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Some Matrix Representations of Fibonacci Quaternions and Octonions

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Abstract. In this study, starting with the usual definition of octonions, we introduce some matrix representations for Fibonacci quaternions and octonions. Then we give Cassini identity for Fibonacci octonions via matrices. Furthermore, we consider some fundamental properties of these algebraic structures.

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1. Introduction

Real and complex numbers are commutative and associative. Quaternions are associative but not commutative, whereas octonions are neither commutative nor associative. Quaternions and octonions are extensively used in physics and mathematics. For example, octonions have played an important role in various physical problems in higher dimensions. The octonions, $\mathbb{O}$, form an eight-dimensional real algebra with a basis $\{1, e_1, e_2, \ldots, e_7\}$. The most elementary method to construct the octonions is to give their multiplication table. Their multiplication is given in the following table which describes the result of multiplying the element in the ith row by the element in the jth column:

Furthermore, some other important properties of $\mathbb{O}$ are systematically given in the following. At first, any element of $\mathbb{O}$ is $x = \sum_{\mu=0}^{7} x_\mu e_\mu$ (where $e_0 = 1$), and octonionic conjugation is given by reversing the sign of the imaginary basis units

$$\overline{x} = x_0 e_0 - \sum_{\mu=1}^{7} x_\mu e_\mu. \tag{1.1}$$

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It should be noted that, conjugation is an antiautomorphism since it satisfies $\overline{x \circ y} = \overline{y} \circ \overline{x}$. Moreover, an octonion can be decomposed in terms of its scalar and vector parts as

$$\begin{aligned} x &= x_0 e_0 + \vec{X} \\ x_0 e_0 &= \frac{x + \overline{x}}{2}; \quad \vec{X} = \frac{x - \overline{x}}{2} = \sum_{\mu=1}^{7} x_\mu e_\mu. \end{aligned} \tag{1.2}$$

Secondly, by using the multiplication table, the product of octonions, x and y is expressed as

$$x \circ y = x_0 y_0 + x_0 \overrightarrow{Y} + \overrightarrow{X} y_0 - \overrightarrow{X}.\overrightarrow{Y} + \overrightarrow{X} \times \overrightarrow{Y} \tag{1.3}$$

Here, $(\cdot)$ is the dot product and $(\times)$ is the cross product of two elements in $\mathbb{R}^7$. Cross product of two vectors exists only in $\mathbb{R}^3$ and $\mathbb{R}^7$. For detailed knowledge reader should see [7]. Furthermore, the inner product on $\mathbb{O}$ is the one inherited from $\mathbb{R}^8$.

Thirdly and the last, the norm of octonion x is given by

$$\begin{aligned} \mathrm{N}(x) &= x \circ \overline{x} = \overline{x} \circ x \\ &= {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + {x_5}^2 + {x_6}^2 + {x_7}^2. \end{aligned} \tag{1.4}$$

Octonions form a composition algebra since the norm is multiplicative:

$$\mathrm{N}(x \circ y) = \mathrm{N}(x)\mathrm{N}(y). \tag{1.5}$$

In this note, we investigate Fibonacci quaternions and octonions and we derive different matrix representations of them. The rest of the paper is structured as follows. Some definitions and notation required in the analysis are given in Sect. 2. In Sect. 3, we present some matrix representations which are another way for the multiplication of Fibonacci quaternions and octonions. In the conclusions, we list a few directions for the future research.

2. Some Preliminaries

Firstly, we introduce some definitions and notation that will help us greatly in the statement of the results. Since the matrix multiplication is associative, octonions cannot be represented by ordinary matrices. Zorn found a new way to represent octonions in terms of 2×2 matrices containing both scalars and vectors using a modified version of matrix multiplication [1,8–10]. The Zorn matrix can be considered over any field $\mathbb{F}$, yet in this paper we set $\mathbb{F} = \mathbb{C}$:

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \text{where } \alpha, \beta \in \mathbb{F} \text{ and } a, b \in \mathbb{F}^3. \tag{2.1}$$

As it is well known, the classic Fibonacci $\{F_n\}_{n \in N}$ sequence is defined by

$$F_0 = 0, \;\; F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2,$$

where F_n denotes the nth classic Fibonacci number [6]. In a similar manner, we can construct a new matrix similar to the matrix z given in equation (2.1) as follows

$$z_{\mathbb{F}} = \begin{bmatrix} \alpha & A \\ B & \beta \end{bmatrix}, \text{where } \alpha, \beta \in \mathbb{F} \tag{2.2}$$

and A, B are the Fibonacci vectors. From [9], we know that a Fibonacci vector is a triple of Fibonacci numbers. That is, the triple $f_n = (F_{n-1}, F_n, F_{n+1})$ is called the *n-th* Fibonacci vector. It is observed that all Fibonacci vectors lie in the plane $z = x + y$. Hence, we can call the sequence $\{\mathcal{F}_n\}_{n\geq 1}$, where

$$\{\mathcal{F}_n\}_{n\geq 1} = \{f_1, f_2, f_3, \ldots, f_n, \ldots\},$$

as the basic Fibonacci vector sequence. That is, the elements of this sequence are given by

$$\{\mathcal{F}_n\} = \{(0,1,1), (1,1,2), (1,2,3), \ldots, (F_{n-1}, F_n, F_{n+1}), \ldots\}.$$

Although the domain of the vector geometry is $\mathbb{Z}^3$, it is convenient to extend it to $\mathbb{F}^3$. So, we refer to $z_{\mathbb{F}}$ as the *Zorn–Fibonacci matrix*. The trace of matrix $z_{\mathbb{F}}$ is

$$\mathrm{tr}(z_{\mathbb{F}}) = \alpha + \beta.$$

The conjugate matrix and the norm of $z_{\mathbb{F}}$ in equation (2.2) are given by

$$\overline{z}_{\mathbb{F}} = \begin{bmatrix} \beta & -A \\ -B & \alpha \end{bmatrix} \text{ and} \tag{2.3}$$

$$\mathrm{N}(z_{\mathbb{F}}) = \begin{vmatrix} \alpha & A \\ B & \beta \end{vmatrix} = \alpha\beta - A.B, \tag{2.4}$$

respectively. Here the $A.B$ is dot product. The addition and multiplication operations for two Zorn–Fibonacci matrices $z_{\mathbb{F}}$ and $w_{\mathbb{F}}$ are

$$\begin{bmatrix} \alpha & A \\ B & \beta \end{bmatrix} + \begin{bmatrix} \gamma & C \\ D & \delta \end{bmatrix} = \begin{bmatrix} \alpha+\gamma & A+C \\ B+D & \beta+\delta \end{bmatrix} \tag{2.5}$$

and

$$\begin{bmatrix} \alpha & A \\ B & \beta \end{bmatrix} \begin{bmatrix} \gamma & C \\ D & \delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma + A.D & \delta A + \alpha C - B \times D \\ \gamma B + \beta D + A \times C & \beta\delta + B.C \end{bmatrix}, \tag{2.6}$$

respectively. Then, for $\lambda \in \mathbb{R}$ we get

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \alpha & A \\ B & \beta \end{bmatrix} = \begin{bmatrix} \alpha & A \\ B & \beta \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}. \tag{2.7}$$

The matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ can be called the *Zorn unit matrix.*

The multiplication operation in the Zorn matrix representation is defined as to be isomorphic to the octonion multiplication which is defined in Table 1. Since multiplication operation for octonions has 480 possible definitions, the multiplication in the Zorn matrix representation should be rearranged accordingly to the defined octonion table.

Table 1. Multiplication of octonions

$\circ$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

3. Two Matrix Representations of Fibonacci Octonions

Quaternions are related to both real and complex numbers. Since complex numbers can be represented by matrices, quaternions can also be represented by matrices. One of the approaches to matrices representing quaternions is to convert a given real quaternion into a 2×2 complex matrix and the other is to use a 4×4 real matrix. These are injective homomorphisms from $\mathbb{H}$ to the matrix rings $M_2(\mathbb{C})$ and $M_4(\mathbb{R})$, respectively. Using linear algebra, the matrix representations permit us to explain certain properties of complex numbers, quaternions, and octonions, i.e: the norm is the determinant of the corresponding matrix, the multiplicative property of the absolute value and the inverse of an element. In [3], Halici introduced the real representation for Fibonacci quaternion Q_n, $Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$, by some special matrices which are as follows:

$$Q_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ -F_{n+1} & F_n & -F_{n+3} & F_{n+2} \\ -F_{n+2} & F_{n+3} & F_n & -F_{n+1} \\ -F_{n+3} & -F_{n+2} & F_{n+1} & F_n \end{bmatrix}. \tag{3.1}$$

Based upon [4,5] the Fibonacci octonion p can be expressed as a set of consecutive eight Fibonacci numbers,

$$p = (F_n, F_{n+1}, F_{n+2}, \ldots, F_{n+7}) = F_n e_0 + \sum_{\mu=1}^{7} F_{n+\mu} e_\mu = p' + p'' e, \tag{3.2}$$

where p' and p'' are the Fibonacci quaternions and e_μ, $\mu = 1, 2, \ldots, 7$ are imaginary octonion units. Therefore, using the Pauli spin matrices, Fibonacci octonion p can be represented by 8×8 real matrices. For further details of them, readers are referred to [1,2,4,5]. Note that e_0 is the multiplicative unit element and $e_4 = e$. The conjugate of Fibonacci octonion p can be defined as

$$\overline{p} = F_n e_0 - \sum_{\mu=1}^{7} F_{n+\mu} e_\mu; \quad \mu = 1, 2, \ldots, 7 \tag{3.3}$$

So, the norm of Fibonacci octonion p is $\mathrm{N}(p) = p \circ \overline{p} = \overline{p} \circ p = \sum_{n=0}^{7} F_n^2 e_0$. Since p' and p'' are not zero, the norm of Fibonacci octonion p is always positive.

In the algebra $\mathrm{Zorn}_{\mathbb{F}}$, we have:

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_4 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$
$$e_j = \begin{bmatrix} 0 & -u_j \\ u_j & 0 \end{bmatrix}, \quad e_{j+4} = \begin{bmatrix} 0 & iu_j \\ iu_j & 0 \end{bmatrix}, \tag{3.4}$$

where j = 1, 2, 3 and $u_1 = (1,0,0),\ u_2 = (0,1,0),\ u_3 = (0,0,1)$ are three dimensional unit vectors in the field $\mathbb{R}$. Since $e_k^2 = -1,\ k = 1, 2, \ldots, 7$ and $\{e_0, e_1, \ldots, e_7\}$ is a standard basis for Fibonacci octonions, we can represent the Fibonacci octonions and Fibonacci quaternions by using the elements of this basis.

Considering the bases given in equation (3.4), we present the following proposition without proof.

Proposition 3.1. *For the nth Fibonacci quaternion and octonion two identical representations are given as follows*

$$\text{(i)}\ \ Q_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3 = \begin{bmatrix} F_n & -f_{n+2} \\ f_{n+2} & F_n \end{bmatrix}, \tag{3.5}$$

where f_{n+2} is the (n+2)th Fibonacci vector sequence.

$$\text{(ii)}\ \ p_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + \cdots + F_{n+7} e_7 = \begin{bmatrix} W_n & -\overline{w}_{n+2} \\ w_{n+2} & \overline{W}_n \end{bmatrix}, \tag{3.6}$$

where $w_{n+k} = F_{n+k} + iF_{n+k+4};\ \ k = 0, 1, 2, 3$ and $W_n = F_n + iF_{n+4}$.

Taking into account matrix operations, we obtain the following equations:

$$Q_n + Q_{n+1} = Q_{n+2},\ \ p_n + p_{n+1} = p_{n+2} \tag{3.7}$$

$$\mathrm{tr}(Q_n) = 2F_n,\ \ \det(Q_n) = F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 = \sum_{k=0}^{3} F_{n+k}^2 \tag{3.8}$$

and

$$\begin{aligned} \mathrm{N}(p_n) &= W_n \overline{W}_n + (w_{n+1}\overline{w}_{n+1} + w_{n+2}\overline{w}_{n+2} + w_{n+3}\overline{w}_{n+3}) \\ &= \sum_{k=0}^{7} F_{n+k}^2. \end{aligned} \tag{3.9}$$

For the octonions, another representation, which we will call a *modified Zorn matrix representation*, is given in [1]. In that representation, three dimensional basis vectors are replaced with Pauli matrices. The Pauli matrices are denoted by $\sigma_k, (k = 1, 2, 3)$ and these three matrices can be reduced to only one matrix as follows:

$$\sigma_k = \begin{bmatrix} \delta_{k,3} & \delta_{k,1} - i\delta_{k,2} \\ \delta_{k,1} + i\delta_{k,2} & -\delta_{k,3} \end{bmatrix}, \tag{3.10}$$

where δ is the Kronecker delta. By recalling the most prominent features of Pauli matrices such as $\sigma_k^2 = I$, $\det(\sigma_k) = -1$ and $\mathrm{tr}(\sigma_k) = 0$, the base elements of modified Zorn matrix representation are represented in [1] as follows.

$$\begin{aligned} e_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}, \\ e_4 &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_{k+4} = \begin{bmatrix} 0 & i\sigma_k \\ i\sigma_k & 0 \end{bmatrix}. \end{aligned} \tag{3.11}$$

Furthermore, the multiplication operation in the modified Zorn matrix representation has been given in [1] as follows:

$$\begin{bmatrix} \alpha & \mathbf{A} \\ \mathbf{B} & \beta \end{bmatrix} \begin{bmatrix} \gamma & \mathbf{C} \\ \mathbf{D} & \delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma + \frac{1}{2}\mathrm{tr}(\mathbf{AD}) & T \\ \Upsilon & \beta\delta + \frac{1}{2}\mathrm{tr}(\mathbf{BC}) \end{bmatrix}, \tag{3.12}$$

where $T = \delta\mathbf{A} + \alpha\mathbf{C} + \frac{i}{2}[\mathbf{B}, \mathbf{D}]$, $\Upsilon = \gamma\mathbf{B} + \beta\mathbf{D} - \frac{i}{2}[\mathbf{A}, \mathbf{C}]$ and $\mathbf{A}$ is a matrix which is analogous to the vector part of Zorn matrix representation, and $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is a commutator.

Now, by considering (3.11) and (3.12), in the following proposition, second representation for Fibonacci quaternion and octonions is given.

Proposition 3.2. *For the nth Fibonacci quaternion and octonion, we have a different representation as follows.*

$$\text{(i)} \;\; Q_n = \begin{bmatrix} F_n & M_{n+3} \\ -M_{n+3} & F_n \end{bmatrix}, \tag{3.13}$$

where

$$M_{n+3} = \begin{bmatrix} -F_{n+3} & -F_{n+1} + iF_{n+2} \\ -F_{n+1} - iF_{n+2} & F_{n+3} \end{bmatrix}. \tag{3.14}$$

$$\text{(ii)} \;\; p_n = \begin{bmatrix} W_n & N_{n+3} \\ -\overline{N}_{n+3} & \overline{W}_n \end{bmatrix}, \tag{3.15}$$

where

$$N_{n+3} = \begin{bmatrix} -\overline{w}_{n+3} & -\overline{w}_{n+1} + i\overline{w}_{n+2} \\ -\overline{w}_{n+1} - i\overline{w}_{n+2} & \overline{w}_{n+3} \end{bmatrix}. \tag{3.16}$$

Proof. For the nth Fibonacci quaternion and octonion, if necessary calculations are made, then these representations can be easily obtained. □

Note that the trace and norm of the matrix p_n is

$$\begin{aligned} \mathrm{tr}(p_n) &= F_n + iF_{n+4} + F_n - iF_{n+4} = 2F_n, \;\text{ and} \\ \mathrm{N}(p_n) &= \sum_{k=0}^{7} F_{n+k}^2. \end{aligned} \tag{3.17}$$

From equation (3.17), it can be easily identified, $\mathrm{N}(p) \neq 0$ for any Fibonacci octonion p. Since usage of the Binet formula in the calculations are more convenient, in the following proposition, we employ the Binet formula for Fibonacci sequence and we give a new representation for Fibonacci octonions.

Proposition 3.3. *For the Fibonacci octonions, the Zorn and the modified Zorn matrix representation are given, respectively, as:*

$$p_n = \begin{bmatrix} B_n & -\bar{b}_{n+2} \\ b_{n+2} & \overline{B}_n \end{bmatrix} \tag{3.18}$$

and

$$P_m = \begin{bmatrix} B_m & C_{m+3} \\ -\overline{C}_{m+3} & \overline{B}_m \end{bmatrix}, \tag{3.19}$$

where

$$C_{m+3} = \begin{bmatrix} -\overline{B}_{m+3} & -\overline{B}_{m+1} + i\overline{B}_{m+2} \\ -\overline{B}_{m+1} - i\overline{B}_{m+2} & \overline{B}_{m+3} \end{bmatrix}.$$

Proof. The Binet style form of Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. If the Binet formula in Zorn matrix representation is used, then

$$p_n = \begin{bmatrix} B_n & -\bar{b}_{n+2} \\ b_{n+2} & \overline{B}_n \end{bmatrix}, \tag{3.20}$$

where

$$B_n = \frac{(\alpha^n - \beta^n) + i(\alpha^{n+4} - \beta^{n+4})}{\alpha - \beta} \tag{3.21}$$

and

$$b_n = [B_{n-1}, B_n, B_{n+1}]. \tag{3.22}$$

Furthermore, as in the Zorn matrix representation, the modified Zorn matrix representation by Binet's formula is expressed as

$$P_m = \begin{bmatrix} B_m & C_{m+3} \\ -\overline{C}_{m+3} & \overline{B}_m \end{bmatrix}. \tag{3.23}$$

It should be noted that quaternion versions of these representations can be calculated in a similar manner. □

Thus, we can obtain Cassini identity by the aid of matrix representations as follows.

Corollary 3.4. *From different matrix representations, the following Cassini identity is obtained:*

(i) $p_{n+1} \circ p_{n-1} - {p_n}^2 = (-1)^n A_1$,
(ii) $p_{n-1} \circ p_{n+1} - {p_n}^2 = (-1)^n A_2$,
(iii) $p_{n+1} \circ p_{n-1} - {p_n}^2 = (-1)^n B_1$,
(iv) $p_{n-1} \circ p_{n+1} - {p_n}^2 = (-1)^n B_2$,

where

$$A_1 = \begin{bmatrix} 2+10i & (-2+2i, -4+12i, -6+35i) \\ (2+2i, 4+12i, 6+35i) & 2-10i \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2+4i & (20i, -2+24i, -2+23i) \\ (20i, 2+24i, 2+23i) & 2-4i \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 2+10i & \begin{bmatrix} -6+35i & -14-2i \\ 10+6i & 6+35i \end{bmatrix} \\ \begin{bmatrix} 6+35i & 14-2i \\ -10+6i & -6-35i \end{bmatrix} & 2-10i \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2+4i & \begin{bmatrix} -2+23i & -24+18i \\ 24+22i & 2+23i \end{bmatrix} \\ \begin{bmatrix} 2+23i & 24+18i \\ -24+22i & -2-23i \end{bmatrix} & 2-4i \end{bmatrix}.$$

In the following example, we demonstrate representations described in Propositions 3.1 and 3.2.

Example. Let us take two Fibonacci octonions, such as, the 4th Fibonacci octonion p_4 and the 1st Fibonacci octonion p_1:

$$p_4 = 3e_0 + 5e_1 + 8e_2 + 13e_3 + 21e_4 + 34e_5 + 55e_6 + 89e_7,$$
$$p_1 = 1e_0 + 1e_1 + 2e_2 + 3e_3 + 5e_4 + 8e_5 + 13e_6 + 21e_7.$$

Firstly, let us start with the Zorn matrix representation of p_4 and p_1,

$$p_4 = \begin{bmatrix} W_4 & -\overline{w}_6 \\ w_6 & \overline{W}_4 \end{bmatrix}$$
$$p_4 = \begin{bmatrix} 3+21i & -(5-34i, 8-55i, 13-89i) \\ (5+34i, 8+55i, 13+89i) & 3-21i \end{bmatrix},$$
$$p_1 = \begin{bmatrix} W_1 & -\overline{w}_3 \\ w_3 & \overline{W}_1 \end{bmatrix}$$
$$p_1 = \begin{bmatrix} 1+5i & -(1-8i, 2-13i, 3-21i) \\ (1+8i, 2+13i, 3+21i) & 1-5i \end{bmatrix}.$$

Using these representations and the Zorn multiplication defined in equation (2.6) p_4p_1 can be calculated as

$$p_4p_1 = \begin{bmatrix} -3018+30i & -(6-76i, 12-106i, 18-140i) \\ (6+76i, 12+106i, 18+140i) & -3018-30i \end{bmatrix}.$$

In fact, the resulting matrix is the Zorn matrix representation of $p_4 \circ p_1$:

$$p_4 \circ p_1 = -3018e_0 + 6e_1 + 12e_2 + 18e_3 + 30e_4 + 76e_5 + 106e_6 + 140e_7.$$

Second representation, which uses Pauli matrices in the Zorn matrix representation [1], is as follows:

$$P_4 = \begin{bmatrix} 3+21i & \begin{bmatrix} -(13-89i) & 50+42i \\ -60+26i & 13-89i \end{bmatrix} \\ \begin{bmatrix} 13+89i & 60+26i \\ -50+42i & -13-89i \end{bmatrix} & 3-21i \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 1+5i & \begin{bmatrix} -(3-21i) & 12+10i \\ -14+6i & 3-21i \end{bmatrix} \\ \begin{bmatrix} 3+21i & 14+6i \\ -12+10i & -3-21i \end{bmatrix} & 1-5i \end{bmatrix}.$$

If we multiply these Fibonacci octonions in a similar way as defined in [1], then we obtain the matrix given below.

$$P_4P_1 = \begin{bmatrix} -3018+30i & \begin{bmatrix} -18+140i & 100+88i \\ -112+64i & 18-140i \end{bmatrix} \\ \begin{bmatrix} 18+140i & 112+64i \\ -100+88i & -81-140i \end{bmatrix} & -3018-30i \end{bmatrix}.$$

This representation is also the modified Zorn matrix representation of $p_4 \circ p_1$.

At this point, it is crucial to consider definitions of two representations for a better insight of the current problem and identify which one seems to be advantageous in calculations. By recalling the definition of Zorn matrix multiplication, which is similar to the ordinary matrix multiplication, one observes that the Zorn matrix multiplication involves additional computations of the dot and the vector products as shown in (2.6). From the definition of the modified version of the multiplication, one can easily see that the multiplication in this representation needs more calculations than the Zorn matrix representation such as the extra matrix multiplication, commutator and trace functions (3.12). It is better and convenient to use the standard version of the octonion algebra. If one is restricted to use the complex algebra rather than the quaternion and octonion algebras, then the reader can use both of them. Advantage of these representations is to calculate multiplication of Fibonacci octonions without knowing the octonion multiplication table.

4. Conclusion

In conclusion, we have introduced two different matrix representations for Fibonacci quaternions and Fibonacci octonions by means of the Zorn matrix and the modified Zorn matrix. As a future research, we will try to generalize these representations over the sedenion algebra which generalizes the octonion algebra.

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