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Characterization of the compact operators on the class $\left(bv, bv_k^{\theta}\right)$

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Abstract. The space bv, the set of all bounded variation sequences, has an important role in the summability theory. In recent study, this spaces has been extended to the space bv_k^{θ} and some matrix class on this space has been characterized [2]. In the present paper, for $1 \leq k < \infty$, computing Hausdorff measure of non-compactness, we characterize compact operators in the class $\left(bv, bv_k^{\theta}\right)$, where θ is a sequence of positive numbers.
Keywords: Sequence spaces, matrix transformations, bv_k^{θ} spaces
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INTRODUCTION

Let ω be the set of all complex sequences, l_k $(1 \leq k < \infty)$ and c be the set of all k-absolutely convergent series and convergent sequences, respectively. We write $bv = \left\{x = (x_k) \in w : \sum_{n=0}^{\infty} |\triangle x_n| < \infty\right\}$ for the set of all sequences of bounded variation. In [2], extending the space bv, the space bv_k^{θ} has been defined by

$$bv_k^{\theta} = \left\{x = (x_k) \in w : \sum_{n=0}^{\infty} \theta_n^{k-1} |\triangle x_n|^k < \infty,\ x_{-1} = 0\right\},$$

which is a BK space for $1 \leq k < \infty$, where (θ_n) is a sequence of nonnegative terms and $\triangle x_n = x_n - x_{n-1}$ for all n.

Also, it is reduced to bv^k in the special case $\theta_n = 1$ for all n, studied by Malkowsky, V.Rakočević and Živković [1], and $bv_1^{\theta} = bv$ for $k = 1$.

Let X and Y be subspaces of w and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series are convergent for $v, n \geq 0$. Then, A defines a matrix transformation from X into Y, denoted by $A \in (X, Y)$, if the sequence $Ax = (A_n(x)) \in Y$ for all sequence $x \in X$.

Lemma 1 *Let $1 \leq k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if*

$$\|A\|_{(\ell,\ell_k)} = \sup_{v} \left\{\sum_{n=0}^{\infty} |a_{nv}|^k\right\}^{1/k} < \infty$$

[1].

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If S and H are subsets of a metric space (X,d) and $\varepsilon > 0$, then S is called an ε-net of H, if, for every $h \in H$, there exists an $s \in S$ such that $d(h,s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite *ε-net* of H. By $\mathscr{M}_X$, we denote the collection of all bounded subsets of X. If $Q \in \mathscr{M}_X$, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathscr{M}_X \to [0,\infty)$ is called the Hausdorff measure of noncompactness [4].

If X and Y are normed spaces, $\mathscr{B}(X,Y)$ states the set of all bounded linear operators from X to Y and is also a normed space the norm

$$\|L\| = \sup_{x \in S_X} \|L(x)\|,$$

where S_X is a unit sphere in X, *i.e.*, $S_X = \{x \in X : \|x\| = 1\}$. Further, a lineer operator $L : X \to Y$ is said to be compact if its domain is all of X and the sequence $(L(x_n))$ has convergent subsequence in Y for every bounded sequence $x = (x_n) \in X$. We denote the set of such operators by $\mathscr{C}(X,Y)$.

The following results are important tool to compute Hausdorff measure of noncompactness.

Lemma 2 *Let X and Y be Banach spaces, $L \in \mathscr{B}(X,Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined*

$$\|L\|_\chi = \chi(L(S_X)),$$

and

$$L \in \mathscr{C}(X,Y) \text{ iff } \|L\|_\chi = 0$$

[3].

Lemma 3 *Let Q be a bounded subset of the normed space X where $X = \ell_k$ for $1 \le k < \infty$. If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, ...)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r\to\infty} \sup_{x\in Q} \|(I - P_r)(x)\|,$$

where I is the identity operator on X [4].

Lemma 4 *Let X be normed sequence space, χ_T and χ denote the Hausdorff measures of noncompactness on $\mathscr{M}_{X_T}$ and $\mathscr{M}_X$, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q)$ for all $Q \in \mathscr{M}_{X_T}$, where T is an infinite triangle matrix [3].*

Compact operators in the class $\left(bv, bv_k^\theta\right)$

The class $\left(bv, bv_k^\theta\right)$ of infinite matrices has more recently been characterized by Hazar and Sarıgöl [2] as follows. In the present paper, by computing Hausdorff measure of noncompactness, we characterize compact operators in the same class.

Theorem 1 *Let $1 \le k < \infty$ and $\theta = (\theta_n)$ be a sequence of positive numbers. Further let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \ge 0$. Then, $A \in \left(bv, bv_k^\theta\right)$ if and only*

$$\lim_{n\to\infty} \sum_{j=v}^{\infty} a_{nj} \text{ exists} \tag{1}$$

$$\sup_{n,v} \left| \sum_{j=v}^{\infty} a_{nj} \right| < \infty \tag{2}$$

$$\sup_{v} \sum_{n=0}^{\infty} \left| \theta_n^{1-1/k} \sum_{j=v}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right|^k < \infty \tag{3}$$

For $\theta_v = 1$, this theorem also includes the following result of [1].

Corollary 1 *Let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \geq 0$ and $1 < k < \infty$. Then, $A \in (bv, bv^k)$ if and only if $(1), (2)$ hold and*

$$\sup_v \sum_{n=0}^{\infty} \left| \sum_{j=v}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right|^k < \infty.$$

Now we give the following theorem.

Theorem 2 *Let $1 \leq k < \infty$ and $\theta = (\theta_n)$ be a sequence of positive numbers. If $A \in \left(bv, bv_k^{\theta}\right)$, then*

$$\|A\|_{\chi} = \lim_{r\to\infty} \left(\sum_{n=r+1}^{\infty} \left| \theta_n^{1-1/k} \sum_{j=v}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right|^k \right)^{1/k}$$

and

$$A \in \mathscr{C}\left(bv_k^{\theta}, bv\right) \text{ iff } \lim_{r\to\infty} \sum_{n=r+1}^{\infty} \left| \theta_n^{1-1/k} \sum_{j=v}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right|^k = 0.$$

Proof Let $T' : bv \to \ell$ and $T'' : bv_k^{\theta} \to \ell_k$ be given by $T'(x) = x_v - x_{v-1}$, $x_{-1} = 0$ and $T''(x) = \theta_v^{1/k^*}\left(x_v - x_{v-1}\right)$, $x_{-1} = 0$. Then, it is easy to show that T' and T" are linear bijection preseving norms, i.e., $\|x\|_{bv} = \|x\|_{\ell}$ and $\|x\|_{bv_k^{\theta}} = \|x\|_{\ell_k}$. So, bv and bv_k^{θ} are norm isometric to the spaces ℓ and ℓ_k, respectively, i.e., $bv \simeq \ell$ and $bv_k^{\theta} \simeq \ell_k$. Note that $T'(x) = y$ for $x \in bv$. Then, $x = T'^{-1}(y) \in S_{bv}$ if and only if $y \in S_{\ell}$, where S_X is unit sphere. Further, we define the infinite matrix D by

$$d_{nv} = \theta_n^{1-1/k} \sum_{j=v}^{\infty} \left(a_{nj} - a_{n-1,j}\right).$$

Note that we get

$$\begin{array}{ccc} bv & \xrightarrow{A} & bv_k^{\theta} \\ T' \downarrow & & T'' \downarrow \\ \ell & \xrightarrow{D} & \ell_k \end{array}$$

and so, $T''AT'^{-1} = D$ and $A \in \left(bv, bv_k^{\theta}\right)$ iff $D \in (\ell, \ell_k)$. Also,

$$\begin{aligned} \|A\|_{\left(bv, bv_k^{\theta}\right)} &= \sup_{x\neq 0} \frac{\|A(x)\|_{bv_k^{\theta}}}{\|x\|_{bv}} = \sup_{x\neq 0} \frac{\left\|T''^{-1}DT'(x)\right\|_{bv_k^{\theta}}}{\|x\|_{bv}} \\ &= \sup_{y\neq 0} \frac{\|D(y)\|_{\ell_k}}{\|y\|_{\ell}} = \|D\|_{(\ell,\ell_k)}. \end{aligned}$$

So, it follows from Lemma 2, Lemma 3 and Lemma 4 that

$$\begin{aligned} \|A\|_{\chi} &= \chi\left(AS_{bv}\right) = \chi(T''AS_{bv}) \\ &= \chi(DT'S_{bv}) = \lim_{r\to\infty} \sup_{y\in S_{\ell}} \left\|(I - P_r)\, D(y)\right\|_{\ell_k} \\ &= \lim_{r\to\infty} \sup_{y\in S_{\ell}} \left\|D^{(r)}(y)\right\|_k = \lim_{r\to\infty} \left\|D^{(r)}\right\|_{(\ell,\ell_k)} \\ &= \lim_{r\to\infty} \sup_v \left(\sum_{n=r+1}^{\infty} |d_{nv}|^k \right)^{1/k} \end{aligned}$$

where $P_r : \ell \to \ell$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$, and

$$d_{nv}^{(r)} = \begin{cases} 0, & 0 \le n \le r \\ d_{nv}, & n > r \end{cases}$$

which complete the proof together with Lemma 2.

In the special case, the following result is immediate.

Corollary 2 *Let* $1 \le k < \infty$. *If* $A \in (bv^k, bv)$, *then*

$$\|A\|_{\chi} = \lim_{r\to\infty} \sup_{v} \left(\sum_{n=r+1}^{\infty} \left| \sum_{j=v}^{\infty} (a_{nj} - a_{n-1,j}) \right|^k \right)^{1/k},$$

and

$$A \in \mathscr{C}\left(bv, bv^k\right) \text{ iff } \lim_{r\to\infty} \sup_{v} \sum_{n=r+1}^{\infty} \left| \sum_{j=v}^{\infty} (a_{nj} - a_{n-1,j}) \right|^k = 0.$$

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