Çankaya Üniversitesi Fen-Edebiyat Fakültesi,
Journal of Arts and Sciences Sayı: 11 / Mayıs 2009

Single and Multiple Positive Solutions for Nonlinear Discrete Problems

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Abstract

In this paper, we discuss the existence of single and multiple positive solutions to the nonlinear second order discrete three-point boundary value problems. To prove the main results, we will use the well-known result of the fixed point theorems in cones.

Key words: Boundary value problems; Difference equations; Fixed point theorems; Positive solutions.

Özet

Bu çalışmada, lineer olmayan ikinci mertebe diskret üç-nokta sınır değer problemleri için tek ve birden fazla pozitif çözümlerin varlığı incelenecek. Ana sonuçları ispatlamak için konilerde sabit nokta teoremlerinin iyi bilinen sonuçlarını kullanacağız.

Anahtar Kelimeler: Sınır değer problemleri; Fark denklemleri; Sabit nokta teoremleri; Pozitif çözümler.

1. Introduction

There are many authors who studied the existence of solutions to second order two-point boundary value problems on difference equations. For some recent results, we refer the reader to [2], [3], [4], [5], [7], [9], [10]. However, to the best of the author's knowledge, there are few results for the existence of solutions to second order three-point boundary value problems on difference equations.

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In paper [1], Agarwal, Thompson and Tisdell studied existence results for solutions to second-order discrete boundary value problem

$$\Delta^2 y_{k+1} = f(x_k, y_k, \Delta y_k), \quad k = 1,2,...,n-1$$

$$G(y_0, y_n, y_d) = (0,0), \quad d \in \{1,...,n-1\}$$

where f and G are continuous.

In paper [11], Zhang and Medina studied existence of positive solutions for the nonlinear discrete three-point boundary value problem

$$\Delta^2 x_{k-1} + f(x_k) = 0, \quad k = 1,2,...,n$$

$$x_0 = 0, \ x_{n+1} - ax_l = b$$

where $n \in \{2,3,...\}$, $l \in \{1,2,...,n\}$, a and b are positive numbers and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$.

We are interested in the existence of one or two positive solutions of the following three-point boundary value problem:

$$\begin{cases} \Delta^2 y(n) + h(n) f(n, y(n+1)) = 0, \ n \in [a,c] \\ \Delta y(a) = 0, \ \alpha y(c+1) + \beta \Delta y(c+1) = \Delta y(b) \end{cases} \tag{1}$$

where $a < b < c$ are distinct integers, $\alpha > 0$, $\beta > 1$ and Δ denotes the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n)$. Additionally, throughout the paper we assume $h:[a,c] \to [0,\infty)$ is continuous such that $h(n_0) > 0$ for at least one $n_0 \in [b,c]$, $f:[a,c] \times [0,\infty) \to [0,\infty)$ is continuous.

Problem (1) is equivalent to the problem

$$y(n+2) - 2y(n+1) + y(n) + h(n) f(n, y(n+1)) = 0, \ n \in [a,c]$$

$$y(a+1) = y(a), \ \beta y(c+2) + (\alpha - \beta) y(c+1) = y(b+1) - y(b).$$

2. Preliminaries

Let $G(n,s)$ be Green's function for the boundary value problem

$$\Delta^2 y(n) = 0, \ n \in [a,c] \subset \mathbb{Z}$$

$$\Delta y(a) = 0, \ \alpha y(c+1) + \beta \Delta y(c+1) = \Delta y(b).$$

A direct calculation gives

$$G(n,s) = \begin{cases} s \in [a,b] \begin{cases} c + \dfrac{\beta - 1}{\alpha} + 1 - n & , \quad s+1 \le n \\ c - s + \dfrac{\beta - 1}{\alpha} & , \quad n \le s \end{cases} \\ s \in [b,c] \begin{cases} c + \dfrac{\beta}{\alpha} + 1 - n & , \quad s+1 \le n \\ c - s + \dfrac{\beta}{\alpha} & , \quad n \le s \end{cases} \end{cases} \tag{2}$$

To state and prove the main results of this paper, we need the following lemmas.

Lemma 1 Let $\alpha > 0,\ \beta > 1$. Then the Green's function $G(n,s)$ in (2) satisfies the following inequality

$$G(n,s) \geq \frac{n-a}{c-a+1} G(c+1,s)$$

For $(n,s) \in [a,c+1] \times [a,c]$.

Proof. We proceed sequentially on the branches of the Green's function $G(n,s)$ in (2).
(i) For $s \in [a,b]$ and $s+1 \leq n$, we obtain

$$G(n,s) = c + \frac{\beta - 1}{\alpha} + 1 - n$$

and

$$\frac{G(n,s)}{G(c+1,s)} = \frac{c-n+1}{\frac{\beta-1}{\alpha}} + 1 > \frac{n-a}{c-a+1}.$$

(ii) Let $s \in [a,b]$ and $n \leq s$. Then

$$G(n,s) = c - s + \frac{\beta - 1}{\alpha}$$

and

$$\frac{G(n,s)}{G(c+1,s)} = 1 > \frac{n-a}{c-a+1}.$$

(iii) Take $s \in [b,c]$ and $s+1 \leq n$. Then

$$G(n,s) = c + \frac{\beta}{\alpha} + 1 - n$$

and

$$\frac{G(n,s)}{G(c+1,s)} = \frac{c-n+1}{\frac{\beta}{\alpha}} + 1 \geq \frac{n-a}{c-a+1}.$$

(iv) Fix $s \in [b,c]$ and $n \leq s$. Then

$$G(n,s) = c - s + \frac{\beta}{\alpha}$$

and

$$\frac{G(n,s)}{G(c+1,s)} = 1 > \frac{n-a}{c-a+1}.$$

Lemma 2 Assume $\alpha > 0,\ \beta > 1$. Then the Green's function $G(n,s)$ in (2) satisfies

$$0 < G(n,s) \leq G(s+1,s)$$

for $(n,s) \in [a,c+1] \times [a,c]$.

Proof. Since for $s \in [a,b]$

$$G(c+1,s) = \frac{\beta - 1}{\alpha} > 0$$

and for $s \in (b, c+1]$

$$G(c+1,s) = \frac{\beta}{\alpha} > 0,$$

we obtain $G(n,s) > 0$ from Lemma 1. To show that $G(n,s) \le G(s+1,s)$, we again deal with the branches of the Green's function $G(n,s)$ in (2).

(i) For $s \in [a,b]$ and $s+1 \le n \le c+1$, $G(n,s)$ is decreasing in n and $G(n,s) \le G(s+1,s)$.

(ii) Let $s \in [a,b]$ and $a \le n \le s$. Then it is obvious that $G(n,s) = G(s+1,s)$.

(iii) Take $s \in (b,c]$ and $s+1 \le n \le c+1$. Since $G(n,s)$ is decreasing in n, $G(n,s) \le G(s+1,s)$.

(iv) Fix $s \in (b,c]$ and $a \le n \le s$. Then it is clear that $G(n,s) = G(s+1,s)$.

Lemma 3 Suppose $\alpha > 0$, $\beta > 1$ and $s \in [a,c]$. Then the Green's function $G(n,s)$ in (2) satisfies

$$\min_{n \in [b,c+1]} G(n,s) \ge k \left\| G(.,s) \right\|,$$

where

$$k = \frac{\beta - 1}{\alpha(c-a) + \beta - 1} \tag{3}$$

and $\|.\|$ is defined by $\|x\| = \max_{n \in [a,c+1]} |x(n)|$.

Proof. Since the Green's function $G(n,s)$ in (2) is nonincreasing in n, we get $\min_{n \in [b,c+1]} G(n,s) \ge G(c+1,s)$. Moreover, it is obvious that $\left\| G(.,s) \right\| = G(s+1,s)$ for $s \in [a,c]$ by Lemma 2. Then we have from the branches in (2) that

$$G(c+1,s) \ge kG(s+1,s).$$

Let the Banach space $\mathcal{B} = \{y : [a,c+1] \to \mathbb{R}\}$ with the norm $\|y\| = \max_{n \in [a,c+1]} |y(n)|$ and define the cone $P \subset \mathcal{B}$ by

$$P = \left\{ y \in \mathcal{B} : y(n) \ge 0, \ \min_{n \in [b,c+1]} y(n) \ge k \|y\| \right\} \tag{4}$$

where k is given in (3).

(1) is equivalent to the nonlinear integral equation

$$y(n) = \sum_{s=a}^{c} G(n,s) h(s) f(s, y(s+1)). \tag{5}$$

We can define the operator $A : P \to \mathcal{B}$ by

$$Ay(n) = \sum_{s=a}^{c} G(n,s) h(s) f(s, y(s+1)), \tag{6}$$

where $y \in P$. If $y \in P$, then by Lemma 3 we have

$$\min_{n\in[b,c+1]} Ay(n) = \sum_{s=a}^{c} \min_{n\in[b,c+1]} G(n,s)h(s)f(s,y(s+1))$$

$$= k\sum_{s=a}^{c} \max_{n\in[a,c+1]} \left|G(n,s)\right| h(s)f(s,y(s+1))$$

$$= k\left\|Ay\right\|.$$

Thus $Ay \in P$ and therefore $AP \subset P$. In addition, $A: P \to P$ is completely continuous by a standard application of the Arzela-Ascoli Theorem.

We will apply the following well-known result of the fixed point theorems to prove the existence of one or two positive solutions to the (1).

Lemma 4 [6],[8] Let P be a cone in a Banach space $\mathcal{B}$, and let D be an open, bounded subset of $\mathcal{B}$ with $D_P := D \cap P \neq \emptyset$ and $\overline{D}_P \neq P$. Assume that $A: \overline{D}_P \to P$ is a compact map such that $y \neq Ay$ for $y \in D_P$. The following results hold.

(i) If $\|Ay\| \leq \|y\|$ for $y \in D_P$, then $i_P(A, D_P) = 1$.

(ii) If there exists an $q \in P\setminus\{0\}$ such that $y \neq Ay + \lambda q$ for all $y \in D_P$ and all $\lambda > 0$, then $i_P(A, D_P) = 0$.

(iii) Let U be open in P such that $\overline{U}_P \subset D_P$. If $i_P(A, D_P) = 1$ and $i_P(A, U_P) = 0$, then A has a fixed point in $D_P \setminus \overline{U}_P$. The same result holds if $i_P(A, D_P) = 0$ and $i_P(A, U_P) = 1$.

3. Single and Multiple Positive Solutions

For the cone P given in (4) and any positive real number r, define the convex set

$$P_r := \left\{ y \in P : \|y\| < r \right\}$$

and the set

$$\Omega_r := \left\{ y \in P : \min_{n\in[b,c+1]} y(n) < kr \right\}.$$

The following results are proved in [3].

Lemma 5 The set Ω_r has the following properties.

(i) Ω_r is open relative to P.

(ii) $P_{kr} \subset \Omega_r \subset P_r$.

(iii) $y \in \partial\Omega_r$ if and only if $\min_{n\in[b,c+1]} y(n) = kr$.

(iv) If $y \in \partial\Omega_r$, then $kr \leq y(n) \leq r$ for $n \in [b, c+1]$.

For convenience, we introduce the following notations. Let

$$f_{kr}^{r} := \min\left\{ \min_{n\in[b,c+1]} \frac{f(n,y)}{r} : y \in [kr, r]\right\}$$

$$f_{0}^{r} := \max\left\{ \max_{n\in[a,c+1]} \frac{f(n,y)}{r} : y \in [0, r]\right\}$$

$$f^{a} := \limsup_{y\to a} \max_{n\in[a,c+1]} \frac{f(n,y)}{y}$$

$$f_{a} := \liminf_{y\to a} \min_{n\in[b,c+1]} \frac{f(n,y)}{y} \ \left(a := 0^{+}, \infty\right).$$

In the next two lemmas, we give conditions on f guaranteeing that $i_P(A,P_r) = 1$ or $i_P(A,\Omega_r) = 0$.

Lemma 6 Let

$$L := \sum_{s=a}^{c} G(s+1,s)h(s). \tag{7}$$

If the conditions

$$f_0^r \le \frac{1}{L} \text{ and } y \ne Ay \text{ for } y \in \partial P_r,$$

hold, then $i_P(A,P_r) = 1$.

Proof. If $y \in \partial P_r$, then using Lemma 2, we have

$$\begin{aligned} Ay(n) &= \sum_{s=a}^{c} G(n,s)h(s)f(s,y(s+1)) \\ &\le \|f(.,y)\| \sum_{s=a}^{c} G(s+1,s)h(s) \\ &\le r = \|y\| \end{aligned}$$

It follows that $\|Ay\| \le \|y\|$ for $y \in \partial P_r$. By Lemma 4(i), we get $i_P(A,P_r) = 1$.

Lemma 7 Let

$$N := \left(\sum_{s=b}^{c} G(c+1,s)h(s) \right)^{-1} \tag{8}$$

If the conditions

$$f_{kr}^r \ge Nk \text{ and } y \ne Ay \text{ for } y \in \partial\Omega_r,$$

hold, then $i_P(A,\Omega_r) = 0$.

Proof. Let $q(n) \equiv 1$ for $n \in [a,c+1]$, then $q \in \partial P_1$. Assume there exist $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$ such that $y_0 = Ay_0 + \lambda_0 q$. Then for $n \in [b,c+1]$ we have

$$y_0(n) = Ay_0(n) + \lambda_0 q(n)$$

$$\geq \sum_{s=b}^{c} G(n,s)h(s)f(s,y_0(s+1))+\lambda_0$$

$$\geq Nkr\sum_{s=b}^{c} G(c+1,s)h(s)+\lambda_0$$

$$=\text{kr}+\lambda_0$$

But this implies that $\text{kr} \geq \text{kr}+\lambda_0$, a contradiction. Hence, for $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$, so by Lemma 4(*ii*), we get $i_P(A,\Omega_r)=0$.

Theorem 8 Let *k, L*, and *N* be as in (3), (7), and (8), respectively. Suppose that one of the following conditions holds.

(C1) There exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < kc_2$ and $c_2 < c_3$ such that

$$f_0^{c_1}, f_0^{c_3} \leq \frac{1}{L},\ f_{kc_2}^{c_2} \geq Nk, \text{ and } y \neq Ay \text{ for } y \in \partial\Omega_{c_2}.$$

(C2) There exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < c_2 < kc_3$ such that

$$f_{kc_1}^{c_1}, f_{kc_3}^{c_3} \geq Nk,\ f_0^{c_2} \leq \frac{1}{L}, \text{ and } y \neq Ay \text{ for } y \in \partial P_{c_2}.$$

Then (1) has two positive solutions. Additionally, if in (C1) the condition $f_0^{c_1} \leq \frac{1}{L}$ is replaced by $f_0^{c_1} < \frac{1}{L}$, then (1) has a third positive solution in P_{c_1}.

Proof. Assume that (C1) holds. We show that either *A* has a fixed point in ∂P_{c_1} or in $\Omega_{c_2} \setminus \overline{P}_{c_1}$. If $y \neq Ay$ for $y \in \partial P_{c_1}$, then by Lemma 6, we have $i_P(A,P_{c_1})=1$. Since $f_{kc_2}^{c_2} \geq Nk$ and $y \neq Ay$ for $y \in \partial\Omega_{c_2}$, from Lemma 7 we get $i_P(A,\Omega_{c_2})=0$. By Lemma 5(*ii*) and $c_1 < kc_2$, we have $\overline{P}_{c_1} \subset \overline{P}_{kc_2} \subset \Omega_{c_2}$. From Lemma 4(*iii*), *A* has a fixed point in $\Omega_{c_2} \setminus \overline{P}_{c_1}$. If $y \neq Ay$ for $y \in \partial P_{c_3}$, then from Lemma 6 $i_P(A,P_{c_3})=1$. By Lemma 5(*ii*) and Lemma 4(*iii*), *A* has a fixed point in $P_{c_3} \setminus \overline{\Omega}_{c_2}$. The proof is similar when (C2) holds and we omit it here.

Corollary 9 If there exist a constant $c > 0$ such that one of the following conditions holds:

(H1) $0 \leq f^0, f^\infty < \frac{1}{L}$, $f_{kc}^c \geq Nk$, and $y \neq Ay$ for $y \in \partial\Omega_c$.

(H2) $N < f_0, f_\infty \leq \infty$, $f_0^c \leq \frac{1}{L}$, and $y \neq Ay$ for $y \in \partial P_c$.

Then (1) has two positive solutions.

Proof. We show that (H1) implies (C1). It is easy to verify that $0 \leq f^0 < \frac{1}{L}$ implies that there exists $c_1 \in (0,kc)$ such that $f_0^{c_1} < \frac{1}{L}$. Let $m \in \left(f^\infty, \frac{1}{L}\right)$. Then there exists $c_2 > c$ such that

$\max\limits_{n\in[a,c+1]} f(n,y) \le ky$ for $y\in[c_2,\infty)$ because $0 \le f^{\infty} < \dfrac{1}{L}$. Let

$n = \max\left\{ \max\limits_{n\in[a,c+1]} f(n,y) : 0 \le y \le c_2 \right\}$ and $c_4 > \max\left\{ \dfrac{n}{\frac{1}{L}-m}, \dfrac{1}{L} \right\}$.

Then we have

$\max\limits_{n\in[a,c+1]} f(n,y) \le my + n \le mc_4 + n < \dfrac{1}{L}c_4$ for $y\in[0,c_4]$.

This implies that $f_0^{c_4} < \dfrac{1}{L}$ and (C1) holds. Similarly, (H2) implies (C2).

As a special case of Theorem 8 and Corollary 9, we have the following two results.

Theorem 10 Assume that one of the following conditions holds.

(C3) There exist constants $c_1, c_2 \in \mathbb{R}$ with $0 < c_1 < kc_2$ such that

$f_0^{c_1} \le \dfrac{1}{L}$ and $f_{kc_2}^{c_2} \ge Nk$.

(C4) There exist constants $c_1, c_2 \in \mathbb{R}$ with $0 < c_1 < c_2$ such that

$f_{kc_1}^{c_1} \ge Nk$ and $f_0^{c_2} \le \dfrac{1}{L}$.

Then (1) has a positive solution.

Corollary 11 Assume that one of the following conditions holds:

(H3) $0 \le f^0 < \dfrac{1}{L}$ and $N < f_{\infty} \le \infty$.

(H4) $0 \le f^{\infty} < \dfrac{1}{L}$ and $N < f_0 \le \infty$.

Then (1) has a positive solution.

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