Boundary Value Problems for Second Order Nonlinear Differential Equations on Infinite Intervals

G. Sh. Guseinov[a], I. Yaslan[b]

[a] *Department of Mathematics, Atilim University*

06836 Incek, Ankara, Turkey

E-mail : guseinov@atilim.edu.tr

[b] *Department of Mathematics, Ege University*

35100 Bornova, Izmir, Turkey

E-mail : iyaslan@sci.ege.edu.tr

Abstract In this paper, we consider boundary value problems for nonlinear differential equations on the semi axis $(0,\infty)$ and also on the whole axis $(-\infty,\infty)$, under the assumption that the left-hand side being a second order linear differential expression belongs to the Weyl limit-circle case. The boundary value problems are considered in the Hilbert spaces $L^2(0,\infty)$ and $L^2(-\infty,\infty)$, and include boundary conditions at infinity. The existence and uniqueness results for solutions of the considered boundary value problems are established.

Keywords: Weyl limit circle case; completely continuous operator; fixed point theorems.

1 INTRODUCTION

Consider the second order nonlinear differential equation

$$-[p(x)y']' + q(x)y = f(x,y), \quad 0 \le x < \infty, \tag{1.1}$$

where $y = y(x)$ is a desired solution.

Let $L^2(0,\infty)$ be the Hilbert space of real-valued functions $y(x)$ on $[0,\infty)$ such that $\int_0^\infty |y(x)|^2 dx < \infty$ with the inner product $(y,z) = \int_0^\infty y(x)z(x)dx$ and norm $\|y\| = \{\int_0^\infty |y(x)|^2 dx\}^{1/2}$.

We will assume that the following conditions are satisfied.

(C1) The coefficients $p(x)$ and $q(x)$ are real-valued measurable functions on $[0,\infty)$ such that

$$\int_0^b \frac{dx}{|p(x)|} < \infty, \qquad \int_0^b |q(x)|dx < \infty$$

for each finite positive number b. Moreover, the functions $p(x)$ and $q(x)$ are such that all solutions of the second order linear differential equation

$$-[p(x)y']' + q(x)y = 0, \quad 0 \le x < \infty, \tag{1.2}$$

belong to $L^2(0,\infty)$.

(C2) The function $f(x,\xi)$ is real-valued and continuous in $(x,\xi) \in [0,\infty) \times \mathbf{R}$, and

$$|f(x,\xi)| \le g(x) + d|\xi| \tag{1.3}$$

for all (x,ξ) in $[0,\infty) \times \mathbf{R}$, where $g(x) \ge 0$, $g \in L^2(0,\infty)$, and d is positive constant.

Remark 1.1 *A function $y = y(x)$ defined on $[0,\infty)$ is called a solution of equation* (1.1) *if y has the first derivative y', the product $p(x)y'(x)$ is absolutely continuous on each finite interval $[0,b] \subset [0,\infty)$, and* (1.1) *holds almost everywhere on $[0,\infty)$.*

Remark 1.2 *The condition $(C1)$ means that for the differential expression $Ly = -[p(x)y']' + q(x)y$ the so-called Weyl limit-circle case holds (see* [3, 11]*). If $p(x) \equiv 1$ and $q(x) = -e^{2x}$, then the condition $(C1)$ holds (see [11, p.93]).*

Remark 1.3 *If we define the operator F taking each function $y(x)$ to the function $f(x,y(x))$, then the condition* (1.3) *is necessary and sufficient for F to map $L^2(0,\infty)$ into itself (see [9, Chapter 1]).*

Let D be the linear manifold of all elements $y \in L^2(0,\infty)$ such that $Ly = -[p(x)y']' + q(x)y$ is defined and $Ly \in L^2(0,\infty)$. Denote $y^{[1]}(x) = p(x)y'(x)$, the quasi-derivative of $y(x)$. For given two differentiable functions $y = y(x)$ and $z = z(x)$ we define the Wronskian of y and z by

$$\begin{aligned} W_x(y,z) &= y(x)z^{[1]}(x) - y^{[1]}(x)z(x) \\ &= p(x)[y(x)z'(x)] - y'(x)z(x)], \quad x \in [0,\infty). \end{aligned}$$

It follows from the Green's formula

$$\textstyle\int_0^b [(Ly)z - y(Lz)](x)dx = W_b(y,z) - W_0(y,z) \tag{1.4}$$

that, for all $y, z \in D$ the limit

$$W_\infty(y,z) = \lim_{b\to\infty} W_b(y,z)$$

exists as a finite number.

Let $u = u(x)$ and $v = v(x)$ be solutions of (1.2) satisfying the initial conditions

$$u(0) = 0,\ u^{[1]}(0) = 1;\ v(0) = -1,\ v^{[1]}(0) = 0. \tag{1.5}$$

By the constancy of the Wronskian of any two solutions of (1.2) we have $W_x(u,v) = 1$. Consequently, u and v are linearly independent and they form a fundamental system of solutions of (1.2). It follows from the condition $(C1)$ that $u, v \in L^2(0,\infty)$ and moreover $u, v \in D$. Therefore for each $y \in D$ the values $W_\infty(y,u)$ and $W_\infty(y,v)$ exist and are finite. For these values we can get, by using Green's formula (1.4) and the initial conditions (1.5), the formulas

$$W_\infty(y,u) = y(0) + \int_0^\infty u(x)Ly(x)dx, \quad W_\infty(y,v) = y^{[1]}(0) + \int_0^\infty v(x)Ly(x)dx. \quad (1.6)$$

Our boundary value problem (BVP) consists of finding a function $y = y(x),\ x \in [0,\infty)$, such that $y \in L^2(0,\infty)$ and y satisfies the equation (1.1) [consequently $y \in D$ by (1.3)] and the boundary conditions

$$\alpha y(0) + \beta y^{[1]}(0) = d_1, \quad \gamma W_\infty(y,u) + \delta W_\infty(y,v) = d_2, \quad (1.7)$$

where $\alpha, \beta, \gamma,$ and δ are given real numbers satisfying the condition

(C3) $\omega := \alpha\delta - \beta\gamma \neq 0,$
and d_1, d_2 are arbitrary given real numbers.

Notice that the first boundary condition in (1.7) can be written as

$$\alpha W_0(y,u) + \beta W_0(y,v) = d_1.$$

Also notice that since in (1.7) the function y satisfies the equation (1.1), taking into account (1.6) we have the following formulas for the values $W_\infty(y,u)$ and $W_\infty(y,v)$:

$$W_\infty(y,u) = y(0) + \int_0^\infty u(x)f(x,y(x))dx, \quad W_\infty(y,v) = y^{[1]}(0) + \int_0^\infty v(x)f(x,y(x))dx.$$

In Section 2 we construct an appropriate Green's function by means of which the BVP (1.1), (1.7) is reduced to a fixed point problem.

In Section 3 by using the Contraction Mapping Theorem (Banach Fixed Point Theorem) we show that there is a unique solution of the BVP $(1.1),(1.7)$ if $f(x,\xi)$ satisfies a Lipschitz condition.

In Section 4 a theorem based on the Schauder Fixed Point Theorem is proved which gives a result that yields existence of solutions without the implication that solutions must be unique.

Finally, in Section 5, we deal with BVPs on the whole axis.

For the other formulations of BVPs on infinite intervals we refer to $[1,2]$. The way giving "boundary conditions at infinity" used in the present paper was employed earlier in $[4-8,10]$.

2 GREEN'S FUNCTION AND THE OPERATOR A

For $h \in L^2(0,\infty)$ consider the linear BVP

$$-[p(x)y']' + q(x)y = h(x), \quad 0 \leq x < \infty, \quad (2.1)$$

$$\alpha y(0) + \beta y^{[1]}(0) = 0, \ \gamma W_\infty(y,u) + \delta W_\infty(y,v) = 0, \quad (2.2)$$

where $y \in L^2(0,\infty)$ is a desired solution, u and v are solutions of (1.2) under the initial conditions (1.5).

Let us set

$$\varphi(x) = \alpha u(x) + \beta v(x), \quad \psi(x) = \gamma u(x) + \delta v(x). \tag{2.3}$$

These functions (together with u and v) are solutions of (1.2) and are in $L^2(0,\infty)$. Besides, we have

$$\varphi(0) = W_x(\varphi, u) = -\beta, \quad \varphi^{[1]}(0) = W_x(\varphi, v) = \alpha; \tag{2.4}$$

$$\psi(0) = W_x(\psi, u) = -\delta, \quad \psi^{[1]}(0) = W_x(\psi, v) = \gamma. \tag{2.5}$$

Therefore φ satisfies the boundary condition at zero in (2.2), and ψ satisfies the boundary condition at infinity. The Wronskian of φ and ψ is

$$W_x(\varphi, \psi) = \alpha\delta - \beta\gamma = \omega \tag{2.6}$$

and hence it is different from zero by the condition (C3). Let

$$G(x,s) = -\tfrac{1}{\omega}\begin{cases} \varphi(x)\psi(s), & \text{if } 0 \le x \le s < \infty, \\ \varphi(s)\psi(x), & \text{if } 0 \le s \le x < \infty. \end{cases} \tag{2.7}$$

Since φ,
$psi \in L^2(0,\infty)$, we have

$$\textstyle\int_0^\infty \int_0^\infty |G(x,s)|^2 dxds < \infty. \tag{2.8}$$

Theorem 2.1 *The nonhomogeneous BVP* (2.1), (2.2) *has a unique solution* $y \in L^2(0,\infty)$ *for which the formula*

$$y(x) = \textstyle\int_0^\infty G(x,s)h(s)ds, \quad 0 \le x < \infty \tag{2.9}$$

holds, where the function $G(x,s)$ *is defined by* (2.7) , *and* $G(x,s)$ *is called the Green's function of the BVP* (2.1), (2.2).

Proof. Under the condition (C3), by (2.6), the solutions $\varphi(x)$ and $\psi(x)$ of the homogeneous equation (1.2) are linearly independent and therefore by a variation of constants formula the general solution of the nonhomogeneous equation (2.1) has the form

$$y(x) = c_1\varphi(x) + c_2\psi(x) + \tfrac{1}{\omega}\textstyle\int_0^x [\varphi(x)\psi(s) - \varphi(s)\psi(x)]h(s)ds, \tag{2.10}$$

where c_1 and c_2 are arbitrary constants. Now we try to choose the constants c_1 and c_2 so that the function $y(x)$ satisfies also the boundary conditions (2.2).

From (2.10) we have

$$y^{[1]}(x) = c_1\varphi^{[1]}(x) + c_2\psi^{[1]}(x) + \tfrac{1}{\omega}\textstyle\int_0^x [\varphi^{[1]}(x)\psi(s) - \varphi(s)\psi^{[1]}(x)]h(s)ds. \tag{2.11}$$

Consequently,

$$y(0) = c_1\varphi(0) + c_2\psi(0) = -c_1\beta - c_2\delta,$$

$$y^{[1]}(0) = c_1\varphi^{[1]}(0) + c_2\psi^{[1]}(0) = c_1\alpha + c_2\gamma.$$

Substituting these values of $y(0)$ and $y^{[1]}(0)$ into the first condition of (2.2) we get

$$-c_2(\alpha\delta - \beta\gamma) = 0, \quad \text{that is,} \quad -c_2\omega = 0.$$

Therefore $c_2 = 0$ and (2.10), (2.11) become

$$y(x) = c_1\varphi(x) + \tfrac{1}{\omega}\textstyle\int_0^x [\varphi(x)\psi(s) - \varphi(s)\psi(x)]h(s)ds, \tag{2.12}$$

$$y^{[1]}(x) = c_1\varphi^{[1]}(x) + \tfrac{1}{\omega}\textstyle\int_0^x [\varphi^{[1]}(x)\psi(s) - \varphi(s)\psi^{[1]}(x)]h(s)ds.$$

Hence,

$$\begin{aligned} W_x(y,u) &= y(x)u^{[1]}(x) - y^{[1]}(x)u(x) \\ &= c_1W_x(\varphi,u) + \tfrac{1}{\omega}\textstyle\int_0^x [W_x(\varphi,u)\psi(s) - \varphi(s)W_x(\psi,u)]h(s)ds \\ &= -c_1\beta + \tfrac{1}{\omega}\textstyle\int_0^x [-\beta\psi(s) + \delta\varphi(s)]h(s)ds \\ &= -c_1\beta + \textstyle\int_0^x u(s)h(s)ds. \end{aligned}$$

Therefore

$$W_\infty(y,u) = -c_1\beta + \int_0^\infty u(s)h(s)ds.$$

Likewise

$$W_\infty(y,v) = c_1\alpha + \int_0^\infty v(s)h(s)ds.$$

Substituting these values of $W_\infty(y,u)$ and $W_\infty(y,v)$ into the second condition of (2.2) we get

$$c_1(-\beta\gamma + \alpha\delta) + \int_0^\infty [\gamma u(s) + \delta v(s)]h(s)ds = 0.$$

Hence,

$$c_1 = -\frac{1}{\omega}\int_0^\infty \psi(s)h(s)ds.$$

Putting this value of c_1 in (2.12) we obtain

$$y(x) = -\frac{1}{\omega}\int_x^\infty \varphi(x)\psi(s)h(s)ds - \frac{1}{\omega}\int_0^x \varphi(s)\psi(x)h(s)ds,$$

that is, the formulas (2.9), (2.7) hold. The theorem is proved.

Corollary 2.1 *The unique solution $y(x)$ of the nonhomogeneous equation* (2.1) *under the nonhomogeneous boundary conditions*

$$\alpha y(0) + \beta y^{[1]}(0) = d_1, \quad \gamma W_\infty(y,u) + \delta W_\infty(y,v) = d_2 \tag{2.13}$$

is given by the formula

$$y(x) = w(x) + \int_0^\infty G(x,s)h(s)ds,$$

where the function $G(x,s)$ *is defined by* (2.7), *and*

$$w(x) = \tfrac{d_2}{\omega}\varphi(x) - \tfrac{d_1}{\omega}\psi(x). \tag{2.14}$$

Proof. The function $w(x)$ defined by (2.14) is a unique solution of the homogeneous equation (1.2) satisfying, by (2.4) and (2.5), the nonhomogeneous boundary conditions (2.13), while the function $\int_0^\infty G(x,s)h(s)ds$ is, by Theorem 2.1, a unique solution of the nonhomogeneous equation (2.1) satisfying the homogeneous boundary conditions (2.2). Hence the desired result holds.

By Corollary 2.1 the nonlinear BVP (1.1), (1.7) in $L^2(0,\infty)$ is equivalent to the nonlinear integral equation

$$y(x) = w(x) + \int_0^\infty G(x,s)f(s,y(s))ds, \quad 0 \le x < \infty, \tag{2.15}$$

where the functions $w(x)$ and $G(x,s)$ are defined by (2.14) and (2.7), respectively.

So we have to investigate the equation (2.15) in $L^2(0,\infty)$. By (1.3), (2.8), and $w \in L^2(0,\infty)$ we can define the operator $A : L^2(0,\infty) \to L^2(0,\infty)$ by the formula

$$Ay(x) = w(x) + \int_0^\infty G(x,s)f(s,y(s))ds, \quad 0 \le x < \infty, \tag{2.16}$$

where $y \in L^2(0,\infty)$. Then the equation (2.15) can be written as $y = Ay$. Therefore solving equation (2.15) in $L^2(0,\infty)$ is equivalent to finding fixed points of the operator A.

3 THE LIPSCHITZ CASE

In this section we will use the following well-known Contraction Mapping Theorem named also as the Banach Fixed Point Theorem: *Let* $\mathcal{B}$ *be a Banach space and* $\mathcal{S}$ *a nonempty closed subset of* $\mathcal{B}$. *Assume* $A : \mathcal{S} \to \mathcal{S}$ *is a contraction, i.e. there is a* λ, $0 < \lambda < 1$, *such that* $\|Au - Av\| \le \lambda\|u - v\|$ *for all* u, v *in* $\mathcal{S}$. *Then* A *has a unique fixed point in* $\mathcal{S}$.

Theorem 3.1 *Assume conditions* $(C1)$, $(C2)$, *and* $(C3)$ *are satisfied. In addition, let the function* $f(x,\xi)$ *satisfy the following Lipschitz condition: there is a constant* $K > 0$ *so that*

$$\int_0^\infty |f(x,y(x)) - f(x,z(x))|^2 dx \le K^2 \int_0^\infty |y(x) - z(x)|^2 dx \tag{3.1}$$

for all y *and* z *in* $L^2(0,\infty)$. *If*

$$K\{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} < 1, \tag{3.2}$$

then the BVP (1.1), (1.7) *has a solution in* $L^2(0,\infty)$.

Proof. It will be sufficient to show that under the conditions of the theorem, the operator $A : L^2(0,\infty) \to L^2(0,\infty)$ defined by (2.16) is a contraction mapping. For $y, z \in L^2(0,\infty)$ we have

$|Ay(x) - Az(x)|^2 = | \int_0^\infty G(x,s)[f(s,y(s)) - f(s,z(s))]ds|^2$

$\le \int_0^\infty |G(x,s)|^2 ds \int_0^\infty |f(s,y(s)) - f(s,z(s))|^2 ds \le K^2 \int_0^\infty |y(s) - z(s)|^2 \int_0^\infty |G(x,s)|^2 ds$

$= K^2 \|y - z\|^2 \int_0^\infty |G(x,s)|^2 ds$

for x in $[0, \infty)$. Hence $\|Ay - Az\| \le \lambda \|y - z\|$, where

$$\lambda = K\{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} < 1$$

by (3.2). So, A is a contraction mapping and the theorem is proved.

Remark 3.1 *The condition* (3.1) *is satisfied if* $|f(x,\xi_1) - f(x,\xi_2)| \le K|\xi_1 - \xi_2|$ *for all* x *in* $[0,\infty)$ *and all* ξ_1, ξ_2 *in* $\mathbf{R}$.

In the next theorem, the function $f(x,\xi)$ satisfies a Lipschitz condition not on whole $L^2(0,\infty)$ but on a subset.

Theorem 3.2 *Assume conditions* $(C1)$, $(C2)$, *and* $(C3)$ *are satisfied. In addition, let there exist a number* $R > 0$ *such that*

$$\int_0^\infty |f(x,y(x)) - f(x,z(x))|^2 dx \le K^2 \int_0^\infty |y(x) - z(x)|^2 dx \tag{3.3}$$

for all y *and* z *in* $\mathcal{S} = \{u \in L^2(0,\infty) : \|u\| \le R\}$, *where* $K > 0$ *is a constant which may depend on* R. *If*

$$\{\int_0^\infty |w(x)|^2 dx\}^{1/2} + \{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} \{\sup_{y \in \mathcal{S}} \int_0^\infty |f(s,y(s))|^2 ds\}^{1/2} \le R \tag{3.4}$$

and

$$K\{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} < 1, \tag{3.5}$$

then the BVP (1.1), (1.7) *has a unique solution* $y \in L^2(0,\infty)$ *with*

$$\int_0^\infty |y(x)|^2 dx \le R^2.$$

Proof. Obviously, $\mathcal{S}$ is a closed subset of $L^2(0,\infty)$. Let $A : L^2(0,\infty) \to L^2(0,\infty)$ be the operator defined by (2.16). For y and z in $\mathcal{S}$, taking into account (3.3) and (3.5), in exactly the same way as in the proof of Theorem 3.1, we can get $\|Ay - Az\| \le \lambda \|y - z\|$, where $\lambda < 1$.

It remains to show that A maps $\mathcal{S}$ into itself. For y in $\mathcal{S}$, we have

$$\|Ay\| = \|w(x) + \int_0^\infty G(x,s) f(s,y(s)) ds\| \le \|w\| + \| \int_0^\infty G(x,s) f(s,y(s)) ds\|$$

$$\le \|w\| + \{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} \{\int_0^\infty |f(s,y(s))|^2 ds\}^{1/2} \le R \tag{3.6}$$

by (3.4). Therefore $A : \mathcal{S} \to \mathcal{S}$.

Now the contraction mapping theorem can be applied to obtain a unique solution of equation (2.15) in $\mathcal{S}$, and the proof is complete.

4 EXISTENCE OF SOLUTIONS

An operator (nonlinear, in general) acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

To get an existence theorem without uniqueness of solution, we will apply in this section the following Schauder Fixed Point Theorem: *Let $\mathcal{B}$ be a Banach space and $\mathcal{S}$ a nonempty bounded, convex, and closed subset of $\mathcal{B}$. Assume $A : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator. If the operator A leaves the set $\mathcal{S}$ invariant, i.e. if $A(\mathcal{S}) \subset \mathcal{S}$, then A has at least one fixed point in $\mathcal{S}$.*

Passing on to the BVP (1.1), (1.7) let $A : L^2(0,\infty) \to L^2(0,\infty)$ be the operator defined by (2.16).

Lemma 4.1 *Under the conditions* $(C1)$, $(C2)$, *and* $(C3)$ *the operator A is completely continuous.*

Proof. Consider $\epsilon > 0$ and $y_0 \in L^2(0,\infty)$. We want to show that there exists $\delta > 0$ such that

$$y \in L^2(0,\infty) \text{ and } \|y - y_0\| < \delta \text{ imply } \|Ay - Ay_0\| < \epsilon. \tag{4.1}$$

We have

$$|Ay(x) - Ay_0(x)|^2 \le \int_0^\infty |G(x,s)|^2 ds \int_0^\infty |f(s,y(s)) - f(s,y_0(s))|^2 ds.$$

Hence

$$\|Ay - Ay_0\|^2 \le M \textstyle\int_0^\infty |f(s,y(s)) - f(s,y_0(s))|^2 ds$$

$$= M \textstyle\int_0^N |f(s,y(s)) - f(s,y_0(s))|^2 ds + M \int_N^\infty |f(s,y(s)) - f(s,y_0(s))|^2 ds, \tag{4.2}$$

where

$$M = \int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds$$

and N is an arbitrary positive number. Further, by the condition (1.3) and the elementary inequalities

$$(a+b)^2 \le 2(a^2+b^2), \quad (a+b+c)^2 \le 3(a^2+b^2+c^2),$$

we have

$$\textstyle\int_N^\infty |f(s,y(s)) - f(s,y_0(s))|^2 ds \le \int_N^\infty [|f(s,y(s))| + |f(s,y_0(s))|]^2 ds$$

$$\le \textstyle\int_N^\infty [2g(s) + d|y(s)| + d|y_0(s)|]^2 ds \le \int_N^\infty [12g^2(s) + 3d^2|y(s)|^2 + 3d^2|y_0(s)|^2] ds$$

$$\le 12 \textstyle\int_N^\infty g^2(s) ds + 6d^2 \int_0^\infty |y(s) - y_0(s)|^2 ds + 9d^2 \int_N^\infty |y_0(s)|^2 ds.$$

Choose N such that

$$\int_N^\infty g^2(s) ds < \frac{\epsilon^2}{48M}, \quad \int_N^\infty |y_0(s)|^2 ds < \frac{\epsilon^2}{36d^2 M}.$$

Then we get

$$\int_N^\infty |f(s,y(s))-f(s,y_0(s))|^2 ds < \tfrac{\epsilon^2}{4M} + 6d^2\delta^2 + \tfrac{\epsilon^2}{4M}. \tag{4.3}$$

It is known (see [9, Chapter 1]) that under the condition $(C2)$ the operator F defined by $Fy(x) = f(x, y(x))$ is continuous in $L^2(0,\infty)$. Therefore after choosing N as above, we can find a $\delta_0 > 0$ such that $\|y - y_0\| < \delta_0$ implies

$$\int_0^N |f(s,y(s))-f(s,y_0(s))|^2 ds < \tfrac{\epsilon^2}{4M}. \tag{4.4}$$

Now setting

$$\delta^2 = min\{\frac{\epsilon^2}{24d^2M}, \delta_0^2\},$$

we get from (4.2), (4.3), and (4.4) the desired result (4.1). Thus, the operator A is continuous.

Next, let $Y \subset L^2(0,\infty)$ be a bounded set:

$$\|y\| \le c_1 \quad \text{for all} \quad y \in Y.$$

We must show that $A(Y)$ is a relatively compact set in $L^2(0,\infty)$, that is, every infinite subset of $A(Y)$ has a limit point in $L^2(0,\infty)$. To this end, we use the following known criterion for relatively compactness in $L^2(0,\infty)$: *A set $\mathcal{S} \subset L^2(0,\infty)$ is relatively compact if and only if $\mathcal{S}$ is bounded and for every $\epsilon > 0$ (i) there exists a $\delta > 0$ such that $\int_0^\infty |y(x+h) - y(x)|^2 dx < \epsilon$ for all $y \in \mathcal{S}$ and all $h \ge 0$ with $h < \delta$, (ii) there exists a number $N > 0$ such that $\int_N^\infty |y(x)|^2 dx < \epsilon$ for all $y \in \mathcal{S}$.*

For all $y \in Y$, we have [see (3.6)]

$$\|Ay\| \le \|w\| + \{M \int_0^\infty \int_0^\infty |f(s,y(s))|^2 ds\}^{1/2}.$$

On the other hand using (1.3) we have

$$\begin{aligned}\int_0^\infty |f(s,y(s))|^2 ds &\le \int_0^\infty [g(s) + d|y(s)|]^2 ds \le 2\int_0^\infty [g^2(s) + d^2|y(s)|^2] ds \\ &= 2(\|g\|^2 + d^2\|y\|^2) \le 2(\|g\|^2 + d^2c_1^2).\end{aligned}$$

Therefore $\|Ay\| \le \|w\| + \{2M(\|g\|^2 + d^2c_1^2)\}^{1/2}$ for all $y \in Y$, that is, $A(Y)$ is a bounded set in $L^2(0,\infty)$.

Further, for all $y \in Y$, we have

$$\begin{aligned}\int_0^\infty |Ay(x+h) - Ay(x)|^2 dx &= \int_0^\infty |\int_0^\infty [G(x+h,s) - G(x,s)] f(s,y(s)) ds|^2 dx \\ &\le \int_0^\infty \{\int_0^\infty |G(x+h,s) - G(x,s)|^2 ds \int_0^\infty |f(s,y(s))|^2 ds\} dx \\ &\le 2(\|g\|^2 + d^2c_1^2) \int_0^\infty \int_0^\infty |G(x+h,s) - G(x,s)|^2 dx ds.\end{aligned}$$

Hence we get by (2.8) that for given $\epsilon > 0$ there exists a $\delta > 0$, depending only on ϵ, such that

$$\int_0^\infty |Ay(x+h) - Ay(x)|^2 dx < \epsilon^2$$

for all $y \in Y$ and all $h \geq 0$ with $h < \delta$.

We also have, for all $y \in Y$,

$$\int_N^\infty |Ay(x)|^2 dx \leq 2(\|g\|^2 + d^2 c_1^2) \int_N^\infty \int_0^\infty |G(x,s)|^2 dx ds.$$

Hence we get again by (2.8) that for given $\epsilon > 0$ there exists a positive number N, depending only on ϵ, such that $\int_N^\infty |Ay(x)|^2 dx < \epsilon^2$ for all $y \in Y$.

Thus, $A(Y)$ is a relatively compact set in $L^2(0,\infty)$. The lemma is proved.

Theorem 4.1 *Assume conditions* $(C1)$, $(C2)$, *and* $(C3)$ *are satisfied. In addition, let there exist a number* $R > 0$ *such that*

$$\{\int_0^\infty |w(x)|^2 dx\}^{1/2} + \{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} \{\sup_{y \in \mathcal{S}} \int_0^\infty |f(s,y(s))|^2 ds\}^{1/2} \leq R, \quad (4.5)$$

where $\mathcal{S} = \{y \in L^2(0,\infty) : \|y\| \leq R\}$. *Then the BVP* (1.1), (1.7) *has at least one solution* $y \in L^2(0,\infty)$ *with*

$$\int_0^\infty |y(x)|^2 dx \leq R^2.$$

Proof. Let $A : L^2(0,\infty) \to L^2(0,\infty)$ be the operator defined by (2.16). By Lemma 4.1, the operator A is completely continuous. Using (4.5), as in the proof of Theorem 3.2, we can see that A maps the set $\mathcal{S}$ into itself. Besides, it is obvious that the set $\mathcal{S}$ is bounded, convex, and closed. Therefore, the Schauder fixed point theorem can be applied to obtain a solution of the equation (2.15) in $\mathcal{S}$. The theorem is proved.

Remark 4.1 *Since for all* y *in* $\mathcal{S}$ *the left-hand side of* (4.5) *is less than or equal to*

$$\|w\| + \{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} \{2(\|g\|^2 + d^2 R^2)\}^{1/2},$$

it follows that for given $R > 0$ *the condition* (4.5) *will be satisfied if the numbers* $d_1,\ d_2$ *in the boundary conditions* (1.7), *and the numbers* $\|g\|,\ d$ *in the condition* (1.3) *are sufficiently small.*

5 BOUNDARY VALUE PROBLEMS ON THE WHOLE AXIS

Consider the equation

$$-[p(x)y']' + q(x)y = f(x,y), \quad -\infty < x < \infty. \quad (5.1)$$

We will assume that the following conditions are satisfied.

(H1) The coefficients $p(x)$ and $q(x)$ are real-valued measurable functions on $\mathbf{R} = (-\infty,\infty)$ such that

$$\int_a^b \frac{dx}{|p(x)|} < \infty, \quad \int_a^b |q(x)| dx < \infty$$

for all finite real numbers a and b with $a < b$. Moreover, the functions $p(x)$ and $q(x)$ are such that all solutions of the second order linear differential equation

$$-[p(x)y']' + q(x)y = 0, \quad -\infty < x < \infty, \tag{5.2}$$

belong to $L^2(-\infty, \infty)$.

(H2) The function $f(x, \xi)$ is real-valued and continuous in $(x, \xi) \in \mathbf{R} \times \mathbf{R}$, and

$$|f(x, \xi)| \leq g(x) + d|\xi|$$

for all (x, ξ) in $\mathbf{R} \times \mathbf{R}$, where $g(x) \geq 0,\ \ g \in L^2(-\infty, \infty)$, and d is a positive constant.

Denote by D the linear manifold of all elements $y \in L^2(-\infty, \infty)$ such that $Ly = -[p(x)y']' + q(x)y$ is defined and $Ly \in L^2(-\infty, \infty)$. It follows from the Green's formula

$$\textstyle\int_a^b [(Ly)z - y(Lz)](x)dx = W_b(y, z) - W_a(y, z) \tag{5.3}$$

that, for all $y, z \in D$ the limits

$$W_{-\infty}(y, z) = \lim_{a \to -\infty} W_a(y, z), \qquad W_{\infty}(y, z) = \lim_{b \to \infty} W_b(y, z)$$

exist as finite numbers.

Let $u = u(x)$ and $v = v(x)$ be solutions of (5.2) satisfying the initial conditions

$$u(0) = 0,\ u^{[1]}(0) = 1; \quad v(0) = -1,\ v^{[1]}(0) = 0. \tag{5.4}$$

By condition $(H1)$ the solutions u and v belong to $L^2(-\infty, \infty)$, moreover, they belong to D. Therefore for each $y \in D$ the values $W_{\pm\infty}(y, u)$ and $W_{\pm\infty}(y, v)$ exist and are finite. Note that for these values we have, from (5.3) and (5.4)

$$W_{-\infty}(y, u) = y(0) - \int_{-\infty}^{0} u(x)Ly(x)dx, \quad W_{-\infty}(y, v) = y^{[1]}(0) - \int_{-\infty}^{0} v(x)Ly(x)dx,$$

$$W_{\infty}(y, u) = y(0) + \int_{0}^{\infty} u(x)Ly(x)dx, \quad W_{\infty}(y, v) = y^{[1]}(0) + \int_{0}^{\infty} v(x)Ly(x)dx.$$

Now we supplement equation (5.1) with the following boundary conditions at $-\infty$ and ∞:

$$\alpha W_{-\infty}(y, u) + \beta W_{-\infty}(y, v) = d_1, \qquad \gamma W_{\infty}(y, u) + \delta W_{\infty}(y, v) = d_2, \tag{5.5}$$

where α, β, γ and δ are given real numbers satisfying the condition

(H3) $\quad \omega := \alpha\delta - \beta\gamma \neq 0$,
and d_1, d_2 are arbitrary given real numbers.

So we look for solutions $y \in L^2(-\infty, \infty)$ of equation (5.1) satisfying the boundary conditions (5.5).

Let us set

$$\varphi(x) = \alpha u(x) + \beta v(x), \quad \psi(x) = \gamma u(x) + \delta v(x)$$

and define the function

$$G(x,s) = -\frac{1}{\omega}\begin{cases} \varphi(x)\psi(s), & \text{if } -\infty \le x \le s < \infty, \\ \varphi(s)\psi(x), & \text{if } -\infty \le s \le x < \infty, \end{cases} \tag{1}$$

$$w(x) = \frac{d_2}{\omega}\varphi(x) - \frac{d_1}{\omega}\psi(x).$$

We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dx ds \ < \infty, \qquad \int_{-\infty}^{\infty} |w(x)|^2 dx < \infty.$$

The BVP (5.1), (5.5) in $L^2(-\infty,\infty)$ is equivalent to the integral equation

$$y(x) = w(x) + \int_{-\infty}^{\infty} G(x,s) f(s, y(s)) ds, \qquad -\infty < x < \infty.$$

Reasoning as in the previous sections we can prove the following theorems.

Theorem 5.1 *Assume conditions* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. In addition, let the function* $f(x,\xi)$ *satisfy the following Lipschitz condition: there is a constant* $K > 0$ *so that*

$$\textstyle\int_{-\infty}^{\infty} |f(x,y(x)) - f(x,z(x))|^2 dx \le K^2 \int_{-\infty}^{\infty} |y(x) - z(x)|^2 dx$$

for all y *and* z *in* $L^2(-\infty,\infty)$*. If*

$$K\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dx ds\}^{1/2} < 1,$$

then the BVP $(5.1), (5.5)$ *has a unique solution in* $L^2(-\infty,\infty)$.

Theorem 5.2 *Assume conditions* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. In addition, let there exist a number* $R > 0$ *such that*

$$\int_{-\infty}^{\infty} |f(x,y(x)) - f(x,z(x))|^2 dx \le K^2 \int_{-\infty}^{\infty} |y(x) - z(x)|^2 dx$$

for all y *and* z *in* $\mathcal{S} = \{u \in L^2(-\infty,\infty) : \|u\| \le R\}$, *where* $K > 0$ *is a constant which may depend on* R. *If*

$$\{\int_{-\infty}^{\infty} |w(x)|^2 dx\}^{1/2} + \{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dx ds\}^{1/2}\{\sup_{y\in\mathcal{S}}\int_{-\infty}^{\infty} |f(s,y(s))|^2 ds\}^{1/2} \le R$$

and

$$K\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dx ds\}^{1/2} < 1,$$

then the BVP $(5.1), (5.5)$ *has a unique solution* $y \in L^2(-\infty,\infty)$ *with* $\|y\| \le R$.

Theorem 5.3 *Assume conditions* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. In addition, let there exist a number* $R > 0$ *such that*

$$\{\int_{-\infty}^{\infty} |w(x)|^2 dx\}^{1/2} + \{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dx ds\}^{1/2}\{\sup_{y\in\mathcal{S}}\int_{-\infty}^{\infty} |f(s,y(s))|^2 ds\}^{1/2} \le R,$$

where $\mathcal{S} = \{y \in L^2(-\infty,\infty) : \|y\| \le R\}$. *Then the BVP* $(5.1), (5.5)$ *has at least one solution* $y \in L^2(-\infty,\infty)$ *with* $\|y\| \le R$.

References

[1] R.P. Agarwal, M. Bohner, and D. O'Regan, Time scale systems on infinite intervals, *Nonlinear Anal.*, **47** (2001), 837-848.

[2] R.P. Agarwal, M. Bohner, and D. O'Regan, Time scale boundary value problems on infinite intervals, *J. Comput. Appl. Math.*, **141** (2002), 27-34.

[3] E.A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.

[4] C.T. Fulton, Parametrization of Titchmarsh's $m(\lambda)$-functions in the limit circle case, *Trans. Amer. Math. Soc.*, **229** (1977), 51-63.

[5] M.G. Gasymov and G.Sh. Guseinov, Uniqueness theorems for inverse spectral-analysis problems for Sturm-Liouville operators in the Weyl limit-circle case, *Differentsialnye Uravneniya*, **25** (1989), 588-599; English transl. in *Differential Equations* **25** (1989), 394-402.

[6] G.Sh.Guseinov, Completeness theorem for the dissipative Sturm-Liouville operator, *Turkish J. Math.*, **17** (1993), 48-53.

[7] G.Sh.Guseinov, A boundary value problem for second order nonlinear difference equations on the semi-infinite interval, *J. Difference Equ. Appl.*, **8** (2002), 1019-1032.

[8] G.Sh.Guseinov and H. Tuncay, The determinants of perturbation connected with a dissipative Sturm-Liouville operator , *J. Math. Anal. Appl.*, **194** (1995), 39-49.

[9] M.A. Krasnosel'skii, "Topological Methods in the Theory of Nonlinear Integral Equations", Gostekhteoretizdat, Moscow, 1956; English transl., Pergamon Press, New York, 1964.

[10] M.G. Krein, On the indeterminate case of the Sturm-Liouville boundary value problem in the interval $(0, \infty)$, *Izv. Akad. Nauk SSSR Ser. Mat.*, **16** (1952),293-324. (Russian)

[11] E.C. Titchmarsh, "Eigenfunction Expansions Associated with Second-Order Differential Equations", Vol. 1, 2nd ed., Oxford Univ. Press, Oxford, 1962.