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THREE–POINT BOUNDARY VALUE PROBLEMS WITH DELTA RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE ON TIME SCALES

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Abstract. In this paper, we establish the criteria for the existence and uniqueness of solutions of a three-point boundary value problem for a class of fractional differential equations on time scales. By using some well known fixed point theorems, sufficient conditions for the existence of solutions are established. An illustrative example is also presented.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, [18, 23, 24, 25]. Among all the researches on the theory of the fractional differential equations, the study of the boundary value problems for fractional differential equations recently has attracted a great deal of attention from many researchers. Some results have been obtained on the existence of positive solutions of the boundary value problems for some specific fractional differential equations (see [11, 17, 20, 22, 27, 28, 31, 32] and references therein).

Miller and Ross [21] has been done a pioneering work in discrete fractional calculus. Particularly, Atici and Eloe [5, 7, 8, 9] contributed to the improvement of the discrete fractional calculus. The existence problems of discrete fractional difference equations have been investigated by a several authors (see [1, 7, 8, 14, 15, 16] and references therein). Then, to unify the fractional differential equations with both continuous and discrete forms, fractional calculus on time scales was used, see [3, 4, 6, 10, 29]. Some basic definitions and theorems on time scales can be found in the books [12, 13], which are excellent references for calculus of time scales.

Recently, existence problems of initial value problems for fractional differential equations on time scales has been studied by a few authors [2, 30, 33]. However, to the best of our knowledge, there exists no literature devoted to three-point boundary value problems of fractional differential equations on time scales. Boundary value problems are an important class of dynamic equations, because of their striking applications to almost all area of science, engineering and technology. By researching boundary value

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problems of fractional differential equations on time scales, the results unify the theory of fractional differential and fractional difference equations (and removes obscurity from both areas) and provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time.

We shall consider the following nonlinear three-point boundary value problem with delta Riemann-Liouville fractional derivative on time scales of order $\alpha-1$:

$$\begin{cases} \Delta_{a^*}^{\alpha-1}x(t)=f(t,x(t)),\ t\in J:=[a,b]\cap\mathbb{T},\ 2<\alpha<3 \\ x(a)=x^{\Delta}(b)=0,\ x^{\Delta}(a)=x^{\Delta}(c), \end{cases} \tag{1}$$

where $\mathbb{T}$ is any time scale, $c\in(a,b)\cap\mathbb{T}$, $f\in C([a,b]\times\mathbb{R},\mathbb{R})$ and $\Delta_{a^*}^{\alpha-1}$ denotes the delta fractional derivative on time scale $\mathbb{T}$ of order $\alpha-1$ which will be defined later.

2. Preliminaries

To state the main results of this paper, we will give some definitions of delta Riemann-Liouville type fractional integral and delta fractional derivative on time scales and auxiliary lemmas which are needed later.

Let us consider the rd-continuous functions $h_\alpha:\mathbb{T}\times\mathbb{T}\to\mathbb{R},\ \alpha\geqslant 0$, such that

$$h_{\alpha+1}(t,s)=\int_s^t h_\alpha(\tau,s)\Delta\tau,\ \ h_0(t,s)=1\ \ \forall s,t\in\mathbb{T}, \tag{2}$$

where $\mathbb{T}$ is a time scale such that $\mathbb{T}^k=\mathbb{T}$. Also, we suppose

$$\int_{\sigma(u)}^t h_{\alpha-1}(t,\sigma(\tau))h_{\beta-1}(\tau,\sigma(u))\Delta\tau=h_{\alpha+\beta-1}(t,\sigma(u)),\ \ \alpha,\beta>1,\ u<t,\ u,t\in\mathbb{T}, \tag{3}$$

where σ is the forward jump operator.

If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and $h_k(t,s)=\frac{(t-s)^k}{k!}$, $\forall k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and we define

$$h_\alpha(t,s)=\frac{(t-s)^\alpha}{\Gamma(\alpha+1)},\ \alpha>0$$

which satisfies the properties in (2) and (3) (see [3]).

If $\mathbb{T}=\mathbb{Z}$, then $\sigma(t)=t+1$ and $h_k(t,s)=\frac{(t-s)^{(k)}}{k!}$, $\forall k\in\mathbb{N}_0$, where $t^{(0)}=1$, $t^{(k)}=\prod_{i=0}^{k-1}(t-i)$. We define

$$h_\alpha(t,s)=\frac{(t-s)^{(\alpha)}}{\Gamma(\alpha+1)},\ \alpha>0$$

where $t^{(\alpha)}=\frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$ which satisfies the properties in (2) and (3) (see [3]).

DEFINITION 1. [3] For $\alpha \geqslant 1$, a time scale delta Riemann-Liouville type fractional integral is defined by

$$K_a^\alpha f(t) = \int_a^t h_{\alpha-1}(t,\sigma(\tau))f(\tau)\Delta\tau, \quad K_a^0 f = f,$$

where $f \in L_1([a,b]\cap\mathbb{T})$ and $t \in [a,b]\cap\mathbb{T}$.

If $\alpha = 1$, then we have $K_a^1 f(t) = \int_a^t f(\tau)\Delta\tau$.

DEFINITION 2. [3] For $\alpha \geqslant 2$, $m-1<\alpha\leqslant m\in\mathbb{N}$, i.e., $m = \ulcorner \alpha \urcorner$ (ceiling of the number) and $v = m-\alpha$, the $\Delta-$ fractional derivative on time scale $\mathbb{T}$ of order $\alpha-1$ is defined by

$$\Delta_{a*}^{\alpha-1} f(t) = (K_a^{v+1} f^{\Delta^m})(t) = \int_a^t h_v(t,\sigma(\tau))f^{\Delta^m}(\tau)\Delta\tau, \ \forall t\in[a,b]\cap\mathbb{T},$$

where $f \in C_{rd}^m([a,b]\cap\mathbb{T})$ and f^{Δ^m} is a Lebesgue Δ-integrable function.

If we take $\alpha = m$, then we have $\Delta_{a*}^{\alpha-1} f(t) = (K_a^1 f^{\Delta^m})(t) = f^{\Delta^{m-1}}$.

LEMMA 1. [3] *Let* $\alpha > 2$, $m-1<\alpha<m\in\mathbb{N}$, $v=m-\alpha$, $f\in C_{rd}^m(\mathbb{T})$, $a,b\in\mathbb{T}$, $\mathbb{T}^k=\mathbb{T}$. *Suppose* $h_{\alpha-2}(s,\sigma(t))$, $h_v(s,\sigma(t))$ *to be continuous on* $([a,b]\cap\mathbb{T})^2$. *Then, we have*

$$K_a^{\alpha-1}\Delta_{a*}^{\alpha-1} f(t) = f(t) + E(f^{\Delta^m},\alpha-1,v+1,\mathbb{T},t) - \sum_{k=0}^{m-1} h_k(t,a)f^{\Delta^k}(a),$$

where $E(f^{\Delta^m},\alpha-1,v+1,\mathbb{T},t) = \int_a^t f^{\Delta^m}(u)\mu(u)h_{\alpha-2}(t,\sigma(u))h_v(u,\sigma(u))\Delta u$ *and* $\mu(t) = \sigma(t)-t$.

LEMMA 2. *Assume that* $2<\alpha<3$, $\beta = 3-\alpha$, $x\in C_{rd}^3(\mathbb{T})$, $g\in C_{rd}([a,b]\cap\mathbb{T})$, $a,b\in\mathbb{T}$, $a<b$ *and* $\mathbb{T}^k=\mathbb{T}$, $h_{\alpha-1}(t,\sigma(s))$ *is continuous on* $J\times J$. *Then, a function* $x\in C_{rd}^3(\mathbb{T})$ *is a solution of the boundary value problem*

$$\begin{cases} \Delta_{a^*}^{\alpha-1} x(t) = g(t), \ t\in[a,b]\cap\mathbb{T}, \ 2<\alpha<3 \\ x(a) = x^\Delta(b) = 0, \ x^\Delta(a) = x^\Delta(c), \end{cases} \tag{4}$$

if and only if x is a solution of the following integral equation

$$
\begin{aligned}
x(t) = & \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau \\
& + \frac{(t-a)(b-a)-h_2(t,a)}{c-a}\int_a^c h^\Delta_{\alpha-2}(c,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau \\
& -(t-a)\int_a^b h^\Delta_{\alpha-2}(b,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau. \qquad (5)
\end{aligned}
$$

Proof. Let x be a solution of the BVP (4). By Lemma 1, we have

$$
\begin{aligned}
K_a^{\alpha-1}g(t) &= K_a^{\alpha-1}\Delta_{a^*}^{\alpha-1}x(t) \\
&= x(t)+\int_a^t x^{\Delta^3}(\tau)\mu(\tau)h_{\alpha-2}(t,\sigma(\tau))h_\beta(\tau,\sigma(\tau))\Delta\tau-\sum_{k=0}^{2}h_k(t,a)x^{\Delta^k}(a).
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
x(t) = & \int_a^t h_{\alpha-2}(t,\sigma(\tau))g(\tau)\Delta\tau-\int_a^t h_{\alpha-2}(t,\sigma(\tau))x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\Delta\tau \\
& +x(a)+h_1(t,a)x^\Delta(a)+h_2(t,a)x^{\Delta^2}(a) \qquad (6)
\end{aligned}
$$

and using the differentiation formula [12, Theorem 1.117], we find

$$
x^\Delta(t)=\int_a^t h^\Delta_{\alpha-2}(t,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau+x^\Delta(a)+(t-a)x^{\Delta^2}(a).
$$

From $x^\Delta(a)=x^\Delta(c)$, we have

$$
x^{\Delta^2}(a)=-\frac{1}{c-a}\int_a^c h^\Delta_{\alpha-2}(c,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau.
$$

From $x^\Delta(b)=0$, we get

$$
\begin{aligned}
x^\Delta(a) = & \frac{b-a}{c-a}\int_a^c h^\Delta_{\alpha-2}(c,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau \\
& -\int_a^b h^\Delta_{\alpha-2}(b,\sigma(\tau))\left(g(\tau)-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau.
\end{aligned}
$$

Thus, we obtain (5) by using (6).

The converse of the lemma follows by a direct computation. This completes the proof. $\square$

$C^3_{rd}(\mathbb{T})$ is a Banach space with the norm $\|x\| = \max_{t\in J}|x(t)| + \max_{t\in J}|x^{\Delta^3}(t)|$. The solutions of the BVP (1) are the fixed points of the operator $A : C^3_{rd}(\mathbb{T}) \to C^3_{rd}(\mathbb{T})$ defined by

$$\begin{aligned} Ax(t) = & \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(f(\tau,x(\tau)) - x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau \\ & + \frac{(t-a)(b-a)-h_2(t,a)}{c-a} \\ & \times \int_a^c h^{\Delta}_{\alpha-2}(c,\sigma(\tau))\left(f(\tau,x(\tau)) - x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau \\ & -(t-a)\int_a^b h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\left(f(\tau,x(\tau)) - x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau. \end{aligned} \tag{7}$$

For the sake of convenience, we set

$$\begin{aligned} M = & \max_{t\in J}\left(\int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\Delta\tau + \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))|\Delta\tau \right. \\ & \left. +(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))|\Delta\tau\right) + \max_{t\in J}\int_a^t |h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))|\Delta\tau \end{aligned} \tag{8}$$

and

$$\begin{aligned} N = & \max_{t\in J}\left(\int_a^t |h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau \right. \\ & + \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau \\ & \left. +(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau\right) \\ & + \max_{t\in J}\int_a^t |h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau. \end{aligned} \tag{9}$$

LEMMA 3. *Assume the following two conditions hold:*

$(H1)$ $|f(t,x)| \leqslant \phi(t)\psi(|x|)$ *for all* $t \in J,\ x \in C^3_{rd}(\mathbb{T})$, *where* $\phi : J \to [0,\infty)$ *and* $\psi : [0,\infty) \to [0,\infty)$ *are continuous and nondecreasing.*

$(H2)$ *The functions* $h_{\alpha-2}(t,\sigma(\tau))$, $h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))$, $h_2(t,a)$ *and* $\mu(t)h_\beta(t,\sigma(t))$ *are continuous for* $t \in J$ *and* $\tau \in J$.

Then, the operator $A : C^3_{rd}(\mathbb{T}) \to C^3_{rd}(\mathbb{T})$ *is completely continuous.*

Proof. We divide the proof into two steps.

Step 1. We show that A is continuous. Let (x_n) be a sequence such that $x_n \to x \in C^3_{rd}(\mathbb{T})$. Then, we obtain that

$$\begin{aligned}
&|(Ax_n)(t)-(Ax)(t)| \\
&\leqslant \int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,|f(\tau,x_n(\tau))-f(\tau,x(\tau))|\,\Delta\tau \\
&\quad+\int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\left|x_n^{\Delta^3}(\tau)-x^{\Delta^3}(\tau)\right|\Delta\tau \\
&\quad+\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))|\,|f(\tau,x_n(\tau))-f(\tau,x(\tau))|\,\Delta\tau \\
&\quad+\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\left|x_n^{\Delta^3}(\tau)-x^{\Delta^3}(\tau)\right|\Delta\tau \\
&\quad+(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))|\,|f(\tau,x_n(\tau))-f(\tau,x(\tau))|\,\Delta\tau \\
&\quad+(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\left|x_n^{\Delta^3}(\tau)-x^{\Delta^3}(\tau)\right|\Delta\tau
\end{aligned}$$

and

$$\begin{aligned}
\left|(Ax_n)^{\Delta^3}(t)-(Ax)^{\Delta^3}(t)\right| &\leqslant \int_a^t \left|h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\right|\,|f(\tau,x_n(\tau))-f(\tau,x(\tau))|\,\Delta\tau \\
&\quad+\int_a^t \left|h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\left|x_n^{\Delta^3}(\tau)-x^{\Delta^3}(\tau)\right|\Delta\tau.
\end{aligned}$$

From $f \in C([a,b]\times\mathbb{R},\mathbb{R})$, $(H2)$ and $\|x_n - x\| \to 0$ as $n \to \infty$, it follows that $\|Ax_n - Ax\| \to 0$ as $n \to \infty$. Hence, A is continuous.

Step 2. We show that the image of any bounded subset Ω of $C_{rd}^3(\mathbb{T})$ under A is relatively compact in $C_{rd}^3(\mathbb{T})$. For each $x \in \Omega = \left\{x \in C_{rd}^3(\mathbb{T}) : \|x\| \leqslant r\right\}$, we obtain

$$\begin{aligned}
|(Ax)(t)| \leqslant & \int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,|f(\tau,x(\tau))|\,\Delta\tau \\
& + \int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|x^{\Delta^3}(\tau)\right| \Delta\tau \\
& + \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a} \int_a^c |h_{\alpha-2}^{\Delta}(c,\sigma(\tau))|\,|f(\tau,x(\tau))|\,\Delta\tau \\
& + \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a} \int_a^c |h_{\alpha-2}^{\Delta}\left(c,\sigma(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|x^{\Delta^3}(\tau)\right| \Delta\tau \\
& + (t-a)\int_a^b |h_{\alpha-2}^{\Delta}(b,\sigma(\tau))|\,|f(\tau,x(\tau))|\,\Delta\tau \\
& + (t-a)\int_a^b |h_{\alpha-2}^{\Delta}\left(b,\sigma(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|x^{\Delta^3}(\tau)\right| \Delta\tau \\
\leqslant\ & \phi(b)\psi(r)\int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,\Delta\tau + \|x\| \int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \Delta\tau \\
& + \phi(b)\psi(r)\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a} \int_a^c |h_{\alpha-2}^{\Delta}(c,\sigma(\tau))|\,\Delta\tau \\
& + \|x\| \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a} \int_a^c |h_{\alpha-2}^{\Delta}(c,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\,\Delta\tau \\
& + \phi(b)\psi(r)(t-a)\int_a^b |h_{\alpha-2}^{\Delta}(b,\sigma(\tau))|\,\Delta\tau \\
& + \|x\| \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a} \int_a^b |h_{\alpha-2}^{\Delta}(b,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\,\Delta\tau
\end{aligned}$$

and

$$\begin{aligned}
\left|(Ax)^{\Delta^3}(t)\right| \leqslant & \int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\right| |f(\tau,x(\tau))|\,\Delta\tau \\
& + \int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|x^{\Delta^3}(\tau))\right| \Delta\tau
\end{aligned}$$

$$\leqslant \phi(b)\psi(r)\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\right|\Delta\tau$$
$$+\|x\|\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau.$$

Therefore,

$$\begin{aligned}\|Ax\| &\leqslant \phi(b)\psi(r)M+\|x\|N \\ &\leqslant \phi(b)\psi(r)M+rN,\end{aligned} \tag{10}$$

that is $A\Omega$ is a bounded set.

Now we show that $A\Omega$ is equicontinuous on J. For each $t_1,t_2\in J$, without loss of generality we may assume that $t_1<t_2$, and for all $x\in\Omega$ one can see that

$$\begin{aligned}
&|Ax(t_2)-Ax(t_1)| \\
\leqslant &\left|\int_a^{t_2} h_{\alpha-2}(t_2,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right. \\
&\left.-\int_a^{t_1} h_{\alpha-2}(t_1,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right| \\
&+\left|\frac{(t_2-a)(b-a)-h_2(t_2,a)}{c-a}\int_a^c h_{\alpha-2}^{\Delta}(c,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right. \\
&\left.-\frac{(t_1-a)(b-a)-h_2(t_1,a)}{c-a}\int_a^c h_{\alpha-2}^{\Delta}(c,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right| \\
&+\left|(t_2-a)\int_a^b h_{\alpha-2}^{\Delta}(b,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right. \\
&\left.-(t_1-a)\int_a^b h_{\alpha-2}^{\Delta}(b,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right| \\
\leqslant &\int_a^{t_1} |h_{\alpha-2}(t_2,\sigma(\tau))-h_{\alpha-2}(t_1,\sigma(\tau))|\left|f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau \\
&+\int_{t_1}^{t_2} |h_{\alpha-2}(t_2,\sigma(\tau))|\left|f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau \\
&+\frac{|t_2-t_1|(b-a)+|h_2(t_2,a)-h_2(t_1,a)|}{c-a}\int_a^c h_{\alpha-2}^{\Delta}(c,\sigma(\tau))
\end{aligned}$$

$$\times\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau$$
$$+(t_2-t_1)\int_a^b\left|h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\right|\Delta\tau$$
$$\to 0\ \ (t_1\to t_2)$$

and

$$\left|(Ax)^{\Delta^3}(t_2)-(Ax)^{\Delta^3}(t_1)\right|$$
$$\leqslant\left|\int_a^{t_2}h^{\Delta^3}_{\alpha-2}(t_2,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right.$$
$$\left.-\int_a^{t_1}h^{\Delta^3}_{\alpha-2}(t_1,\sigma(\tau))\left(f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right)\Delta\tau\right|$$
$$\leqslant\int_a^{t_1}\left|h^{\Delta^3}_{\alpha-2}(t_2,\sigma(\tau))-h^{\Delta^3}_{\alpha-2}(t_1,\sigma(\tau))\right|\left|f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau$$
$$+\int_{t_1}^{t_2}\left|h^{\Delta^3}_{\alpha-2}(t_2,\sigma(\tau))\right|\left|f(\tau,x(\tau))-x^{\Delta^3}(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau$$
$$\to 0\ \ (t_1\to t_2)$$

by using $(H2)$. It yields that $A\Omega$ is equicontinuous in $C^3_{rd}(\mathbb{T})$.

As a consequence of this steps, we obtain that A is completely continuous operator. □

3. Existence and uniqueness of solutions

In this section, we will use the following well-known contraction mapping theorem named also as the Banach fixed point theorem: Let $\mathscr{B}$ be a Banach space and S a nonempty closed subset of $\mathscr{B}$. Assume $A: S\to S$ is a contraction, i.e., there is a λ $(0<\lambda<1)$ such that $\|Ax-Ay\|\leqslant\lambda\|x-y\|$ for all x, y in S. Then A has a unique fixed point in S.

THEOREM 1. *Suppose that $(H2)$ holds. Also, we assume that*

$(H3)$ Let the function $f(t,x)$ satisfy the following Lipschitz condition: there is a constant $L>0$ such that

$$|f(t,x)-f(t,y)|\leqslant L|x-y|\ \textit{for each}\ t\in J, \tag{11}$$

for all x and y in $C^3_{rd}(\mathbb{T})$. Moreover, $LM+N<1$, where M and N are defined in (8) and (9), respectively.

Then, the BVP (1) has a unique solution in $C^3_{rd}(\mathbb{T})$.

Proof. For $x, y \in C_{rd}^3(\mathbb{T})$ and $t \in J$, we have

$$
\begin{aligned}
&|(Ax)(t)-(Ay)(t)| \\
\leqslant &\int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,|(f(\tau,x(\tau))-f(\tau,y(\tau)))|\,\Delta\tau \\
&+\int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|\left(x^{\Delta^3}(\tau)-y^{\Delta^3}(\tau)\right)\right|\Delta\tau \\
&+\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))|\,|f(\tau,x(\tau))-f(\tau,y(\tau))|\,\Delta\tau \\
&+\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))| \\
&\left|x^{\Delta^3}(\tau)-y^{\Delta^3}(\tau)\right|\Delta\tau \\
&+(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))|\,|f(\tau,x(\tau))-f(\tau,y(\tau))|\,\Delta\tau \\
&+(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))| \left|x^{\Delta^3}(\tau)-y^{\Delta^3}(\tau)\right|\Delta\tau \\
\leqslant &L\|x-y\|\int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,\Delta\tau \\
&+\|x-y\|\int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau \\
&+L\|x-y\|\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))|\,\Delta\tau \\
&+\|x-y\|\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau \\
&+L\|x-y\|(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))|\,\Delta\tau \\
&+\|x-y\|(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau
\end{aligned}
$$

and

$$\begin{aligned}\left|(Ax)^{\Delta^3}(t)-(Ay)^{\Delta^3}(t)\right| &\leqslant \int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\left(f(\tau,x(\tau))-f(\tau,y(\tau))\right)\right|\Delta\tau \\ &\quad+\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\left(x^{\Delta^3}(\tau)-y^{\Delta^3}(\tau)\right)\right|\Delta\tau \\ &\leqslant L\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\right||x(\tau)-y(\tau)|\Delta\tau \\ &\quad+\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\left|x^{\Delta^3}(\tau)-y^{\Delta^3}(\tau)\right|\Delta\tau \\ &\leqslant L\|x-y\|\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\right|\Delta\tau \\ &\quad+\|x-y\|\int_a^t \left|h_{\alpha-2}^{\Delta^3}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|Ax-Ay\| &= \max_{t\in J}|(Ax)(t)-(Ay)(t)|+\max_{t\in J}\left|(Ax)^{\Delta^3}(t)-(Ay)^{\Delta^3}(t)\right| \\ &\leqslant (LM+N)\|x-y\| \\ &= \lambda\|x-y\|,\end{aligned}$$

where $\lambda = LM+N \in (0,1)$. Hence, A is a contraction mapping and the theorem is proved. □

In the next theorem, the function $f(t,x)$ satisfies a Lipschitz condition on a subset of $C_{rd}^3(\mathbb{T})$.

THEOREM 2. *Suppose that $(H2)$ holds. Besides, we assume that*
$(H4)$ *Let there a number $r>0$ such that*

$$|f(t,x)-f(t,y)| \leqslant L|x-y| \text{ for each } t\in J, \tag{12}$$

for all x and y in $S=\{x\in C_{rd}^3(\mathbb{T}) : \|x\|\leqslant r\}$, where $L>0$ is a constant which may depend on r. Moreover, $LM+N<1$, where M and N are defined in (8) and (9), respectively.

$(H5)$ $\lim\limits_{x\to 0}\frac{f(t,x)}{x}=0$.

Then, the BVP (1) has a unique solution $x\in C_{rd}^3(\mathbb{T})$ with $\max\limits_{t\in J}|x(t)|+\max\limits_{t\in J}|x^{\Delta^3}(t)|\leqslant r$.

Proof. Since $\lim\limits_{x\to 0}\frac{f(t,x)}{x}=0$ from $(H5)$, there exists a constant $r>0$ such that $|f(t,x)|\leqslant\delta|x|$ for $0<|x|\leqslant r$, where $\delta>0$ is a constant satisfying $\delta M+N\leqslant 1$. Let us take $S=\{x\in C_{rd}^3(\mathbb{T}):\|x\|\leqslant r\}$. Obviously, S is a closed subset of $C_{rd}^3(\mathbb{T})$. Let $A:C_{rd}^3(\mathbb{T})\to C_{rd}^3(\mathbb{T})$ be the operator defined by (7). For x and y in S, taking into account $(H4)$, in exactly the same way in the proof of Theorem 1 we can get $\|Ax-Ay\|\leqslant\lambda\|x-y\|$, where $0<\lambda<1$.

It remains to show that A maps S into itself. If $x\in S$, then we obtain

$$
\begin{aligned}
|(Ax)(t)| \leqslant & \int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,|f(\tau,x(\tau))|\,\Delta\tau \\
& + \int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|x^{\Delta^3}(\tau)\right|\Delta\tau \\
& + \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))|\,|f(\tau,x(\tau))|\,\Delta\tau \\
& + \frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c \left|h^{\Delta}_{\alpha-2}\left(c,\sigma(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\right. \left|x^{\Delta^3}(\tau)\right|\Delta\tau \\
& + (t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))|\,|f(\tau,x(\tau))|\,\Delta\tau \\
& + (t-a)\int_a^b \left|h^{\Delta}_{\alpha-2}\left(b,\sigma(\tau)\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\right. \left|x^{\Delta^3}(\tau)\right|\Delta\tau \\
\leqslant & \ \delta r\int_a^t |h_{\alpha-2}(t,\sigma(\tau))|\,\Delta\tau + r\int_a^t \left|h_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right|\Delta\tau \\
& + \delta r\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))|\,\Delta\tau \\
& + r\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^c |h^{\Delta}_{\alpha-2}(c,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau \\
& + \delta r(t-a)\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))|\,\Delta\tau \\
& + r\frac{|(t-a)(b-a)-h_2(t,a)|}{c-a}\int_a^b |h^{\Delta}_{\alpha-2}(b,\sigma(\tau))\,\mu(\tau)h_\beta(\tau,\sigma(\tau))|\Delta\tau
\end{aligned}
$$

and

$$
\begin{aligned}
|(Ax)^{\Delta^3}(t)| \leqslant & \int_a^t \left|h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\right| |f(\tau,x(\tau))| \Delta\tau \\
& + \int_a^t \left|h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \left|x^{\Delta^3}(\tau))\right| \Delta\tau \\
& \leqslant \delta r \left(\int_a^t \left|h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\right| \Delta\tau \right) \\
& + r \left(\int_a^t \left|h^{\Delta^3}_{\alpha-2}(t,\sigma(\tau))\mu(\tau)h_\beta(\tau,\sigma(\tau))\right| \Delta\tau \right).
\end{aligned}
$$

Since $\|Ax\| = \max_{t\in J}|(Ax)(t)| + \max_{t\in J}|(Ax)^{\Delta^3}(t)| \leqslant \delta rM + rN \leqslant r$, we have $A : S \to S$.

From the contraction mapping theorem, the BVP (1) has a unique solution in $C^3_{rd}(\mathbb{T})$. □

4. Existence of solutions

THEOREM 3. [19, 26] *Let E be a Banach space. Assume that $A : E \to E$ is completely continuous operator and the set $V = \{u \in E : u = \lambda Au, 0 < \lambda < 1\}$ is bounded. Then A has a fixed point in E.*

THEOREM 4. *If the conditions $(H1)$ and $(H2)$ satisfy, then the BVP (1) has at least one solution in $C^3_{rd}(\mathbb{T})$.*

Proof. From Lemma 3, $A : C^3_{rd}(\mathbb{T}) \to C^3_{rd}(\mathbb{T})$ is completely continuous operator. Now, we will show that the set $V = \left\{x \in C^3_{rd}(\mathbb{T}) : x = \lambda Ax \text{ for some } 0 < \lambda < 1\right\}$ is bounded. For all $x \in V$, we get

$$
\begin{aligned}
\|x\| &= \|\lambda Ax\| \\
&\leqslant \lambda\phi(b)\psi(r)M + \lambda\|x\|N
\end{aligned}
$$

by using (10). Then we obtain $\|x\| \leqslant \frac{\lambda\phi(b)\psi(r)M}{1-\lambda N}$. From Theorem 3, the BVP (1) has at least one solution in $C^3_{rd}(\mathbb{T})$. □

THEOREM 5. [19, 26] *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $A : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$
\|Au\| \leqslant \|u\| \ \forall x \in \partial\Omega, \tag{13}
$$

then A has a fixed point in $\overline{\Omega}$.

THEOREM 6. *If the conditions $(H2)$ and $(H5)$ satisfy, then the BVP (1) has at least one solution.*

Proof. Since $\lim_{x\to 0}\frac{f(t,x)}{x}=0$, there exists a constant $r>0$ such that $|f(t,x)|\leqslant \delta|x|$ for $0<|x|<r$, where $\delta>0$ is a constant satisfying $\delta M+N<1$. Let us take $\Omega=\left\{x\in C^3_{rd}(\mathbb{T}) : \|x\|<r\right\}$. Since the function f satisfies the condition $(H1)$ by taking $\phi(t)=\delta$ and $\psi(|x|)=|x|$, $A:\overline{\Omega}\to C^3_{rd}(\mathbb{T})$ is completely continuous operator from Lemma 3. If we take $x\in\partial\Omega$, then we obtain $\|Ax\|\leqslant r$ as in the proof of Theorem 2. It follows that $\|Ax\|\leqslant\|x\|$, $\forall x\in\partial\Omega$. Therefore, by means of Theorem 5 the operator A has at least one fixed point in $\overline{\Omega}$. Thus, the BVP (1) has at least one solution $u\in\overline{\Omega}$. □

COROLLARY 1. *Assume that $(H1)$ and $(H2)$ hold. If $\phi(b)M+N\leqslant 1$ and $\psi(z)\leqslant z$, $\forall z\in[0,\infty)$, then the BVP (1) has at least one solution.*

EXAMPLE 1. Let $\mathbb{T}=q\mathbb{Z}=\{qk : k\in\mathbb{Z}\}$ and define

$$h_\alpha(t,s)=\frac{q^\alpha\left(\frac{t-s}{q}\right)^{(\alpha)}}{\Gamma(\alpha+1)},$$

where $t^{(\alpha)}=\frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$ which satisfies the properties in (2) and (3). Consider the following boundary value problem

$$\begin{cases}\Delta^{\alpha-1}_{0^*}x(t)=\frac{tx^4}{5+x^4},\ t\in J:=[0,20]\cap\mathbb{T},\ 2<\alpha<3\\ x(0)=x^\Delta(20)=0,\ x^\Delta(0)=x^\Delta(10).\end{cases}\tag{14}$$

Since $f(t,x)=\frac{tx^4}{5+x^4}\in C([0,20]\times\mathbb{R},\mathbb{R})$ holds $f(t,x)\leqslant 20$ and the condition $(H2)$ is satisfied, the BVP (14) has at least one solution by using Theorem 4.

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