{[0,2\pi]}$ will not always be the Fourier series (see, e.g., [53, p.155]).

The *n-th partial sums*, and *de la Vallé e-Poussin sums* [57] of series (1.1) are defined, respectively, as

$$S_n(x, \ f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x, \ f),$$

$$V_{n,m}(x, \ f) = \frac{1}{m+1} \sum_{\nu=n-m}^{n} S_\nu(x, \ f), \quad (0 \le m \le n, \ m, n \in Z_+ := \{1, 2, 3, \ldots\}).$$

The best approximation to $f \in L_M(\mathbb{T}, \omega)$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{M,\omega} := \inf \left\{ \|f - T_n\|_{L_M(\mathbb{T}, \omega)} : T_n \in \prod_n \right\}.$$

Note that the existence of $T_n^* \in \Pi_n$ such that

$$E_n(f)_{M,\omega} = \left\| f - T_n^* \right\|_{L_M(\mathbb{T}, \omega)}$$

follows, for example, from Theorem 1.1 in [10, p.59].

Let $W_M^r(\mathbb{T}, \omega)$, $(r = 1, 2, \ldots)$ be the class of functions such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_M(\mathbb{T}, \omega)$ becomes a Banach space under the consideration of the norm

$$\| f \|_{W_M(\mathbb{T}, \omega)} := \| f \|_{L_M(\mathbb{T}, \omega)} + \| f^{(r)} \|_{L_M(\mathbb{T}, \omega)}.$$

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := \text{ext}\Gamma$. Further let

$$T := \{ w \in \mathbb{C} : |w| = 1 \}, \quad D := \text{int } T \text{ and } D^- := \text{ext } T.$$

Let $w = \varphi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0,$$

and ψ stands for the inverse of φ.

Let $w = \varphi_1(z)$ indicate a function that maps the domain G conformally onto the disk $|w| < 1$. The inverse mapping of φ_1 will be shown by ψ_1. Let Γ_r be the image of the circle $|\varphi_1(z)| = r$, $0 < r < 1$ under the mapping $z = \psi_1(w)$.

Let us denote by E_p, where $p > 0$, the class of all functions $f(z) \neq 0$ that are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p \, |dz|$$

is uniformly bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class*. If the function $f(z)$ belongs to E_p, then $f(z)$ has definite limiting values $f(z')$ almost every where on Γ, over all nontangential paths; $\left| f(z') \right|$ is summable on Γ; and

$$\lim_{r \to 1} \int_{\Gamma_r} |f(z)|^p \, |dz| = \int_{\Gamma} \left| f(z') \right|^p \, |dz'|.$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [12, pp.438–453], and [8, pp.168–185].

Let $L_M(\Gamma, \omega)$ be a weighted Orlicz space defined on Γ. We also define the *ω-weighted Smirnov–Orlicz class* $E_M(G, \omega)$ as

$$E_M(G, \omega) := \{ f \in E_1(G) : f \in L_M(\Gamma, \omega) \}.$$

With every weight function ω on Γ, we associate another weight ω_0 on T defined by

$$\omega_0(t) := \omega(\psi(t)), \quad t \in T.$$

For $f \in L_M(\Gamma, \omega)$ we define the function

$$f_0(t) := f(\psi(t)), \quad t \in T.$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \le t\}, \quad t \ge 0.$$

The curve Γ is called *Dini-smooth* if it has a parameterization

$$\Gamma : \varphi_0(s), \quad 0 \le s \le 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.,

$$\int_0^\pi \frac{\omega(t, \varphi_0')}{t} dt < \infty$$

and $\varphi_0'(s) \ne 0$ [44, p.48].

If Γ is a Dini-smooth curve, then there exist [58] the constants c_1 and c_2 such that

$$0 \le c_1 \le |\psi'(t)| \le c_2 < \infty, \quad |t| > 1. \tag{1.2}$$

Note that if Γ is a Dini-smooth curve, then by (1.2) we have $f_0 \in L_M(\mathbb{T}, \omega_0)$ if $f \in L_M(\Gamma, \omega)$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p-condition on Γ if

$$\sup_{z \in \Gamma} \sup_{r > 0} \left(\frac{1}{r} \int_{\Gamma \cap D(z,r)} |\omega(\tau)|^p |d\tau| \right)^{1/p} \left(\frac{1}{r} \int_{\Gamma \cap D(z,r)} [\omega(\tau)]^{-p'} |d\tau| \right)^{1/p'} < \infty,$$

where $D(z, r)$ is an open disk with radius r and centered z.

Let us denote by $A_p(\Gamma)$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on Γ. For a detailed discussion of Muckenhoupt weights on curves (see, e.g., [4]).

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then, the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(s)ds}{s - z}, \quad z \in G$$

is analytic in G. Note that if $0 < \alpha_M \le \beta_M < 1$, $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$ and $f \in L_M(\Gamma, \omega)$, then by Lemma 1.4 in [20] $f^+ \in E_M(G, \omega)$.

Let $\varphi_k(z)$, $k = 0, 1, 2, \ldots$ be the Faber polynomials for G. The Faber polynomials $\varphi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^\infty \frac{\varphi_k(z)}{t^{k+1}}, \quad z \in G, \quad t \in D^- \tag{1.3}$$

and the equalities

$$\varphi_k(z) = \frac{1}{2\pi i} \int_T \frac{w^k \psi'(w)}{\psi(w) - z} dw, \quad z \in G, \tag{1.4}$$

$$\varphi_k(z) = \varphi^k(z) + \frac{1}{2\pi i} \int_\Gamma \frac{\varphi^k(s)}{s - z} ds, \quad z \in G^-, \quad k = 0, 1, 2, \ldots \tag{1.5}$$

hold [51, p.33–38].

Let $f \in E_M(G, \omega)$. Since $f \in E_1(G)$, we have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)ds}{s - z} = \frac{1}{2\pi i} \int_T \frac{f(\psi(w))\psi'(w)}{\psi(w) - z} dw,$$

for every $z \in G$. Considering this formula and expansion (1.3), we can associate with f the *Faber* series

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\varphi_k(z), \quad z \in G , \qquad (1.6)$$

where

$$a_k(f) := \frac{1}{2\pi i} \int_T \frac{f(\psi(w))}{w^{k+1}} \mathrm{d}w, \quad k = 0, 1, 2, \ldots$$

This series is called the *Faber series* expansion *of* f, and the coefficients $a_k(f)$, $k = 0, 1, 2, \ldots$ are said to be the *Faber coefficients* of f.

The *n-th partial sums and de la Vallée-Poussin sums of the series* (1.6) are defined, respectively, as

$$S_n(z, f) = \sum_{k=o}^{n} a_k(f)\varphi_k(z) ,$$

$$V_{n,m}(z, f) = \frac{1}{m+1} \sum_{v=n-m}^{n} S_v(z, f), \quad (0 \le m \le n, \; m, n \in Z_+) .$$

Let Γ be a Dini-smooth curve. Using the nontangential boundary values of f_0^+ on T we define the r-th modulus of smoothness of $f \in L_M(\Gamma, \omega)$ as

$$\Omega_{\Gamma, M, \omega}^{l}(\delta, f) := \Omega_{M,\omega_0}^{l}(\delta, f_0^+), \quad \delta > 0,$$

for $l = 1, 2, 3, \ldots$

Let $\mathcal{P} := \{$all polynomials (with no restriction on the degree)$\}$, and let $\mathcal{P}(D)$ be the set of traces of members of $\mathcal{P}$ on D. We define the operator T as follows:

$$T := \mathcal{P}(D) \longrightarrow E_M(G, \omega) ,$$

$$T(P)(z) := \frac{1}{2\pi i} \int_T \frac{P(w)\psi'(w)}{\psi(w) - z} \mathrm{d}t, \quad z \in G.$$

Then, taking into account (1.4) and (1.5) we have

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k \varphi_k(z), \quad z \in G.$$

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest

We need the following results.

Theorem 1.1 [18] *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then for every*
    $f \in W_M^r(T, \omega)$ $(r = 0, 1, 2, \ldots)$ *the inequality*

$$E_n(f)_{M,\omega} \le \frac{c_3}{(n+1)^r} E_n(f^{(r)})_{M,\omega}$$

holds with a constant $c_3 > 0$ independent of n.

Theorem 1.2 [18] *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then for every*
    $f \in L_M(T, \omega)$ *the inequality*

$$E_n(f)_{M,\omega} \le c_4 \Omega_{M,\omega}^{l}\left(\frac{1}{n+1}, f\right)$$

holds with a constant $c_4 > 0$ independent of n.

Theorem 1.3 [26] *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and let $f \in L_M(T, \omega)$, $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. If the condition*

$$\sum_{n=1}^{\infty} n^{r-1} E_n(f)_{M,\omega.} < \infty$$

is satisfied for $r \in Z_+$, then $\tilde{f}^{(r)} \in L_M(T, \omega)$ and

$$E_n(\tilde{f}^{(r)})_{M,\omega} \le c_5 \left\{ (n+1)^r E_n(f)_{M,\omega.} + \left(\sum_{\mu=n+1}^{\infty} \mu^{r-1} E_\mu(f)_{M,\omega.} \right) \right\},$$

where the constant c_5 is independent of n.

Lemma 1.4 *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$.*

1. *Then, for $r \in N$ and $f^{(r)} \in L_M(T, \omega)$ the estimate*

$$\| f^{(r)} \|_{L_M(\mathbb{T},\omega)} \le c_6 \left\{ n^r \| f \|_{L_M(\mathbb{T},\omega)} + E_n(f^{(r)})_{M,\omega.} \right\} \tag{1.7}$$

   holds with a constant $c_6 > 0$ independent of n.
2. *If $\tilde{f}^{(r)} \in L_M(T, \omega)$, then the estimate*

$$\| \tilde{f}^{(r)} \|_{L_M(\mathbb{T},\omega)} \le c_7 \left\{ n^r \| f \|_{L_M(\mathbb{T},\omega)} + E_n(\tilde{f}^{(r)})_{M,\omega.} \right\} \tag{1.8}$$

   holds with a constant $c_7 > 0$ independent of n.

Proof The function $f^{(r)}$ can be written in following form:

$$f^{(r)}(x) = \left(f^{(r)}(x) - V_{2n,n}(x, \ f^{(r)}) \right) + V_{2n,n}(x, \ f^{(r)}). \tag{1.9}$$

Then, by (1.9) we obtain

$$\| f^{(r)} \|_{L_M(\mathbb{T},\omega)} \le \| f^{(r)} - V_{2n,n}(., \ f^{(r)}) \|_{L_M(\mathbb{T},\omega)} + \| V_{2n,n}(., \ f^{(r)}) \|_{L_M(\mathbb{T},\omega)}. \tag{1.10}$$

By [18] the following inequality holds:

$$\| f - S_n(., \ f) \|_{L_M(\mathbb{T},\omega)} \le c_8 E_n(f)_{M,\omega}. \tag{1.11}$$

Then, from the inequality (1.11) we conclude that

$$\| f^{(r)} - V_{2n,n}(., \ f^{(r)}) \|_{L_M(\mathbb{T},\omega)} \le E_n(f^{(r)})_{M,\omega}. \tag{1.12}$$

Using the Bernstein inequality for weighted Orlicz spaces [18], we have

$$\| V_{2n,n}(., \ f^{(r)}) \|_{L_M(\mathbb{T},\omega)} = \| \frac{d^r}{dx^r} V_{2n,n}(x, \ f) \|_{L_M(\mathbb{T},\omega)}$$
$$\le c_9 (2n)^r \| V_{2n,n}(., \ f) \|_{L_M(\mathbb{T},\omega)} \le c_{10} n^r \| f \|_{L_M(\mathbb{T},\omega)}.$$

Now combining (1.10), (1.12) and last relation, we obtain the inequality (1.7) of Lemma 2.1. The inequality (1.8) is proved to be similar.

The proof of Lemma 1.4 is completed. □

Theorem 1.5 [20] *Let Γ be a Dini-smooth curve and $L_M(\Gamma)$ be a reflexive Orlicz space. If $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, then the linear operator $T := P(D) \longrightarrow E_M(G, \omega)$ is bounded.*

Theorem 1.6 [20] *If* Γ *is a Dini-smooth curve,* $0 < \alpha_M \le \beta_M < 1$, *and* $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, *then the operator*

$$T : E_M(D, \omega_0) \longrightarrow E_M(G, \omega)$$

is one-to-one and onto.

The problems of approximation theory in weighted and nonweighted Lebesgue spaces, weighted and nonweighted Orlicz spaces have been investigated by several authors (see, e.g., [13–27,30–32,36,37,45,46]).

Note that the approximation problems by trigonometric polynomials in weighted Lebesgue spaces with weights belonging to the Muckenhoupt class $A_p(\mathbb{T})$ were studied in [13,36,37]. Detailed information on the weighted polynomial approximation can be found in the books [9,41].

In the present paper, we investigate the problems of estimating the deviation of functions from their de la Vallée-Poussin sums in weighted Orlicz spaces $L_M(T, \omega)$.

This result is applied to estimate of approximation of de la Vallé e-Poussin sums of Faber series in weighted Smirnov–Orlicz classes defined on simply connected domains of the complex plane in terms of the modulus of smoothness. We also study the approximation of conjugate function by de la Vallée-Poussin sums of the Fourier series of the conjugate function in weighted Orlicz spaces $L_M(\mathbb{T}, \omega)$. Note that the estimates obtained in this work depend on sequence of the best approximation $E_n(f)_{M, \omega}$. Similar problems in different spaces have been investigated by several researchers (see, e.g., [1,2,11,27,42,43,48–50,52–57,59,60]).

Our main results are as follows.

Theorem 1.7 *Let* $L_M(T)$ *be a reflexive Orlicz space and* $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. *Then for* $f \in L_M(T, \omega)$, $0 \le m \le n$, $m, n \in Z_+$ *the inequality*

$$\| f - V_{n,m}(., f) \|_{L_M(\mathbb{T},\omega)} \le \frac{c_{11}}{m+1} \sum_{k=n-m}^{n} E_k(f)_{M,\omega} \qquad (1.13)$$

holds with a constant $c_{11} > 0$ *independent of* n.

Note that this result in the spaces of continuous functions and Lebesgue space L_p $(1 < p < \infty)$ have been investigated in [49,50,60].

Corollary 1.8 *Let* $L_M(T, \omega)$ *be a weighted Orlicz space with Boyd indices* $0 < \alpha_M \le \beta_M < 1$, *and let* $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, $m, n \in Z_+$, $0 \le m \le n$. *Then for every* $f \in L_M(T, \omega)$, *the estimate*

$$\| f - V_{n,m}(., f) \|_{L_M(\mathbb{T},\omega)} \le \frac{c_{12}}{m+1} \sum_{k=n-m}^{n} \Omega_{M,\omega}^l \left(\frac{1}{k+1}, f \right) \qquad (1.14)$$

holds with a constant $c_{12} > 0$ *independent of* n.

Similar result for the other modulus of smoothness in the spaces of continuous functions has been obtained in [49]. Also, similar results for the Cesaro means, Zygmund means of order 2 and Abel–Poisson means in weighted Orlicz spaces can be found in [15].

Theorem 1.9 *Let* $L_M(T, \omega)$ *be a weighted Orlicz space with Boyd indices* $0 < \alpha_M \le \beta_M < 1$, *and let* $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, $m, n \in Z_+$, $0 \le m \le n$. *If the inequality*

$$\sum_{\nu=1}^{\infty} \frac{E_\nu(f)_{M,\omega}}{\nu} < \infty$$

is satisfied for $f \in L_M(T, \omega)$, *then the estimate*

$$\| \widetilde{f} - V_{n,m}(., \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \le c_{13} \left\{ \frac{1}{m+1} \sum_{\nu=0}^{n} E_{n-m+\nu}(f)_{M,\omega.} + \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f)_{M,\omega.}}{\nu} \right\}$$

holds with a constant $c_{13} > 0$ *independent of* n.

This result for the spaces of continuous functions has been obtained in [1].

Theorem 1.10 *Let Γ be a Dini-smooth curve. Also, let $L_M(\Gamma, \omega)$ be a Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$. Then for $f \in E_M(G, \omega)$ the inequality*

$$\| f - V_{n,m}(., \; f) \|_{L_M(T,\omega)} \le \frac{c_{14}}{m+1} \sum_{k=n-m}^{n} \Omega_{\Gamma,M,\omega}^l \left(\frac{1}{k+1}, \; f \right)$$

holds with a constant $c_{14} > 0$ independent of n.

Similar results for the other means of Fourier trigonometric series in the Smirnov classes $E_p(G)$ ($1 < p < \infty$) and weighted Orlicz spaces $E_M(G, \omega)$ can be found in [15,29].

2 Proofs of the main results

Proof of Theorem 1.7 We take the integer j such that the inequality $2^j \le m + 1 < 2^{j+1}$ is satisfied. The following identity holds:

$$\begin{aligned}
f(x) - V_{n,m}(x, \; f) = & \frac{1}{m+1} \left[f(x) - S_{n-m}(x, \; f) \right] \\
& + \frac{1}{m+1} \left\{ \sum_{k=1}^{j} \sum_{i=n-m+2^{k-1}}^{n-m+2^k-1} [f(x) - S_i(x, \; f)] \right\} \\
& + \frac{1}{m+1} \left\{ \sum_{k=n-m+2^j}^{n} [f(x) - S_k(x, \; f)] \right\}.
\end{aligned}$$ (2.1)

Taking into account of (2.1), we have

$$\begin{aligned}
\| f - V_{n,m}(., \; f) \|_{L_M(\mathbb{T},\omega)} \le & \frac{1}{m+1} \| f - S_{n-m}(., \; f) \|_{L_M(\mathbb{T},\omega)} \\
& + \frac{1}{m+1} \left\{ \sum_{k=1}^{j} \sum_{i=n-m+2^{k-1}}^{n-m+2^k-1} \| f - S_i(., \; f) \|_{L_M(\mathbb{T},\omega)} \right\} \\
& + \frac{1}{m+1} \left\{ \sum_{k=n-m+2^j}^{n} \| f - S_k(., \; f) \|_{L_M(\mathbb{T}, \omega)} \right\}.
\end{aligned}$$ (2.2)

Consideration of (1.11) and (2.1) gives us

$$\begin{aligned}
\| f - V_{n,m}(., \; f) \|_{L_M(\mathbb{T},\omega)} \le & c_{15} \frac{1}{m+1} \left\{ E_{n-m}(f)_{M,\omega.} + \sum_{k=1}^{j} 2^{k-1} E_{n-m+2^{k-1}}(f)_{M,\omega} \right\} \\
& + c_{15} \frac{1}{m+1} \left(m - 2^j + 1 \right) E_{n-m+2^j}(f)_{M,\omega}.
\end{aligned}$$ (2.3)

On the other hand, the following inequality holds:

$$\begin{aligned}
& \sum_{k=1}^{j} 2^{k-1} E_{n-m+2^{k-1}}(f)_{M,\omega} \le E_{n-m+1}(f)_{M,\omega} \\
& + 2 \sum_{k=2}^{j} \sum_{i=n-m+2^{k-2}}^{n-m+2^{k-1}-1} E_i(f)_{M,\omega} \le c_{16} \sum_{k=n-m}^{n-m+2^{j-1}} E_k(f)_{M,\omega}.
\end{aligned}$$ (2.4)

From inequality $2^j \leq m + 1 < 2^{j+1}$, we have $2^j > m - 2^j + 1$. Then,

$$\left(m - 2^j + 1\right) E_{n-m+2^j}(f)_{M,\omega} \leq \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{M,\omega} \tag{2.5}$$

Using (2.3), (2.4) and (2.5), we finally conclude that

$$\| f - V_{n,m}(.,\ f) \|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{17}}{m+1} \left\{ E_{n-m}(f)_{M,\omega} + \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{M,\omega} + \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{M,\omega} \right\}$$

$$\leq \frac{c_{18}}{m+1} \sum_{k=n-m}^{n} E_k(f)_{M,\omega}.$$

Thus, the proof of Theorem 1.7 is completed.

Proof of Corollary 1.8 According to Theorem 1.2 and (1.13), we obtain the inequality (1.14) of Corollary 1.8.

Proof of Theorem 1.9 We consider two cases: 1. Take $0 \leq 2m \leq n$. Suppose that $R(x)$ is an antiderivative of the function $f(x) - V_{n,m}(x,\ f)$ and $U(x)$ is an antiderivative of the function $f(x) - a_0/2$. Then, according to Theorem 1.1 we obtain

$$E_\nu(U)_{M,\omega} \leq \frac{c_{20}}{\nu+1} E_\nu(f)_{M,\omega}\ ,\quad \nu \in Z_+. \tag{2.6}$$

Since

$$\sum_{\nu=1}^{\infty} \frac{E_\nu(f)_{M,\omega}}{\nu} < \infty,$$

the inequality (2.6) yields

$$\sum_{\nu=1}^{\infty} E_\nu(U)_{M,\omega.} < \infty.$$

By virtue of Lemma 1.4 we get

$$\| \widetilde{R}' \|_{L_M(\mathbb{T},\omega)} \leq c_{21} \left\{ n \|R\|_{L_M(\mathbb{T},\omega)} + E_n(\widetilde{R}')_{M,\omega} \right\},$$

where $R = U - V_{n,m}(.,\ U) + c_{22}$ and c_{22} is a constant. Then, taking Theorem 1.3 into account, we conclude that

$$E_n(\widetilde{R}')_{M,\omega} \leq c_{23} \left\{ (n+1) E_n(U)_{M,\omega.} + \sum_{\nu=n+1}^{\infty} E_\nu(U)_{M,\omega} \right\}.$$

Then, the last inequality yields

$$\| \widetilde{f} - V_{n,m}(.,\ \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq c_{24} \left\{ (n+1) \left\|f - V_{n,m}(.,\ U)\right\|_{L_M(\mathbb{T},\omega)} + \sum_{\nu=n+1}^{\infty} E_n(U)_{M,\omega} \right\}.$$

From Theorem 1.7 we have

$$\left\|f - V_{n,m}(.,\ U)\right\|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{25}}{m+1} \sum_{\nu=n-m}^{n} E_\nu(U)_{M,\omega}$$

$$\leq \frac{c_{26}}{m+1} \sum_{\nu=n-m}^{n} \frac{E_\nu(f)_{M,\omega}}{\nu+1} = \frac{c_{26}}{m+1} \sum_{\nu=0}^{n} \frac{E_{n-m+\nu}(f)_{M,\omega}}{n-m+\nu+1}.$$

Since $0 \leq 2m \leq n$, this implies $(n+1)/(n-m+1) \leq 3$. Then, using the last inequality we reach

$$\| \widetilde{f} - V_{n,m}(., \ \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq c_{27} \left\{ \frac{1}{m+1} \sum_{\nu=0}^{n} \frac{E_{n-m+\nu}(f)_{M,\omega}}{n-m+\nu+1}(n+1) + \sum_{\nu=n+1}^{\infty} E_\nu(U)_{M,\omega} \right\}$$

$$\leq c_{28} \left\{ \frac{1}{m+1} \sum_{\nu=0}^{n} E_{n-m+\nu}(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} E_\nu(f)_{M,\omega} \right\}.$$

2. Suppose that the inequality $n < 2m \leq 2n$ is satisfied. From Theorem 1.3 we have

$$E_\nu(\widetilde{f})_{M,\omega} \leq c_{29} \left\{ E_\nu(f)_{M,\omega} + \sum_{\mu=\nu+1}^{\infty} \frac{E_\mu(f)_{M,\omega}}{\mu} \right\}.$$

Using the last inequality and Theorem 1.7, we find that

$$\| \widetilde{f} - V_{n,m}(., \ \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{30}}{m+1} \sum_{\nu=n-m}^{n} E_\nu(\widetilde{f})_{M,\omega}$$

$$\leq \frac{c_{31}}{m+1} \sum_{\nu=n-m}^{n} \left\{ E_\nu(f)_{M,\omega} + \sum_{\mu=\nu+1}^{\infty} \frac{E_\mu(f)_{M,\omega}}{\mu} \right\}$$

$$\leq c_{32} \left\{ \frac{1}{m+1} \sum_{\nu=n-m}^{n} E_\nu(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f)_{M,\omega}}{\nu} \right\}.$$

Hence, the proof of Theorem 1.9 is completed.

Proof of Theorem 1.10. Suppose that $f \in E_M(G, \omega)$. By virtue of Theorem 1.6 the operator $T :$ $E_M(D, \omega_0) \longrightarrow E_M(G, \omega)$ is bounded, one-to-one and onto and $T(f_0^+) = f$. For the function f, the following Faber series holds:

$$f(z) \backsim \sum_{k=0}^{\infty} a_k(f)\varphi_k(z).$$

Since $\omega_0 \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, considering Lemma 1.4 given in Ref. [20] we get $f_0^+ \in E_M(D, \omega_0)$. Then, function f_0^+ has the following Taylor expansion

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)w^k.$$

Note that $f_0^+ \in E_1(D)$ and boundary function $f_0^+ \in L_M(T, \omega_0)$. Then, using Theorem 3.4 [8, p.38] for the function $f_0^+(w)$ we get Fourier expansion

$$f_0^+(t) \backsim \sum_{k=0}^{\infty} a_k(f)e^{itk}.$$

Using the boundedness of the operator T, Theorems 1.7 and 1.2 we reach

$$\| f - V_{n,m}(., \ f) \|_{L_M(\Gamma,\omega)} = \| T(f_0^+) - T(V_{n,m}(., \ f_0^+)) \|_{L_M(\Gamma,\omega)}$$

$$\leq c_{33} \| f_0^+ - V_{n,m}(., \ f_0^+) \|_{L_M(T,\omega_0)} \leq \frac{c_{34}}{m+1} \sum_{k=n-m}^{n} E_k(f_0^+)_{M,\omega}$$

$$\leq \frac{c_{35}}{m+1} \sum_{k=n-m}^{n} \Omega_{M,\omega_0}^l \left(\frac{1}{k+1}, \ f_0^+ \right) = \frac{c_{35}}{m+1} \sum_{k=n-m}^{n} \Omega_{\Gamma,M,\omega}^l \left(\frac{1}{k+1}, \ f \right).$$

Thus, the theorem is proved.

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