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EXISTENCE OF POSITIVE SOLUTIONS FOR HIGHER ORDER THREE-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract

In this paper, by using the four functionals fixed point theorem, Avery-Henderson fixed point theorem and the five functionals fixed point theorem, respectively, we investigate the conditions for the existence of at least one, two and three positive solutions to nonlinear higher order three-point boundary value problems on time scales.

Keywords: Boundary value problems, fixed point theorems, positive solutions, time scales

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1] (in his Phd thesis in 1988) and is a rapidly expanding area of research. Time scales theory explains the mathematical structure underpinning the theories of discrete and continuous dynamical systems and allows us to connect them. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. The study of time scales has led to many important applications, e.g. in the study of insect population models, epidemic models, heat transfer and neural networks. Some basic definitions and theorems on time scales can be found in the book [2] and another excellent source on time scales is the book [3].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

Definition 1.1 [2] Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operators $\sigma: \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$ while the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = sup\{s \in \mathbb{T} : s < t\}$.

If $\sigma(t) > t$, t is said to be right scattered, and if If $\sigma(t) = t$, t is said to be right dense. If $\rho(r) < r$, r is said to be left scattered, and if $\rho(r) = r$, r is said to be left dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - M$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.

Definition 1.2 [2] For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

Theorem 1.3 Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + (\sigma(t) - t)\, f^\Delta(t).$$

Definition 1.4 [2] A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right -dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 1.5 [2] If f is rd-continuous, then there is a function F such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, we define

$$\int_a^b f(t)\, \Delta t = F(b) - F(a), \quad \forall a, b \in \mathbb{T}.$$

Theorem 1.6 [2] Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b)} (\sigma(t) - t)f(t), & if\ a < b, \\ 0, & if\ a = b, \\ -\sum_{t\in[b,a)} (\sigma(t) - t)f(t), & if\ a > b. \end{cases}$$

In this paper we are concerned with the existence of single and multiple positive solutions to the following nonlinear higher order three-point boundary value problem (BVP) on time scales:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f\left(t, y(\sigma(t))\right),\ t \in [t_1, t_3] \subset \mathbb{T},\ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(\sigma(t_3)) = 0,\ \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = y^{\Delta^{2i+1}}(t_2) \end{cases} \tag{1}$$

for $0 \le i \le n - 1$, where $\alpha > 0$ and $\beta > 0$ are given constants. We assume that $f: [t_1, \sigma(t_3)] \times [0, \infty) \to [0, \infty)$ is continuous. Throughout this paper we suppose $\mathbb{T}$ is any time scale and $[t_1, t_3]$ is a subset of $\mathbb{T}$ such that $[t_1, t_3] = \{t \in \mathbb{T}: t_1 \le t \le t_3\}$.

The study of three-point boundary value problems was initiated by Neuberger [4] in 1966. The first result concerning existence of positive solutions for higher order three-point boundary value problems was given by Eloe and McKelwey [5] in 1997. Since then, by applying the cone theory techniques, more general nonlinear three point boundary value problems have been studied by several authors. We refer the reader to $[6 - 8]$ and references therein.

In recent years, there has been much research activity concerning the second order three-point boundary value problems for dynamic equations on time scales. We refer the reader to the recent papers $[9 - 17]$ and references cited therein. We would like to mention some results of Anderson and Avery [18], Anderson and Karaca [19], Sang [20], Yaslan [21], Sang [22], and Yaslan [23].

In [18], Anderson and Avery were concerned with the following even-order three-point BVP:

$$\begin{cases} (-1)^n x^{(\Delta\nabla)^n}(t) = \lambda h(t) f(x(t)),\ t \in [a, c] \subset \mathbb{T},\ n \in \mathbb{N} \\ x^{(\Delta\nabla)^i}(a) = 0,\ x^{\Delta\nabla^i}(c) = \beta x^{(\Delta\nabla)^i}(b),\ 0 \le i \le n - 1. \end{cases} \tag{2}$$

They have studied the existence of at least one positive solution to the BVP (2) using the functional-type cone expansion-compression fixed point theorem.

In [19], Anderson and Karaca investigated the following higher-order three-point BVP:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f\left(t, y\big(\sigma(t)\big)\right),\ t \in [a,b] \subset \mathbb{T},\ n \in \mathbb{N} \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a),\ \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)),\ 0 \le i \le n-1. \end{cases} \tag{3}$$

Existence results of bounded solutions of a noneigenvalue problem are first established as a result of the Schauder fixed point theorem. Second, the monotone method is discussed to ensure the existence of solutions of the BVP (3). Third, they established criteria for the existence of at least one positive solution of the eigenvalue problem by using the Krasnosel'skii fixed point theorem. Later, they investigated the existence of at least two positive solutions of the BVP (3) by using the Avery-Henderson fixed point theorem.

In [20], Sang considered the BVP (3). The existence result was first obtained by using a fixed point theorem due to Krasnoselskii and Zabreiko. Later, under certain growth conditions imposed on the nonlinearity, several sufficient conditions for the existence of a nonnegative and nontrivial solution were obtained by using Leray-Schauder nonlinear alternative.

In [21], Yaslan studied the following even-order three-point BVP:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f\left(t, y\big(\sigma(t)\big)\right),\ t \in [t_1, t_3] \subset \mathbb{T},\ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(t_1) = 0,\ \alpha y^{\Delta^{2i}}\big(\sigma(t_3)\big) - \beta y^{\Delta^{2i+1}}\big(\sigma(t_3)\big) = y^{\Delta^{2i+1}}(t_2) \end{cases} \tag{4}$$

The criteria for the existence of at least one solution and of at least one positive solution for the BVP (4) were established by using Schauder fixed point theorem and Krassnoselskii's fixed point theorem, respectively. Later, the existence of multiple positive solutions to the BVP (4) was investigated by using Avery–Henderson fixed point theorem and Legget–Williams fixed point theorem.

In [22], Sang was concerned with the BVP (4). The existence results of at least one positive solution for a noneigenvalue problem and an eigenvalue problem were established by using fixed point theorems, which have extended and improved the famous Guo-Krasnosel'skii fixed point theorem at different aspects.

In [23], we investigated the conditions for the existence of one or two positive solutions for the BVP (1) by using a result from the theory of fixed point index and establish the criteria for the existence of at least three positive solutions for the BVP (1) by using Leggett-Williams fixed point theorem.

We have organized the paper as follows. In Section 2, we give some lemmas which are needed later. In Section 3, first, we use the four functionals fixed point theorem to show the existence of at least one positive solutions for the BVP (1). Second, we apply the Avery-Henderson fixed-point theorem to prove the existence of at least two positive solutions to the BVP (1). Finally, we use the five functional fixed-point theorem to show that the existence of at least three positive solutions to the BVP (1).

2. Preliminaries

From [23], we know the linear boundary value problem

$$-y^{\Delta^2}(t) = h(t), \qquad t \in [t_1, t_3]$$
$$y^{\Delta}(\sigma(t_3)) = 0, \ \ \alpha y(t_1) - \beta y^{\Delta}(t_1) = y^{\Delta}(t_2),$$

has the unique solution

$$y(t) = \int_{t_1}^{\sigma(t_3)} \left(\sigma(s) + \frac{\beta}{\alpha} - t_1\right) h(s)\Delta s + \frac{1}{\alpha}\int_{t_2}^{\sigma(t_3)} h(s)\Delta s + \int_{t}^{\sigma(t_3)} \big(t - \sigma(s)\big)h(s)\Delta s.$$

If $G(t,s)$ is Green's function for the boundary value problem

$$-y^{\Delta^2}(t) = 0, \qquad t \in [t_1, t_3]$$
$$y^{\Delta}(\sigma(t_3)) = 0, \ \ \alpha y(t_1) - \beta y^{\Delta}(t_1) = y^{\Delta}(t_2),$$

then we have

$$G(t,s) = \begin{cases} H_1(t,s), & t_1 \le s \le t_2 \\ H_2(t,s), & t_2 < s \le t_3 \end{cases} \tag{5}$$

where

$$H_1(t,s) = \begin{cases} \sigma(s) + \frac{\beta}{\alpha} - t_1, & \sigma(s) \le t, \\ t + \frac{\beta}{\alpha} - t_1, & t \le s \end{cases} \quad \text{and} \quad H_2(t,s) = \begin{cases} \sigma(s) + \frac{\beta+1}{\alpha} - t_1, & \sigma(s) \le t, \\ t + \frac{\beta+1}{\alpha} - t_1, & t \le s. \end{cases}$$

To state the main results of this paper, we will need the following lemmas.

Lemma 2.1 [23] If $\alpha > 0$ and $\beta > 0$, then the Green's function $G(t,s)$ in (5) satisfies

$$G(t,s) \ge \frac{t - t_1}{\sigma(t_3) - t_1} G(\sigma(t_3), s)$$

for $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Lemma 2.2 [23] Let $\alpha > 0$ and $\beta > 0$. Then the Green's function $G(t,s)$ in (5) satisfies $0 < G(t,s) \le G(\sigma(s), s)$ for $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Lemma 2.3 [23] Assume $\alpha > 0, \beta > 0$ and $s \in [t_1, t_3]$. Then the Green's function $G(t,s)$ in (5) satisfies $\min_{t\in[t_2,\sigma(t_3)]} G(t,s) \ge K\|G(.,s)\|$, where

$$K = \frac{\beta + \alpha(t_2 - t_1)}{\beta + 1 + \alpha(\sigma(t_3) - t_1)} \tag{6}$$

and $\|.\|$ is defined by $\|x\| = \max_{t\in[t_1,\sigma(t_3)]}|x(t)|$.

If we let $G_1(t,s) := G(t,s)$ for G as in (5), then we can recursively define $G_j(t,s) = \int_{t_1}^{\sigma(t_3)} G_{j-1}(t,r)G(r,s)\Delta r$ for $2 \le j \le n$ and $G_n(t,s)$ is Green's function for the homogeneous problem

$$(-1)^n y^{\Delta^{2n}}(t) = 0, \qquad t \in [t_1,t_3] \subset \mathbb{T}, \qquad n \in N$$
$$y^{\Delta^{2i+1}}\big(\sigma(t_3)\big) = 0,\ \ \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = y^{\Delta^{2i+1}}(t_2),\ \ 0 \le i \le n-1$$

Lemma 2.4 [23] Let $\alpha > 0$ and $\beta > 0$. The Green's function $G_n(t,s)$ satisfies the following inequalities

$$0 \le G_n(t,s) \le L^{n-1}\|G(.,s)\|, \qquad (t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3]$$

and

$$G_n(t,s) \ge K^n M^{n-1}\|G(.,s)\|, \qquad (t,s) \in [t_2,\sigma(t_3)] \times [t_1,t_3]$$

where K is given in (6),

$$L = \int_{t_1}^{\sigma(t_3)} \|G(.,s)\|\Delta s > 0 \tag{7}$$

and

$$M = \int_{t_2}^{\sigma(t_3)} \|G(.,s)\|\Delta s > 0. \tag{8}$$

Let $\mathcal{B}$ denote the Banach space $C[t_1,\sigma(t_3)]$ with the norm $\|y\| = \max_{t\in[t_1,\sigma(t_3)]}|y(t)|$. Define the cone $P \subset \mathcal{B}$ by

$$P = \left\{y \in B: y(t) \ge 0,\ \min_{t\in[t_2,\sigma(t_3)]} y(t) \ge \frac{K^n M^{n-1}}{L^{n-1}}\|y\|\right\}. \tag{9}$$

where K, L, M are given in (6), (7), (8), respectively.

(1) is equivalent to the nonlinear integral equation

$$y(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s) f\left(s, y\big(\sigma(s)\big)\right)\Delta s \tag{10}$$

We can define the operator $A: P \to \mathcal{B}$ by

$$Ay(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s) f\left(s, y\big(\sigma(s)\big)\right)\Delta s \tag{11}$$

where $y \in P$. Then (10) can be written as $y = Ay$. Therefore solving (10) in P is equivalent to finding fixed points of the operator A in (11). If $y \in P$, then by Lemma 2.4 we have

$$\min_{t\in[t_2,\sigma(t_3)]} Ay(t) \ge \frac{K^n M^{n-1}}{L^{n-1}} \int_{t_1}^{\sigma(t_3)} \max_{t\in[t_1,\sigma(t_3)]} |G_n(t,s)| f\left(s, y\big(\sigma(s)\big)\right)\Delta s = \frac{K^n M^{n-1}}{L^{n-1}}\|Ay\|$$

Thus $Ay \in P$ and therefore $AP \subset P$. In addition, $A: P \to P$ is completely continuous by a standard application of the Arzela-Ascoli Theorem.

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove main theorems.

We are now in a position to present the four functionals fixed point theorem. Let φ and Ψ be nonnegative continuous concave functionals on the cone P, and let η and θ be nonnegative continuous convex functionals on the cone P. Then for positive numbers r, τ, μ and R, define the sets

$$Q(\varphi, \eta, r, R) = \{x \in P: r \leq \varphi(x),\ \eta(x) \leq R\},$$
$$U(\Psi, \tau) = \{x \in Q(\varphi, \eta, r, R): \tau \leq \Psi(x)\},$$
$$V(\theta, \mu) = \{x \in Q(\varphi, \eta, r, R): \theta(x) \leq \mu\}.$$

The following theorem can be found in [24].

Theorem 2.5 (Four Functionals Fixed Point Theorem) Suppose P is a cone in a real Banach space E, φ and Ψ are nonnegative continuous concave functionals on P, η and θ are nonnegative continuous convex functionals on P, and there exist nonnegative positive numbers r, τ, μ and R, such that $A: Q(\varphi, \eta, r, R) \to P$ is a completely continuous operator, and $Q(\varphi, \eta, r, R)$ is a bounded set. If

(i) $\{x \in U(\Psi, \tau): \eta(x) < R\} \cap \{x \in V(\theta, \mu): r < \varphi(x)\} \neq \emptyset$
(ii) $\varphi(Ax) \geq r$, for all $x \in Q(\varphi, \eta, r, R)$, with $\varphi(x) = r$ and $\mu < \theta(Ax)$,
(iii) $\varphi(Ax) \geq r$, for all $x \in V(\theta, \mu)$, with $\varphi(x) = r$,
(iv) $\eta(Ax) \leq R$, for all $x \in Q(\varphi, \eta, r, R)$, with $\eta(x) = R$ and $\Psi(Ax) < \tau$,
(v) $\eta(Ax) \leq R$, for all $x \in U(\Psi, \tau)$, with $\eta(x) = R$,

then A has a fixed point x in $Q(\varphi, \eta, r, R)$.

Theorem 2.6 [25] (Avery-Henderson Fixed Point Theorem) Let P be a cone in a real Banach space E. Set $P(\phi, r) = \{u \in P: \phi(u) < r\}$. Assume there exist positive numbers r and M, nonnegative increasing continuous functionals η, ϕ on P, and a nonnegative continuous functional θ on P with $\theta(0) = 0$ such that $\phi(u) \leq \theta(u) \leq \eta(u)$ and $\|u\| \leq M\phi(u)$ for all $u \in \overline{P(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that $\theta(\lambda u)) \leq \lambda\theta(u)$ for all $0 \leq \lambda \leq 1$ and $u \in \partial P(\theta, q)$. If $A: \overline{P(\phi, r)} \to P$ is a completely continuous operator satisfying

(i) $\phi(Au) > r$ for all $u \in \partial P(\phi, r)$,
(ii) $\theta(Au) < q$ for all $u \in \partial P(\theta, q)$,
(iii) $P(\eta, p) \neq \emptyset$ and $\eta(Au) > p$ for all $u \in \partial P(\eta, p)$,

then A has at least two fixed points u_1 and u_2 such that $p < \eta(u_1)$ with $\theta(u_1) < q$ and $q < \theta(u_2)$ with $\phi(u_2) < r$.

Now, we will present the five functionals fixed point theorem. Let φ, η, θ be nonnegative continuous convex functionals on the cone P, and γ, Ψ nonnegative continuous concave functionals on the cone P. For nonnegative numbers h, a, b, d and c, define the following convex sets:

$$\begin{cases} P(\varphi, c) = \{x \in P: \varphi(x) < c\}, \\ P(\varphi, \gamma, a, c) = \{x \in P: a \leq \gamma(x), \varphi(x) \leq c\}, \\ Q(\varphi, \eta, d, c) = \{x \in P: \eta(x) \leq d, \varphi(x) \leq c\}, \\ P(\varphi, \theta, \gamma, a, b, c) = \{x \in P: a \leq \gamma(x), \theta(x) \leq b, \varphi(x) \leq c\}, \\ Q(\varphi, \eta, \Psi, h, d, c) = \{x \in P: h \leq \Psi(x), \eta(x) \leq d, \varphi(x) \leq c\}. \end{cases} \tag{12}$$

The following theorem can be found in [26].

Theorem 2.7 (Five Functionals Fixed Point Theorem) Let P be a cone in a real Banach space E. Suppose that there exist nonnegative numbers c and M, nonnegative continuous concave functionals γ and Ψ on P, and nonnegative continuous convex functionals φ, η, θ on P, with $\gamma(x) \leq \eta(x)$, $\|x\| \leq M\varphi(x)$, $\forall x \in \overline{P(\varphi, c)}$. If $A: \overline{P(\varphi, c)} \to \overline{P(\varphi, c)}$ is a completely continuous and there exist nonnegative numbers h, a, k, b with $0 < a < b$ such that

(i) $\{x \in P(\varphi, \theta, \gamma, b, k, c): \gamma(x) > b\} \neq \emptyset$ and $\gamma(Ax) > b$ for $x \in P(\varphi, \theta, \gamma, b, k, c)$,
(ii) $\{ x \in Q(\varphi, \eta, \Psi, h, a, c): \eta\ (x) < a\} \neq \emptyset$ and $\eta(Ax) < a$ for $x \in Q(\varphi, \eta, \Psi, h, a, c)$,
(iii) $\gamma(Ax) > b$ for $x \in P(\varphi, \gamma, b, c)$ with $\theta(Ax) > k$,
(iv) $\eta(Ax) < a$ for $x \in Q(\varphi, \eta, a, c)$ with $\Psi(Ax) < h$,

then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, r)}$ such that $\eta(x_1) < a$, $\gamma(x_2) > b$, $\eta(x_3) > a$ with $\gamma(x_3) < b$.

3. Main results

Now, we will give the sufficient conditions to have at least one positive solution for the BVP (1). Four functionals fixed point theorem will be used to prove the next theorem.

Theorem 3.1 Suppose $\alpha > 0$ and $\beta > 0$. In addition, let there exist constants μ and τ with $0 < \frac{K^n M^{n-1}}{L^{n-1}}\mu < \tau \leq \mu < \frac{\tau L^{n-1}}{K^n M^{n-1}}$ such that the function f satisfies the following conditions:

(i) $f(t, y) \geq \frac{\mu}{ML^{n-1}}$ for all $(t, y) \in [t_2, \sigma(t_3)] \times \left[\frac{K^n M^{n-1}}{L^{n-1}}\mu, \mu\right]$,
(ii) $f(t, y) \leq \frac{\tau}{K^n L M^{n-1}}$ for all $(t, y) \in [t_1, \sigma(t_3)] \times \left[0, \frac{\tau L^{n-1}}{K^n M^{n-1}}\right]$.

Then the BVP (1) has at least one positive solution y such that $\frac{K^n M^{n-1}}{L^{n-1}}\mu \leq y(t) \leq \frac{\tau L^{n-1}}{K^n M^{n-1}}$ for $t \in [t_1, \sigma(t_3)]$.

Proof. Define maps $\varphi(y) = \Psi(y) = \min_{t \in [t_2, \sigma(t_3)]} y(t)$, $\theta(y) = \max_{t \in [t_2, \sigma(t_3)]} y(t)$, $\eta(y) = \max_{t \in [t_1, \sigma(t_3)]} y(t)$. Then φ and Ψ are nonnegative continuous concave functionals on P, and η and θ are nonnegative continuous convex functionals on P. Since $\|y\| = \max_{t \in [t_1, \sigma(t_3)]} |y(t)| = \eta(y) \leq \frac{\tau L^{n-1}}{K^n M^{n-1}}$ for all $y \in Q\left(\varphi, \eta, \frac{K^n M^{n-1}}{L^{n-1}}\mu, \frac{\tau L^{n-1}}{K^n M^{n-1}}\right)$, $Q\left(\varphi, \eta, \frac{K^n M^{n-1}}{L^{n-1}}\mu, \frac{\tau L^{n-1}}{K^n M^{n-1}}\right)$ is a bounded set. The operator $A: Q\left(\varphi, \eta, \frac{K^n M^{n-1}}{L^{n-1}}\mu, \frac{\tau L^{n-1}}{K^n M^{n-1}}\right) \to P$ is completely continuous by a standard application of the Arzela-Ascoli theorem.

Now, we verify that the remaining conditions of Theorem 2.5. We obtain

$$\Psi(\mu) = \mu \geq \tau, \qquad \eta(\mu) = \mu < \tfrac{\tau L^{n-1}}{K^n M^{n-1}}, \qquad \theta(\mu) = \mu, \quad \varphi(\mu) = \mu > \tfrac{K^n M^{n-1}}{L^{n-1}}\mu\,.$$

Then, we have $\mu \in \left\{y \in U(\Psi,\tau): \eta(y) < \frac{\tau L^{n-1}}{K^n M^{n-1}}\right\} \cap \left\{y \in V(\theta,\mu): \frac{K^n M^{n-1}}{L^{n-1}}\mu < \varphi(y)\right\}$, which means that (i) in Theorem 2.5 is fulfilled.

Now, we shall verify that the condition (ii) of Theorem 2.5 is satisfied. By Lemma 2.4, we get

$$\theta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3),s) f\left(s, y(\sigma(s))\right)\Delta s \leq L^{n-1}\int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right)\Delta s.$$

Since $\theta(Ay) > \mu$, we find

$$\int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right)\Delta s > \tfrac{\mu}{L^{n-1}}. \tag{13}$$

Then, we obtain

$$\begin{aligned}\varphi(Ay) &= \int_{t_1}^{\sigma(t_3)} G_n(t_2,s) f\left(s, y(\sigma(s))\right)\Delta s\\ &\geq K^n M^{n-1}\int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right)\Delta s > \frac{K^n M^{n-1}}{L^{n-1}}\mu\end{aligned}$$

using Lemma 2.4 and (13).

Now, we shall show that the condition (iii) of Theorem 2.5 holds. Since $\varphi(y) = \frac{K^n M^{n-1}}{L^{n-1}}\mu$ and $y \in V(\theta,\mu)$, we find $\frac{K^n M^{n-1}}{L^{n-1}}\mu \leq y(t) \leq \mu$ for $t \in [t_2, \sigma(t_3)]$. By Lemma 2.4 and the hypothesis (i), we have

$$\begin{aligned}\varphi(Ay) &= \int_{t_1}^{\sigma(t_3)} G_n(t_2,s) f\left(s, y(\sigma(s))\right)\Delta s\\ &\geq K^n M^{n-1}\int_{t_2}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right)\Delta s \geq \frac{K^n M^{n-1}}{L^{n-1}}\mu.\end{aligned}$$

Now, we shall verify that the condition (iv) of Theorem 2.5 is fulfilled. We get

$$\Psi(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(t_2,s) f\left(s, y(\sigma(s))\right)\Delta s \geq K^n M^{n-1}\int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right)\Delta s$$

using Lemma 2.4. Since $\Psi(Ay) < \tau$,

$$\int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right)\Delta s < \tfrac{\tau}{K^n M^{n-1}}. \tag{14}$$

Then, by Lemma 2.4 and (14) we obtain

$$\eta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3),s)\, f\left(s, y(\sigma(s))\right) \Delta s$$
$$\leq L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.,s)\|\, f\left(s, y(\sigma(s))\right) \Delta s < \frac{\tau L^{n-1}}{K^n M^{n-1}}.$$

Finally, we shall show that the condition (v) of Theorem 2.5 is satisfied. Since $\eta(y) = \frac{\tau L^{n-1}}{K^n M^{n-1}}$, we find $0 \leq y(t) \leq \frac{\tau L^{n-1}}{K^n M^{n-1}}$ for $t \in [t_1, \sigma(t_3)]$. Using Lemma 2.4 and the hypothesis (ii), we have

$$\eta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3),s) f\left(s, y(\sigma(s))\right) \Delta s$$
$$\leq L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s \leq \frac{\tau L^{n-1}}{K^n M^{n-1}}.$$

Hence, by Theorem 2.5, the BVP (1) has at least one positive solution y such that $\frac{K^n M^{n-1}}{L^{n-1}}\mu \leq y(t) \leq \frac{\tau L^{n-1}}{K^n M^{n-1}}$ for $t \in [t_1, \sigma(t_3)]$. This completes the proof.

Now we will use the Avery-Henderson fixed point theorem to prove the next theorem.

Theorem 3.2 Assume $\alpha > 0$ and $\beta > 0$. Suppose there exist numbers $0 < p < q < r$ such that the function f satisfies the following conditions:

(i) $f(t,y) > \frac{r}{K^n M^n}$ for all $(t,y) \in [t_2, \sigma(t_3)] \times \left[r, \frac{r L^{n-1}}{K^n M^{n-1}}\right]$,
(ii) $f(t,y) < \frac{q}{L^n}$ for all $(t,y) \in [t_1, \sigma(t_3)] \times \left[0, \frac{q L^{n-1}}{K^n M^{n-1}}\right]$,
(iii) $f(t,y) > \frac{p}{K^n M^n}$ for all $(t,y) \in [t_2, \sigma(t_3)] \times \left[\frac{K^n M^{n-1}}{L^{n-1}} p, p\right]$,

where K, L, M, are defined in (6), (7), (8), respectively. Then the BVP (1) has at least two positive solutions y_1 and y_2 such that

$$p < \max_{t\in[t_1,\sigma(t_3)]} y_1(t) \text{ with } \max_{t\in[t_2,\sigma(t_3)]} y_1(t) < q$$
$$q < \max_{t\in[t_2,\sigma(t_3)]} y_2(t) \text{ with } \min_{t\in[t_2,\sigma(t_3)]} y_2(t) < r.$$

Proof. Define the cone P as in (9). From Lemma 2.4, $AP \subset$ P and A is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by $\phi(y) := \min_{t\in[t_2,\sigma(t_3)]} y(t)$, $\theta(y) := \max_{t\in[t_2,\sigma(t_3)]} y(t)$, $\eta(y) := \max_{t\in[t_1,\sigma(t_3)]} y(t)$.

For each $y \in P$, we have $\phi(y) \leq \theta(y) \leq \eta(y)$ and from (9)

$$\|y\| \leq \frac{L^{n-1}}{K^n M^{n-1}} \phi(y).$$

Moreover, $\theta(0) = 0$ and for all $y \in P$, $\lambda \in [0,1]$ we get $\theta(\lambda y) = \lambda \theta(y)$.

We now verify that the remaining conditions of Theorem 2.6 hold.

Claim 1: If $y \in \partial P(\phi, r)$, then $\phi(Ay) > r$: Since $y \in \partial P(\phi, r)$ and $\|y\| \leq \frac{L^{n-1}}{K^n M^{n-1}}\phi(y)$, we have $r \leq y(t) \leq \frac{rL^{n-1}}{K^n M^{n-1}}$ for $t \in [t_2, \sigma(t_3)]$. Then, by hypothesis (i) and Lemma 2.4 we find

$$\phi(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(t_2, s) f\left(s, y(\sigma(s))\right) \Delta s$$
$$\geq K^n M^{n-1} \int_{t_2}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s > r.$$

Claim 2: If $y \in \partial P(\theta, q)$, then $\theta(Ay) < q$: Since $y \in \partial P(\theta, q)$ and $\|y\| \leq \frac{L^{n-1}}{K^n M^{n-1}}\phi(y)$, we have $0 \leq y(t) \leq \frac{qL^{n-1}}{K^n M^{n-1}}$ for $t \in [t_1, \sigma(t_3)]$. Thus, using hypothesis (ii) and Lemma 2.4 we get

$$\theta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3), s) f\left(s, y(\sigma(s))\right) \Delta s$$
$$\leq L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s < q.$$

Claim 3: $P(\eta, p) \neq \emptyset$ and $\eta(Ay) > p$ for all $y \in \partial P(\eta, p)$: Since $\frac{p}{2} \in P$ and $p > 0$, $\frac{p}{2} \in P(\eta, p)$. If $y \in \partial P(\eta, p)$ and $\eta(y) \geq \frac{K^n M^{n-1}}{L^{n-1}}\|y\|$, we obtain $\frac{K^n M^{n-1}}{L^{n-1}} p \leq y(t) \leq \|y\| = p$ for $t \in [t_2, \sigma(t_3)]$. Hence, by hypothesis (iii) and Lemma 2.4 we have

$$\eta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3), s) f\left(s, y(\sigma(s))\right) \Delta s$$
$$\geq K^n M^{n-1} \int_{t_2}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s > p.$$

Since the conditions of Theorem 2.6 are satisfied, the BVP (1) has at least two positive solutions y_1 and y_2 such that

$$p < \max_{t \in [t_1, \sigma(t_3)]} y_1(t) \text{ with } \max_{t \in [t_2, \sigma(t_3)]} y_1(t) < q$$
$$q < \max_{t \in [t_2, \sigma(t_3)]} y_2(t) \text{ with } \min_{t \in [t_2, \sigma(t_3)]} y_2(t) < r.$$

Now, we will apply the five functionals fixed point theorem to investigate the existence of at least three positive solutions for the BVP (1).

Theorem 3.3 Let $\alpha > 0$ and $\beta > 0$. Suppose that there exist constants a, b, c with $0 < a < b < \frac{bL^{n-1}}{K^n M^{n-1}} < c$ such that the function f satisfies the following conditions:

(i) $f(t, y) \leq \frac{c}{L^n}$ for $(t, y) \in [t_1, \sigma(t_3)] \times [0, c]$,

(ii) $f(t,y) > \frac{b}{K^n M^n}$ for $(t,y) \in [t_2, \sigma(t_3)] \times \left[b, \frac{bL^{n-1}}{K^n M^{n-1}}\right]$,

(iii) $f(t,y) < \frac{a}{L^n}$ for $(t,y) \in [t_1, \sigma(t_3)] \times [0,a]$,

where K, L, M, are defined in (6), (7), (8), respectively. Then the BVP (1) has at least three positive solutions $y_1,\ y_2$ and y_3 such that

$$\max_{t\in[t_1,\sigma(t_3)]} y_1(t) < a < \max_{t\in[t_1,\sigma(t_3)]} y_3(t), \qquad \min_{t\in[t_2,\sigma(t_3)]} y_3(t) < b < \min_{t\in[t_2,\sigma(t_3)]} y_2(t).$$

Proof. Define the cone P as in (9) and define these maps

$$\gamma(y) = \Psi(y) = \min_{t\in[t_2,\sigma(t_3)]} y(t), \qquad \theta(y) := \max_{t\in[t_2,\sigma(t_3)]} y(t),$$
$$\varphi(y) = \eta(y) = \max_{t\in[t_1,\sigma(t_3)]} y(t).$$

Then γ and Ψ are nonnegative continuous concave functionals on P, and φ, η and θ are nonnegative continuous convex functionals on P. Let $P(\varphi,c)$, $P(\varphi,\gamma,a,c)$, $Q(\varphi,\eta,d,c)$, $P(\varphi,\theta,\gamma,a,b,c)$ and $Q(\varphi,\eta,\Psi,h,d,c)$ be defined by (12). It is clear that $\gamma(y) \le \eta(y)$ and $\|y\| = \varphi(y),\ \forall y \in \overline{P(\varphi,c)}$.

If $y \in \overline{P(\varphi,c)}$, then we have $y(t) \in [0,c]$ for all $t \in [t_1, \sigma(t_3)]$. By Lemma 2.4 and the hypothesis (i), we get

$$\begin{aligned}\varphi(Ay) &= \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3),s) f\left(s, y(\sigma(s))\right) \Delta s \\ &\le L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s \le c.\end{aligned}$$

This proves that $A: \overline{P(\varphi,c)} \to \overline{P(\varphi,c)}$.

Now we verify that the remaining conditions of Theorem 2.7.

Let $y_1 = b + \varepsilon_1$ such that $0 < \varepsilon_1 < \left(\frac{L^{n-1}}{K^n M^{n-1}} - 1\right) b$. Since $\gamma(y_1) = b + \varepsilon_1 > b$, $\theta(y_1) = b + \varepsilon_1 < \frac{bL^{n-1}}{K^n M^{n-1}}$ and $\varphi(y_1) = b + \varepsilon_1 < \frac{bL^{n-1}}{K^n M^{n-1}} < c$, we obtain $\left\{y \in P\left(\varphi, \theta, \gamma, b, \frac{bL^{n-1}}{K^n M^{n-1}}, c\right) : \gamma(y) > b\right\} \ne \emptyset$. If $y \in P\left(\varphi, \theta, \gamma, b, \frac{bL^{n-1}}{K^n M^{n-1}}, c\right)$, then we have $b \le y(t) \le \frac{bL^{n-1}}{K^n M^{n-1}}$ for all $t \in [t_2, \sigma(t_3)]$. By using Lemma 2.4 and the hypothesis (ii), we get

$$\begin{aligned}\gamma(Ay) &= \int_{t_1}^{\sigma(t_3)} G_n(t_2,s) f\left(s, y(\sigma(s))\right) \Delta s \\ &\ge K^n M^{n-1} \int_{t_2}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s > b.\end{aligned}$$

Thus, the condition (i) of Theorem 2.7 holds.

Let $y_2 = a - \varepsilon_2$ such that $0 < \varepsilon_2 < \left(1 - \frac{K^n M^{n-1}}{L^{n-1}}\right) a$. Since $\eta(y_2) = a - \varepsilon_2 < a$, $\Psi(y_2) = a - \varepsilon_2 > \frac{K^n M^{n-1}}{L^{n-1}} a$ and $\varphi(y_2) = a - \varepsilon_2 < c$, we find $\{y \in Q(\varphi, \eta, \Psi\ , \frac{K^n M^{n-1}}{L^{n-1}} a, a, c) : \eta\ (y)\ <\ a\} \neq \emptyset$,. If $y \in Q(\varphi, \eta, \Psi\ , \frac{K^n M^{n-1}}{L^{n-1}} a, a, c)$, then we obtain $0 \leq y(t) \leq a$, for $t \in [t_1, \sigma(t_3)]$. Hence,

$$\begin{aligned}\eta(Ay) &= \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3), s) f\left(s, y(\sigma(s))\right) \Delta s \\ &\leq\ L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.\,, s)\| f\left(s, y(\sigma(s))\right) \Delta s < a\end{aligned}$$

by Lemma 2.4 and the hypothesis (iii). It follows that condition (ii) of Theorem 2.7 is fulfilled.

Now, we shall show that the condition (iii) of Theorem 2.7 is satisfied. We have

$$\theta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3), s) f\left(s, y(\sigma(s))\right) \Delta s \leq\ L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.\,, s)\| f\left(s, y(\sigma(s))\right) \Delta s$$

using Lemma 2.4. Since θ(Ay)> $\frac{bL^{n-1}}{K^n M^{n-1}}$, we get

$$\int_{t_1}^{\sigma(t_3)} \|G(.\,, s)\| f\left(s, y(\sigma(s))\right) \Delta s > \frac{b}{K^n M^{n-1}}. \tag{15}$$

Then, by Lemma 2.4 and (15) we find

$$\begin{aligned}\gamma(Ay) &= \int_{t_1}^{\sigma(t_3)} G_n(t_2, s) f\left(s, y(\sigma(s))\right) \Delta s \\ &\geq\ K^n M^{n-1} \int_{t_2}^{\sigma(t_3)} \|G(.\,, s)\| f\left(s, y(\sigma(s))\right) \Delta s > b.\end{aligned}$$

Finally, we shall verify that the condition (iv) of Theorem 2.7 holds. By Lemma 2.4, we obtain

$$\Psi(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(t_2, s) f\left(s, y(\sigma(s))\right) \Delta s \geq\ K^n M^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.\,, s)\| f\left(s, y(\sigma(s))\right) \Delta s.$$

Since $\Psi(Ay) < \frac{K^n M^{n-1}}{L^{n-1}} a$, we have

$$\int_{t_1}^{\sigma(t_3)} \|G(.\,, s)\| f\left(s, y(\sigma(s))\right) \Delta s < \frac{a}{L^{n-1}}. \tag{16}$$

Then, we find

$$\eta(Ay) = \int_{t_1}^{\sigma(t_3)} G_n(\sigma(t_3), s) f\left(s, y(\sigma(s))\right) \Delta s$$

$$\leq L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(.,s)\| f\left(s, y(\sigma(s))\right) \Delta s < a$$

using Lemma 2.4 and (16).

Since the conditions of Theorem 2.7 are satisfied, the BVP (1) has at least three positive solutions $y_1,\ y_2, y_3 \in \overline{P(\varphi, c)}$ such that

$\max_{t\in[t_1,\sigma(t_3)]} y_1(t) < a < \max_{t\in[t_1,\sigma(t_3)]} y_3(t),$
$\min_{t\in[t_2,\sigma(t_3)]} y_3(t) < b < \min_{t\in[t_2,\sigma(t_3)]} y_2(t).$

This completes the proof.

Example 3.4 Let $\mathbb{T} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup [3,6]$. Consider the following boundary value problem:

$$\begin{cases} -y^{\Delta^2}(t) = \dfrac{y^2}{y^2+1} \\ y^{\Delta}(6) = 0, \quad 3y\left(\dfrac{1}{3}\right) - 2y^{\Delta}\left(\dfrac{1}{3}\right) = y^{\Delta}(3). \end{cases}$$

If we take p = 0.0106, q = 0.0108 and r = 5, then all the conditions in Theorem 3.2 are satisfied. Thus, by Theorem 3.2, the BVP has at least two positive solutions y_1 and y_2 such that

$$0.0106 < \max_{t\in\left[\frac{1}{3},6\right]} y_1(t) \text{ with } \max_{t\in[3,6]} y_1(t) < 0.0108$$
$$0.0108 < \max_{t\in[3,6]} y_2(t) \text{ with } \min_{t\in[3,6]} y_2(t) < 5.$$

If we take a = 0.1, b = 1 and c = 12, then all the conditions in Theorem 3.3 are satisfied. Thus, the BVP has at least three positive solutions $y_1,\ y_2$ and y_3 such that

$$\max_{t\in\left[\frac{1}{3},6\right]} y_1(t) < 0.1 < \max_{t\in\left[\frac{1}{3},6\right]} y_3(t), \quad \min_{t\in[3,6]} y_3(t) < 1 < \min_{t\in[3,6]} y_2(t).$$

References

[1] Hilger, S., “Analysis on measure chains-A unified approach to continuous and discrete calculus”, *Results Math.*, 18 (1990): 18-56.

[2] Bohner, M. and Peterson, A., Dynamic Equations on Time Scales: An Introduction with Applications, Birkhauser, Boston, 2001.

[3] Bohner, M. and Peterson, A. (editors), Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[4] Neuberger, J. W., “The lack of self-adjointness in three point boundary value problems”,Pacific J. Math., 18 (1966): 165-168.

[5] Eloe, P. W. and McKelwey, J., “Positive solutions of three point boundary value problems”, Comm. Appl. Nonlinear Anal., 4 (1997): 45-54.

[6] Agarwal, R. P., O'Regan, D. and Yan, B., “Positive solutions for singular three-point boundary value problems”, Electron. J. Differential Equations, 2008 (2008): 1-20.
[7] Karaca, I. Y., “Discrete third-order three-point boundary value problem”, J. Comput. Appl. Math. 205 (2007): 458–468.
[8] Karaca, I. Y., “Positive solutions of an n th order three-point boundary value problem”, Rocky Mountain J. Math., 43 (2013): 205-224.
[9] Fan, J. and Han, F., “Existence of positive solutions to a three-point boundary value problem for second order dynamic equations with derivative on time scales”, Ann. Differential Equations, 30 (2014): 282–290.
[10] Guo, M., “Existence of positive solutions of p-Laplacian three-point boundary value problems on time scales”, Math. Comput. Modelling, 50 (2009): 248–253.
[11] Murty, K. N., Saryanayana, R. and Gopalarao, Ch., “Three-point boundary value problems on time scale dynamical systems-existence and uniqueness”, Bull. Calcutta Math. Soc., 106 (2014): 369–380.
[12] Prasad, K. R., Murali, P. and Rao, S. N., “Existence of multiple positive solutions to three-point boundary value problems on time scales”, Int. J. Difference Equ., 4 (2009): 219–232.
[13] Prasad, K. R., Sreedhar, N. and Srinivas, M. A. S., “Eigenvalue intervals for iterative systems of nonlinear three-point boundary value problems on time scales”, Nonlinear Stud., 22 (2015): 419–431.
[14] Sun, J., “Existence of positive solution to second-order three-point BVPs on time scales”, Bound. Value Probl,. Art. ID 685040 (2009): 1-6.
[15] Wang, D., “Three positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales”, Nonlinear Analysis, 68 (2008): 2172-2180.
[16] Yaslan, İ. and Liceli, O., “Three-point boundary value problems with delta Riemann-Liouville fractional derivative on time scales”, Fract. Differ. Calc., 6 (2016): 1–16.
[17] Zhao, B. and Sun, H., “Multiplicity results of positive solutions for nonlinear three-point boundary value problems on time scales”, Adv. Dyn. Syst. Appl., 4 (2009): 243–253.
[18] Anderson, D. R. and Avery, R. I., “An even-order three-point boundary value problem on time scales”, J. Math. Anal. Appl., 291 (2004): 514-525.
[19] Anderson, D. R. and Karaca, I. Y., “Higher-order three-point boundary value problem on time scales”, Comput. Math. Appl., 56 (2008): 2429-2443.
[20] Sang, Y., “Solvability of a higher-order three-point boundary value problem on time scales”, Abstract and Applied Analysis, Art. ID 341679 (2009): 1-16.
[21] Yaslan, İ., “Existence results for an even-order boundary value problem on time scales”, Nonlinear Analysis, 70 (2009): 483-491.
[22] Sang, Y., “Some new existence results of positive solutions to an even-order boundary value problem on time scales”, Abstract and Applied Analysis, Art. ID 314382 (2013): 1-9.
[23] Yaslan, İ., “Multiple positive solutions for a higher order boundary value problem on time scales”, Fixed Point Theory, 17 (2016): 201-214.

[24] Avery, R. I., Henderson. J. and O'Regan, D., “Four functionals fixed point theorem”, Math. Comput. Modelling, 48 (2008): 1081-1089.
[25] Avery, R. I. and Henderson, J., “Two positive fixed points of nonlinear operators on ordered Banach spaces”, Comm. Appl. Nonlinear Anal., 8 (2001): 27-36.
[26] Avery, R. I., “A generalization of the Legget-Williams fixed point theorem”, Math. Sci.Research Hot-Line, 3 (1999): 9-14.