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SOME REMARKS ON ORBITAL DIGRAPHS FOR THE FINITE PRIMITIVE GROUPS

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Abstract. In this paper we concern with the relationship between the finite groups $PSL(2,q)$, $q \geq 5$ a prime, and orbital digraphs. And also we explain that for a generator elliptic element in permutation group, there is a hyperbolic circuit in suborbital graph.

1. Introduction

Let F be a field. We denote by $GL(2,F)$ the general linear group of invertible 2×2 matrices with coefficients in F, that is, those matrices with nonzero determinant. And we denote by $SL(2,F)$ the special linear group of matrices with determinant 1, which forms the kernel of the determinant map, $det : GL(2,F) \longrightarrow F\setminus\{0\}$. Thus we can say by $PGL(2,F)$ the quotient group

$$PGL(2,F) = GL(2,F)/\{\lambda I : \lambda \in F \setminus \{0\}\},$$

and by $PSL(2,F)$ the quotient group

$$PSL(2,F) = SL(2,F)/\{\varepsilon I : \varepsilon = \pm 1\}.$$

Let $P^1(F) = F \cup \{\infty\}$ be the projective line over F. We can embed $PGL(2,F)$ and $PSL(2,F)$ into the symmetric group $SymP^1(F)$ of permutations of $P^1(F)$. To every $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,F)$, we associate the fractional linear transformation, $\psi_T : P^1(F) \longrightarrow P^1(F)$, defined by $\psi_T(z) = \dfrac{az+b}{cz+d}$. Here we set if $c \neq 0$ then $\psi_T(\infty) = \frac{a}{c}$, if $c = 0$ then $\psi_T(\infty) = \infty$. And $\psi_T(\frac{-d}{c}) = \infty$. Hence, we get a group homomorphism

$$\psi : GL(2,F) \longrightarrow SymP^1(F),$$

where $\psi(T) = \psi_T$ and $PGL(2,F)$ identifies with $\psi(GL(2,F))$. It is clear that if F is a finite field, $|F| = q$, then $GL(2,F)$ has only finitely many elements. That is, when $F = F_q$, the finite field of order q, we write $PGL(2,q)$ and $PSL(2,q)$. These groups are very important in many mathematical problems. Especially, they are used a lot in graph and combinatoric theory.

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2. Preliminaries

Lemma 2.1. ***(i)*** $|GL(2,q)| = q(q-1)(q^2-1)$

(ii) $|SL(2,q)| = |PGL(2,q)| = q(q^2-1)$

(iii) $|PSL(2,q)| = \begin{cases} q(q^2-1) & \textit{if } q \textit{ is even,} \\ \dfrac{q(q^2-1)}{2} & \textit{if } q \textit{ is odd.} \end{cases}$

Proof. **(i)** A 2×2 matrix in $GL(2,F)$ is obtained by first choosing the first column a nonzero vector in F_q^2, there are q^2-1 possible choices for that; then by choosing the second column, a vector in F_q^2 linearly independent from the first one: there are q^2-q possible choice for that. **(ii)** and **(iii)** follow from elementary group theory. □

Lemma 2.2. *$SL(2,q)$ is a normal subgroup of $GL(2,q)$, and the index $|GL(2,q) : SL(2,q)| = q-1$.*

Proof. The determinant function maps $GL(2,q)$ into $F\setminus\{0\}$, and elementary properties of the determinant imply that it is, in fact, a homomorphism onto the multiplicative group $F\setminus\{0\}$. The kernel of this homomorphism is $SL(2,q)$. Hence $SL(2,q)$ is a normal subgroup and its index in $GL(2,q)$ is $|F\setminus\{0\}| = q-1$. □

Lemma 2.3. *For any field F, the group $SL(2,F)$ is generated by the union of the two subgroups $\left\{\begin{pmatrix}1 & \kappa_1\\ 0 & 1\end{pmatrix} : \kappa_1 \in F\right\}$ and $\left\{\begin{pmatrix}1 & 0\\ \kappa_2 & 1\end{pmatrix} : \kappa_2 \in F\right\}$. Therefore, every matrix in $SL(2,F)$ is a finite product of matrices which are either upper triangular or lower triangular and which have 1's along the diagonal.*

Proof. Let $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,F)$. We distinguish two cases:

(I) if $c \neq 0$ then an immediate computation gives

$$\begin{pmatrix}1 & \frac{a-1}{c}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\begin{pmatrix}1 & \frac{d-1}{c}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}a & \frac{a(d-1)}{c}+\frac{a-1}{c}\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

(II) if $c = 0$ then $d \neq 0$ and the matrix $\begin{pmatrix}a+b & b\\ d & d\end{pmatrix} \in SL(2,F)$ can be treated as in the first case. But then

$$\begin{pmatrix}a+b & b\\ d & d\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix} = \begin{pmatrix}a & b\\ 0 & d\end{pmatrix}.$$

□

Lemma 2.4. *Let q be a prime. Every nonscalar matrix in $SL(2,q)$ has an abelian centralizer.*

Proof. Let $T = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ be a nonscalar matrix in $SL(2,q)$. We consider the fractional linear transformation ψ_T on $P^1(F_{q^2})$, the projective line over the field with

q^2 elements. Since T is nonscalar, we have $\psi_T \neq I$. The fixed point equation $\frac{az+b}{cz+d} = z$ has one or two solutions in $P^1(F_{q^2})$. We separate cases:

(I) ψ_T has a unique fixed point. Conjugating within $PGL(2, q^2)$, we may assume that this fixed point is ∞; then ψ_T is a translation, $\psi_T(z) = z + b$ for $z \in F_{q^2}$, so $T = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. The centralizer of T in $SL(2, q^2)$ is the subgroup $\left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in F_{q^2} \right\}$, which is abelian.

(II) ψ_T has two fixed points; conjugating within $PGL(2, q^2)$, we may assume that these are 0 and ∞. Then $\psi_T(z) = \alpha^2 z$ for some $\alpha \in F_{q^2} \setminus \{0\}$, $\alpha \neq \pm 1$. This means $T = \pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. The centralizer of T inside $SL(2, q^2)$ is then the diagonal subgroup, which is abelian. □

The modular group $PSL(2, \mathbb{Z})$ is the group of all linear fractional transformations $z \longrightarrow \frac{az+b}{cz+d}$, where a, b, c, d are integers and $ad - bc = 1$. It is well known that $PSL(2, \mathbb{Z})$ is generated by the two elements $x : z \longrightarrow -\frac{1}{z}$ and $y : z \longrightarrow \frac{z-1}{z}$, which satisfy the relations $x^2 = y^3 = 1$. The group $PGL(2, \mathbb{Z})$ is the group of all transformations $z \longrightarrow \frac{az+b}{cz+d}$, where a, b, c, d are integers and $ad - bc = \pm 1$. If t is the transformations $z \longrightarrow \frac{1}{z}$, which belongs to $PGL(2, \mathbb{Z})$ but not to $PSL(2, \mathbb{Z})$, then x, y, t satisfy

$$x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1.$$

The group $PSL(2, \mathbb{Z})$ acts faithfully on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Imz} > 0\}$ by Möbius transformations, and moreover when equipped with the hyperbolic metric this action is by orientation preserving isometries. As well as, the space $\mathbb{H}$ can be intrinsically characterized as the unique two dimensional simply connected Riemann manifold with constant curvature.The hyperbolic metric and the Euclidean metric on $\mathbb{H}$ are equivalent, inducing the same topology. However, lengths and geodesics are different.

In this study we will get $F = \mathbb{Z}$. The group $PGL(2, q)$ has a natural permutation representation on the projective line $\mathbb{Z}_q \cup \{\infty\}$, and therefore any homomorphism $\alpha : PGL(2, \mathbb{Z}) \longrightarrow PGL(2, q)$ gives rise to an action of $PGL(2, \mathbb{Z})$ on $\mathbb{Z}_q \cup \{\infty\}$. Therefore the natural ring-epimorphism $\mathbb{Z} \to \mathbb{Z}_q$, $m \to [m]$, induces a group homomorphism $SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}_q)$ and also this in turn induces a group-homomorphism $\phi_q : PSL(2, \mathbb{Z}) \longrightarrow PSL(2, \mathbb{Z}_q)$. If q is a prime then $PSL(2, \mathbb{Z}_q) \cong SL(2, \mathbb{Z}_q)/\{\pm I\}$. Moreover these groups are all isomorphic because they each contain the same matrices. For example, if $q = 2$ then,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

matrices are congruence according to $mod(2)$.

Permutation Groups. Since 1950 interactions between group theory and theory of graphs have greatly stimulated the development of each other, especially the theory of orbital graphs has almost develop in parallel with the theory of permutation groups. The study of orbital graphs has long been one of the main themes in algebraic graph theory.

Let Ω be a nonempty set. A bijection of Ω onto itself is called a permutation of Ω. Under composition of mappings, the set of all permutations of Ω forms a group, which is called the symmetric group on Ω, and is denoted as $Sym(\Omega)$. Any subgroup of $Sym(\Omega)$ is said to be a permutation group on Ω. If Ω is a finite set, say $|\Omega| = n$ then we say that denote $Sym(\Omega)$ by S_n. The set of even permutations of S_n forms an index two subgroup of S_n, which is called the alternating group, and is denoted by A_n. That is $|S_n : A_n| = 2$ and $|A_n| = \dfrac{n!}{2}$. Some of the isomorphism relations between families of small classical groups are as follows: $PSL(2,2) \cong S_3$, $PSL(2,3) \cong A_4$, $SL(2,3) \cong 2A_4$, $PSL(2,4) \cong A_5$, $PGL(2,3) \cong S_4$, $PGL(2,5) \cong S_5$ and so on.

Definition 2.1. *Let G be a group acting on a set Ω. For a point $\alpha \in \Omega$, the set $G\alpha := \{g\alpha \mid g \in G\}$ is called the orbit of α under G.*

A group G acting on a set Ω is said to be transitive on Ω if it has only one orbit, that is $G\alpha = \Omega$ for all $\alpha \in \Omega$, otherwise it is called intransitive.

Now we give the theorem without proof.

Theorem 2.5. *Every group of order n is isomorphic to a subgroup of S_n.*

Definition 2.2. *The set of elements in G that fix the point α constitute a subgroup of G, called the stabilizer of α in G, and is denoted by G_α. Thus*

$$G_\alpha := \{g \in G \mid g\alpha = \alpha\}.$$

If G is a subgroup of $Sym(\Omega)$, then we will say that the pair (G,Ω) is a permutation group of degree $|\Omega|$, and that G acts on Ω.

Remark. *We recall that $|G\alpha| = |G : G_\alpha|$, $\alpha \in \Omega$.*

In order to find the number of orbits of G on Ω, we may use the set $G(g)$ of fixed of g. If η is the number of orbits of (G,Ω), then $\eta|G| = \sum_{g\in G} |G(g)|$.

3. Main Calculation

Let $q \geq 5$ be a prime. The group $PSL(2,q)_0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,q) : c \equiv 0 \, mod(q) \right\}$ is special congruence subgroup of $PSL(2,q)$. We proceed to a description of the special subgroup of $PSL(2,q)$ process and the draw of the orbital graphs in blocks.

Lemma 3.1. *Let $q \geq 5$ be a prime. The degree of any nontrivial representation of $PSL(2,q)$ is at least $\frac{q-1}{2}$.*

Proof. Recall that we denote by $\theta : SL(2,q) \longrightarrow PSL(2,q)$ the canonical map and we let $W = \theta\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in F_q \setminus \{0\}, b \in F_q \right\}$ the stabilizer in $PSL(2,q)$ of $\infty \in P^1(F_q)$. On the other hand $|W| = \frac{q(q-1)}{2}$. Let ξ be a nontrivial representation of $PSL(2,q)$ on $\mathbb{C}^n$. Consider the restriction $\xi|_W$. We may decompose it as a direct sum of irreducible representations of W. since $q \geq 5$, the group $PSL(2,q)$ is simple. So $\xi|_W$ is a faithful representation of W, meaning that $\xi|_W(g) \neq I$ if $g \neq I$. This implies that at least one of the irreducible representations must appear in $\xi|_W$, so that $n \geq \frac{q-1}{2}$. □

Imprimitive Action. Let $\mathbb{Q} \cup \{\infty\}$ element of the extended rational number set and the orbit is $\Delta := \{\frac{a}{b}|a, b \in \mathbb{Z}_q\}$. It is clear that $\Delta \subset \mathbb{Q} \cup \{\infty\}$. Then any element of Δ can be given by as a reduced fraction $\dfrac{x}{y}$ with $x, y \in \mathbb{Z}_q$ and $(x, y) = 1$. Besides, ∞ is represented as $\frac{1}{0} = \frac{-1}{0}$. The action of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, q)$ on $\dfrac{x}{y}$ is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \frac{x}{y} \to \frac{\alpha x + \beta y}{\gamma x + \delta y}.$$

We now explain imprimitivity of the action on $PSL(2, q)$ on Δ. $(PSL(2, q), \Delta)$ is transitive permutation group, comprising of a group $PSL(2, q)$ acting on a set Δ transitively. $\upsilon_1, \upsilon_2 \in \Delta$ satisfy $\upsilon_1 \approx \upsilon_2$ then $\gamma(\upsilon_1) \approx \gamma(\upsilon_2)$ for all $\gamma \in PSL(2, q)$. In this case equivalence relation $\approx$ on Δ is invariant and equivalence classes form blocks. We say $(PSL(2, q), \Delta)$ imprimitive, if Δ accepts some invariant equivalence relation different from the identity relation and the universal relation. Otherwise $(PSL(2, q), \Delta)$ is primitive. These two relations are supposed to be trivial relations. Also $\approx$ relation of equivalence classes are called orbits of action.

Since the following lemma is well known from [10], we only give the statement;

Lemma 3.2. *Let (G, Ω) be a transitive permutation group.(G, Ω) is primitive if and only if G_σ is a maximal subgroup of G for each $\sigma \in \Omega$. That is $G_\sigma \leq H \leq G$ implies $H = G_\sigma$ or $H = G$.*

Theorem 3.3. *If (G, Ω) is primitive and G_σ is simple, then either*
(i) *G is simple, or*
(ii) *G has a normal subgroup N which acts regularly on Ω.*

Proof. Suppose that (i) is false, so that G has a proper normal subgroup N, $N \neq I$. Given σ in Ω, consider $N \cap G_\sigma$. It is normal in G_σ, and since G_σ is simple, it must be either G_σ itself or I. Now $N \cap G_\sigma = G_\sigma$ means that $G_\sigma \leq N$, and we must have $G_\sigma < N$ since N is transitive and G_σ is not. Thus, by the above lemma, $N = G$, which contradictions the assumption that N is proper. It follows that $N \cap G_\sigma = I$, so N acts regularly. □

Consequently we understand that if $G_\sigma < H < G$ then Ω is imprimitive. So we use the transitivity, for all element of Ω has the form $g(\sigma)$ for some $g \in G$. Therefore one of the non trivial G invariant equivalence relation on Ω is given as follows:

$$g_1(\sigma) \approx g_2(\sigma) \text{ if and only if } g_1^{-1} g_2 \in H.$$

The number of the blocks is the index $\Psi = |G : H|$.

We can apply these ideas to the case where G is the $PSL(2, q)$ and Ω is Δ. We have the following lemmas:

Lemma 3.4. *$PSL(2, q)$ acts transitively on Δ.*

Proof. We will show that the orbit containing ∞ is Δ. If $\frac{a}{b} \in \Delta$ then as $(a,b)=1$ there exist $u,v \in \mathbb{Z}_q$ with $ux - vy = 1$. We can state the element $\begin{pmatrix} a & u \\ b & v \end{pmatrix}$ of $PSL(2,q)$ sends ∞ to $\frac{a}{b}$. □

Lemma 3.5. *The stabilizer of ∞ in Δ is the set of $\left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z}_q \right\}$ denoted by $PSL(2,q)_\infty$.*

Proof. Because of the action is transitive, stabilizer of any two points conjugate. Therefore we can only look at the stabilizer of ∞ in $PSL(2,q)$.
Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,q)$.Thus, $A(\infty) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $a=1$, $c=0$, $d=1$ and $b=\lambda$.Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$is obtained.That is,
$PSL(2,q)_\infty = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z}_q \right\}$. Moreover it is easily seen that this inequality $PSL(2,q)_\infty < PSL(2,q)_0 < PSL(2,q)$ is satisfied. □

Let $\approx$ denote the $PSL(2,q)$ invariant equivalence relation on Δ by $PSL(2,q)_0$, let $v = \frac{r}{s}$ and $w = \frac{x}{y}$ be elements of Δ.Then there are the elements $g_1 := \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ and $g_2 := \begin{pmatrix} x & * \\ y & * \end{pmatrix}$ in $PSL(2,q)$ such that $v = g_1(\infty)$ and $w = g_2(\infty)$. So we have

$$v \approx w = g_1(\infty) \approx g_2(\infty) \Leftrightarrow g_1^{-1} g_2 \in PSL(2,q)_0$$

and so from the above we can easily calculate that $g_1^{-1} g_2 = \begin{pmatrix} * & * \\ ry - sx & * \end{pmatrix} \in PSL(2,q)_0$. Hence $ry - sx \equiv 0 \, mod(q)$ is obtained.

It is known that the number of the blocks $\Psi = |\Gamma : \Gamma_0(n)| = n \prod_{p|n} (1 + \frac{1}{p})$ where p is a prime and $n > 1$. In particular, if $n = p$ is a prime, then there are $p+1$ blocks, these being $[0], [1], [2], ..., [p-1], [\infty]$. Because of the number of blocks is $|PSL(2,q) : PSL(2,q)_0| = q+1$. These blocks are
$[\infty] := [\frac{1}{0}] = \{ \frac{x}{y} \in \Delta : (x,y) = 1 \text{ and } \ y \equiv 0 \, mod(q) \}$,

$[j] := [\frac{j}{1}] = \{ \frac{x}{y} \in \Delta : (x,y) = 1 \text{ and } \ x - jy \equiv 0 \, mod(q) \}$ where $j \neq \infty$.

Orbital Digraphs.

Definition 3.1. *Let V be a nonempty set, the elements of which are called vertices. A digraph (or directed graph) Σ is a pair (V,E) where E is a subset of $V \times V$. The elements of E are called edges. The digraph Σ is said to be finite if the vertex set V is finite. If $(\alpha,\beta) \in E$, this is indicated as $\alpha \to \beta$. If $(\alpha,\beta) \in E$ or $(\beta,\alpha) \in E$ then α and β are connected to a edge.*

Definition 3.2. *Let Σ be a graph and $A \subset V$. $(A, E \cap A \times A)$ subgraph is named of Σ, vertex set of which is A.*

Definition 3.3. *Let a sequence $v_1, v_2, \ldots, v_k$ of different vertices. Then the form $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k \longrightarrow v_1$, where $k \in \mathbb{N}$ and $k \geq 3$, is called a directed circuit in Σ.*

If $k = 2$, then we will say the configuration $v_1 \longrightarrow v_2 \longrightarrow v_1$ a self paired edge.

If $k = 3$ or $k = 4$, then the circuit, directed or not, is called a triangle or quadrilateral. In a graph is a finite or infinite sequence of edges which connect a sequence of vertices which are all distinct from one another are called a path.

Let (G, V) be transitive permutation group. Then G acts on $V \times V$ by

$$\Theta : \underset{(g,(\alpha,\beta))}{G \times (V \times V)} \underset{\to}{\longrightarrow} \underset{(g(\alpha),g(\beta))}{V \times V}$$

where $g \in G$ and $\alpha, \beta \in V$. The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* Σ. Its vertices are the elements of V, and if $(\gamma, \delta) \in O(\alpha, \beta)$ there is a directed edge from γ to δ. Moreover $O(\alpha, \alpha)$ is diagonal of $V \times V$. The corresponding suborbital graph called the trivial suborbital graph, it consists of a *loop* based at each vertex.

Since $PSL(2, q)$ acts transitively on Δ, it permutes the blocks transitively. Also there is a disjoint union of isomorphic copies of suborbital graphs. We recall that edges of these graphs can be drawn as hyperbolic geodesic in the upper half-plane.

Let $F_{u,q}$ denote the subgraphs in Σ whose vertices form the block $[\infty]$. Similarly we may write subgraphs and are for other blocks.

Theorem 3.6. *There is an edge $\dfrac{r}{s} \longrightarrow \dfrac{x}{y}$ in $F_{u,q}$ if and only if either*

(i) *$x \equiv ur \, mod(q)$, $y \equiv us \, mod(q)$ and $ry - sx = q$, or*

(ii) *$x \equiv -ur \, mod(q)$, $y \equiv -us \, mod(q)$ and $ry - sx = -q$.*

Proof. Assume that $\dfrac{r}{s} \longrightarrow \dfrac{x}{y}$ be an edge in $F_{u,q}$. Then there is some element $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, q)$ such that $T(\frac{1}{0}) = \dfrac{a}{c} = \dfrac{(-1)^m r}{(-1)^m s}$ gives that $r = (-1)^m a$, $s = (-1)^m c$ for $m = 0, 1$. Again $T(\frac{u}{q}) = \dfrac{au + qb}{cu + qd} = \dfrac{(-1)^n x}{(-1)^n y}$ for $n = 0, 1$. Hence $x \equiv (-1)^{m+n} ur \, mod(q)$ and $y \equiv (-1)^{m+n} us \, mod(q)$. So we have the matrix equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix} = \begin{pmatrix} (-1)^m r & (-1)^n x \\ (-1)^m s & (-1)^n y \end{pmatrix}$ for $m, n = 0, 1$. If we get determinant it is easily seen that $ry - sx = \pm q$.

Conversely we do calculations only for (i). Therefore $x \equiv ur \, mod(q)$, $y \equiv us \, mod(q)$ and $ry - sx = q$. Then there exist integers b and d such that $x = ur + qb$, $y = us + qd$. So $\begin{pmatrix} a & ur + qb \\ c & us + qd \end{pmatrix} = \begin{pmatrix} r & x \\ s & y \end{pmatrix}$. As $ad - bc = 1$ from determinants we have $ry - sx = q$. Consequently we obtain $\begin{pmatrix} a & ur + qb \\ c & us + qd \end{pmatrix} \in PSL(2, q)$ and $\dfrac{r}{s} \longrightarrow \dfrac{x}{y}$ in $F_{u,q}$. $\square$

Theorem 3.7. *The subgraph $F_{u,q}$ contains a hyperbolic triangle if and only if $u^2 \pm u + 1 \equiv 0\, mod(q)$.*

Proof. As $PSL(2,q)$ permutes the vertices transitively of $F_{u,q}$, then we may suppose that hyperbolic triangle has the form $\frac{1}{0} \longrightarrow \frac{k_0}{q} \longrightarrow \frac{x_0}{qy_0} \longrightarrow \frac{1}{0}$. In addition to assume that $\frac{k_0}{q} < \frac{x_0}{qy_0}$. Using Theorem 3.6. from the first edge, we get $k_0 \equiv u\, mod(q)$. The second edge gives $x_0 \equiv -uk_0\, mod(q)$ and $ky_0 - x_0 = -1$. From the last edge we have $1 \equiv -ux_0\, mod(q)$ and $y_0 = 1$. Hence $1 \equiv -u(k_0+1)\, mod(q)$ is obtained. This gives that $u^2 + u + 1 \equiv 0\, mod(q)$.

If $\frac{k_0}{q} > \frac{x_0}{qy_0}$ holds then we conclude that this equation $u^2 - u + 1 \equiv 0\, mod(q)$ is achieved.

On the other hand suppose that $u^2 \pm u + 1 \equiv 0\, mod(q)$. Obviously we obtain the hyperbolic triangle $\frac{1}{0} \longrightarrow \frac{u}{q} \longrightarrow \frac{u \pm 1}{q} \longrightarrow \frac{1}{0}$ form Theorem 3.6. □

Corollary 3.8. *Actually $F_{u,q}$ contains hyperbolic triangle if and only if the group $PSL(2,q)_0$ contains elliptic element $\psi_1 = \begin{pmatrix} -u & \frac{u^2+u+1}{q} \\ -q & u+1 \end{pmatrix}$ of order 3 in $PSL(2,q)_0$. It is clear that $\psi_1(\frac{1}{0}) = \frac{u}{q}$, $\psi_1(\frac{u}{q}) = \frac{u+1}{q}$ and $\psi_1(\frac{u+1}{q}) = \frac{1}{0}$. That is, by the mapping the ψ_1 transform vertices to each other.*

Now we will give examples for $q = 5$ in $[\infty]$ and $[0]$ blocks.

Example 3.1. *For some $u \in \mathbb{Z}_5$ hyperbolic triangles in subgraph $F_{u,5}$ figures are as follows:*

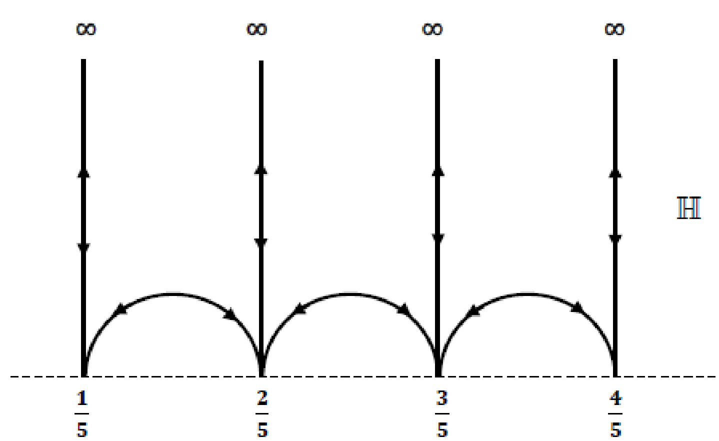

FIGURE 1. Hyperbolic triangle circuits in $[\infty]$

$$\frac{1}{0} \to \frac{1}{5} \to \frac{2}{5} \to \frac{1}{0},\ \frac{1}{0} \to \frac{2}{5} \to \frac{3}{5} \to \frac{1}{0},\ \frac{1}{0} \to \frac{3}{5} \to \frac{4}{5} \to \frac{1}{0},$$
$$\frac{1}{0} \to \frac{2}{5} \to \frac{1}{5} \to \frac{1}{0},\ \frac{1}{0} \to \frac{3}{5} \to \frac{2}{5} \to \frac{1}{0},\ \frac{1}{0} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{0}$$

Remark. *Since $u \in \mathbb{Z}_5$ and $(u,p) = 1$ there is finite number of hyperbolic triangles in suborbital graph.*

Corollary 3.9. *Again we can say that the subgraph $Z_{u,q}$ whose vertices form in the block $[0]$, contains hyperbolic triangle if and only if the group $PSL(2,q)_0$ contains elliptic element $\psi_2 = \begin{pmatrix} u+1 & -q \\ \frac{u^2+u+1}{q} & -u \end{pmatrix}$ of order 3 in $PSL(2,q)_0$. It is clear that $\psi_2(\frac{0}{1}) = \frac{q}{u}$, $\psi_2(\frac{q}{u}) = \frac{q}{u+1}$ and $\psi_2(\frac{q}{u+1}) = \frac{0}{1}$. That is, by the mapping the ψ_2 transform vertices to each other.*

Example 3.2. *Similarly hyperbolic triangles in subgraph $Z_{u,5}$ figures are as follows:*

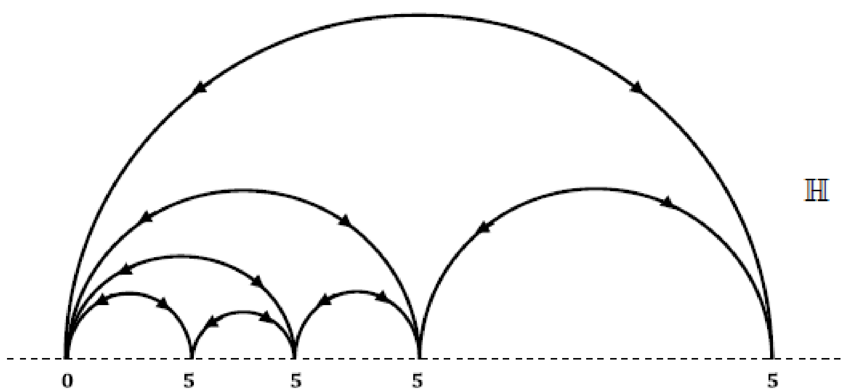

FIGURE 2. Hyperbolic triangle circuits in [0]

$$\frac{0}{1} \to \frac{5}{1} \to \frac{5}{2} \to \frac{0}{1},\ \frac{0}{1} \to \frac{5}{2} \to \frac{5}{3} \to \frac{0}{1},\ \frac{0}{1} \to \frac{5}{3} \to \frac{5}{4} \to \frac{0}{1},$$

$$\frac{0}{1} \to \frac{5}{2} \to \frac{5}{1} \to \frac{0}{1},\ \frac{0}{1} \to \frac{5}{3} \to \frac{5}{2} \to \frac{0}{1},\ \frac{0}{1} \to \frac{5}{4} \to \frac{5}{3} \to \frac{0}{1}$$

Remark. *We know that there are self paired edges in suborbital graphs. If using the group $PSL(2,2)$, then self paired edges reveal in subgraps. For $PSL(2,2)$ there are 3 blocks. Below, we will give a lemma for infinite block.*

Lemma 3.10. *The subgraph $F_{u,2}$ is self paired if and only if $u^2 + 1 \equiv 0 \, mod(2)$.*

Proof. Assume that $O(\frac{1}{0}, \frac{u}{2}) = O(\frac{u}{2}, \frac{1}{0})$. Then there exists $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $T(\infty) = \frac{u}{2}$ and $T(\frac{u}{2}) = \infty$. Hence T must be $\begin{pmatrix} u & -\frac{u^2+1}{2} \\ 2 & -u \end{pmatrix}$ and $detT = 1$. So we have $u^2 + 1 \equiv 0\, mod(2)$. Conversely case is obvious. $\square$

Example 3.3. *We can easily see that there are self paired edges in $F_{u,2}$ and $Z_{u,2}$, and also there exists a loop in $L_{u,2}$ whose vertices form in the block* [1]*. Figures are as follows:*

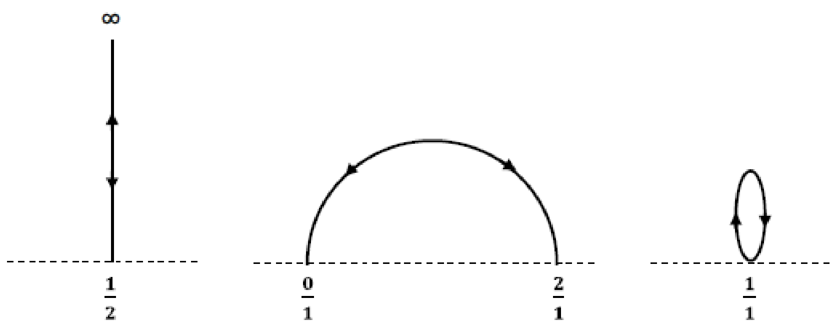

FIGURE 3. $[\infty]$, $[0]$ and $[1]$ blocks

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