China Ocean Eng., 2019, Vol. 33, No. 4, P. 477–483
DOI: 10.1007/s13344-019-0045-1, ISSN 0890-5487
http://www.chinaoceanengin.cn/ E-mail: coe@nhri.cn

New Wave Solutions of Time-Fractional Coupled Boussinesq–Whitham–Broer–Kaup Equation as A Model of Water Waves

Emrah ATILGAN[a], Mehmet SENOL[b], Ali KURT[c, *], Orkun TASBOZAN[d]
[a]Department of Management Informatics Systems, Mustafa Kemal University, Hatay, Turkey
[b]Department of Mathematics, Nevsehir Haci Bektas Veli University, Nevsehir, Turkey
[c]Department of Mathematics, Pamukkale University, Denizli, Turkey
[d]Department of Mathematics, Mustafa Kemal University, Hatay, Turkey

Received October 11, 2018; revised January 7, 2019; accepted February 26, 2019

Abstract
The main purpose of this paper is to obtain the wave solutions of conformable time fractional Boussinesq–Whitham–Broer–Kaup equation arising as a model of shallow water waves. For this aim, the authors employed auxiliary equation method which is based on a nonlinear ordinary differential equation. By using conformable wave transform and chain rule, a nonlinear fractional partial differential equation is converted to a nonlinear ordinary differential equation. This is a significant impact because neither Caputo definition nor Riemann–Liouville definition satisfies the chain rule. While the exact solutions of the fractional partial derivatives cannot be obtained due to the existing drawbacks of Caputo or Riemann–Liouville definitions, the reliable solutions can be achieved for the equations defined by conformable fractional derivatives.

Key words: time fractional coupled Boussinesq–Whitham–Broer–Kaup equation, conformable fractional derivative, auxiliary equation method

Citation: Atilgan, E., Senol, M., Kurt, A., Tasbozan, O., 2019. New wave solutions of time-fractional coupled Boussinesq–Whitham–Broer–Kaup equation as a model of water waves. China Ocean Eng., 33(4): 477–483, doi: 10.1007/s13344-019-0045-1

1 Introduction

In recent years, fractional calculus has attracted many researches in the area of applied mathematics, physics and branches of engineering (Sabatier et al., 2007). Since L'Hospital asked the question, in 1695, what might be a derivative order of 1/2, many researchers tried to find a definition of fractional derivative. Most of the studies focused on an integral form of fractional derivative. Two most famous approaches are the Riemann–Liouville definition and the Caputo definition. However, the two definitions have some drawbacks. For instance,

- Riemann–Liouville definition does not satisfy $D^{\alpha}1 = 0$ when α is not a natural number.
- Caputo definition assumes that the function is differentiable.
- Both definitions do not satisfy the derivative of the product of two functions.
- Both definitions do not satisfy the derivative of the quotient of two functions.
- Both definitions do not satisfy the chain rule.
- Both definitions do not satisfy the index rule.

We overcome these deficiencies of the existing definitions using conformable fractional derivatives. In this paper, we first give the definition and some properties of conformable fractional derivative and integral. Then Boussinesq–Whitham–Broer–Kaup (BWBK) equation and a brief description of the auxiliary equation method are expressed. We illustrate one example that shows reliability and efficiency of the presented method. Also, figures of the different values of α and the parameters in the solutions are presented. To the best of our knowledge, these solutions have not been given in literature before. Recently, Khalil et al. (2014) have introduced a new definition of fractional derivative and integral, called conformable fractional derivative and integral.

Definition 1.1 Let $f:[0,\infty)\rightarrow R$ is a function α-th order "conformable functional derivate" defined by

$$T_{\alpha}(f)(t)=\lim_{\varepsilon\rightarrow 0}\frac{f(t+\varepsilon t^{1-\alpha})-(f)(t)}{\varepsilon}\text{for all } t>0,\ \alpha\in(0,1). \quad (1)$$

Definition 1.2 The conformable integral of a function f starting from $a\geqslant 0$ is defined by Khalil et al. (2014) as:

$$I_{\alpha}^{a}(f)(s)=\int_{a}^{s}\frac{f(t)}{t^{1-\alpha}}\mathrm{d}t. \quad (2)$$

Similarly, the definitions of conformable fractional par-

*Corresponding author. E-mail: pau.dr.alikurt@gmail.com

tial derivative are given by Atangana et al. (2015).

Definition 1.3 Let f be a function with n variables such as $x_1,...,x_n$, and the conformable partial derivatives of f of order $\alpha \in (0,1]$ in x_i are defined as follows:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x_i^\alpha} f(x_1,...,x_n) = \lim_{\varepsilon \to 0} \frac{f(x_1,...,x_{i-1},x_i+\varepsilon x_i^{1-\alpha},...,x_n)-f(x_1,...,x_n)}{\varepsilon}. \quad (3)$$

The authors proved the product rule and showed how to prove the fractional Rolle Theorem and Mean Value Theorem for α-differentiable functions using their conformable definition. Abdeljawad (2015) improved their study by introducing left and right conformable fractional derivatives and provided the fractional versions of chain rule, exponential functions, Laplace transforms, Gronwall's inequality, Taylor power series expansions. Batarfi et al. (2015) applied conformable fractional derivative on a boundary value problem. Eslami and Rezazadeh (2016b) used the first integral method to construct exact solutions of the time-fractional Wu–Zhang system by describing the fractional derivatives using conformable idea. Iyiola and Nwaeze (2016) proved extended mean value theorem and the Racetrack-type principle for α-differentiable functions using conformable fractional derivatives and fractional integral. Tasbozan et al. (2018) used sine-Gordon expansion method to obtain the Drinfeld–Sokolov–Wilson system in shallow water waves. Aminikhah et al. (2016) employed sub-equation method to find the exact solutions to the fractional (1+1) and (2+1) regularized long-wave equations. Eslami and Mirzazadeh (2013) implemented first integral method to obtain the exact solutions of nonlinear Schrödinger equation. Rezazadeh (2018) used the new extended direct algebraic method to construct the exact solutions of the complex Ginzburg–Landau equation. Rezazadeh et al. (2018a) build the exact solutions of Schrödinger–Hirota equation with the help of new extended direct algebraic method. Many different and powerful methods such as the sine–cosine function method (Eslami and Mirzazadeh, 2016a), trial solution method (Eslami, 2016), the extended Fan sub-equation method (Tariq et al., 2018), Liu's extended trial function scheme (Rezazadeh et al., 2018c), modified Kudryashov's method (Biswas et al., 2018b, 2018c), modified simple equation method (Biswas et al., 2018a), Riccati sub equation method (Khodadad et al., 2017), functional variable method (Eslami et al., 2017), sine-Gordon expansion method (Rezazadeh et al., 2018b), and the unified method (Osman et al., 2018) were applied to obtain the exact solutions of various partial differential.

The properties of this new definition (Khalil et al., 2014) are given below.

Theorem 1.4 Let $\alpha \in (0,1]$ and f, g functions are α-differentiable at point $t>0$, then

(1) $T_\alpha(mf+ng) = mT_\alpha(f)+nT_\alpha(g)$ for all $m,n \in \mathbb{R}$;

(2) $T_\alpha(t^p) = pt^{p-\alpha}$ for all p;

(3) $T_\alpha(f.g) = fT_\alpha(g)+gT_\alpha(f)$;

(4) $T_\alpha(\frac{f}{g}) = \dfrac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$;

(5) $T_\alpha(c) = 0$ for all constant functions $f(t)=c$;

(6) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}$.

The Boussinesq estimation for water waves is a suitable approximation for weakly nonlinear and fairly long waves in fluid mechanics. The approximation is named after Joseph Valentin Boussinesq (1842–1929), who first derived them in reply to the investigation by John Scott Russell of the wave of translation (Boussinesq, 1872). Let $\boldsymbol{v} = \partial\varphi/\partial x$ be the horizontal flow velocity component, $\boldsymbol{w} = \partial\varphi/\partial z$ be the vertical flow component and g be the acceleration by gravity, the following equation denotes the boundary conditions at the free surface elevation $z=\eta(x,t)$ for water waves on an incompressible fluid and irrotational flow in the (x,z) plane with reference to Boussinesq's paper (Boussinesq, 1872),

$$\frac{\partial\eta}{\partial t}+\boldsymbol{v}\frac{\partial\eta}{\partial x}-\boldsymbol{w}=0;$$
$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\boldsymbol{v}^2+\boldsymbol{w}^2)+g\eta=0. \quad (4)$$

In Eq. (4) only considered are the linear and quadratic terms with respect to η and $\boldsymbol{v}_\mathrm{b}$ ($\boldsymbol{v}_\mathrm{b}=\partial\varphi_\mathrm{b}/\partial x$, the horizontal velocity at $z=-h$). By neglecting the cubic and the higher order terms the following partial equations are acquired:

$$\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial x}[(h+\eta)\boldsymbol{v}_\mathrm{b}]=\frac{1}{6}h^3\frac{\partial^3\boldsymbol{v}_\mathrm{b}}{\partial x^3};$$
$$\frac{\partial\boldsymbol{v}_\mathrm{b}}{\partial t}+\boldsymbol{v}_\mathrm{b}\frac{\partial\boldsymbol{v}_\mathrm{b}}{\partial x}+g\frac{\partial\eta}{\partial x}=\frac{1}{2}h^2\frac{\partial^3\boldsymbol{v}_\mathrm{b}}{\partial t\partial x^2}. \quad (5)$$

By setting the right side of the equations to zero, they become shallow water equation. By adding some approximations at the same accuracy, Eq. (5) can be written in the form

$$\frac{\partial^2\eta}{\partial t^2}-gh\frac{\partial^2\eta}{\partial x^2}-gh\frac{\partial^2}{\partial x^2}\left(\frac{3\eta^2}{2h}+\frac{h^2}{3}\frac{\partial^2\eta}{\partial x^2}\right)=0.$$

Using the water depth h and gravitational acceleration g for non-dimensionalization yields

$$\frac{\partial^2\psi}{\partial\tau^2}-gh\frac{\partial^2\psi}{\partial\chi^2}-\frac{\partial^2}{\partial\chi^2}\left(\frac{1}{2}\psi^2+\frac{\partial^2\psi}{\partial\chi^2}\right)=0, \quad (6)$$

where $\psi=3\dfrac{\eta}{h}, \tau=\sqrt{3\dfrac{g}{h}}t$ and $\chi=\sqrt{3}\dfrac{x}{h}$. Then Eq. (6) can be written as:

$$u_{tt}-u_{xx}-\left(\frac{1}{2}u^2+qu_{xx}\right)_{xx}=0, \quad (7)$$

where $|q|=1$ is a real parameter. If q is set to -1, we get the well-posed Boussinesq equation. Similarly, if q is set to 1, we get the ill-posed classical Boussinesq equation. In Eq. (7), when the term qu_{xx} is changed to qu_{tt}, the new equation named the improved Boussinesq equation is obtained as follows:

$$u_{tt} - u_{xx} - \left(\frac{1}{2}u^2 + qu_{tt}\right)_{xx} = 0. \tag{8}$$

Finally, variant Boussinesq equation

$$u_t + uu_x - v_x + qu_{xx} = 0;$$
$$v_t + (uv)_x + pu_{xxx} - qv_{xx} = 0, \tag{9}$$

studied by Sachs (1988). Sachs used the new variable $w = 1 + v$ and rewrite the system as:

$$u_t + w_x + uu_x = 0;$$
$$w_t + u_{xxx} + (wu)_x = 0, \tag{10}$$

where $u = u(x,t)$ is the velocity, and $v = v(x,t)$ is the height of the free wave surface for fluid in the trough, and the subscripts denote the partial derivatives (Sachs, 1988).

With superiority of conformable fractional derivative over the other fractional derivative definitions, we can obtain the analytical solutions for nonlinear partial differential equation. For instance, the analytical solution of time fractional coupled Boussinesq–Whitham–Broer–Kaup equation cannot be obtained using other fractional derivative definitions. They do not satisfy the chain rule, in spite of that we can obtain the analytical solution of conformable time fractional coupled Boussinesq–Whitham–Broer–Kaup equation.

2 Time fractional coupled Boussinesq–Whitham– Broer–Kaup equation

The investigation of the exact travelling wave solutions has always been a challenging research area in physics and in applied mathematics since most approximations consist of partial differential equations. In 1870, Boussinesq suggested some equations for the propagation of small amplitude and long waves of water. Whitham used Lagrangian approach to find linear and non-linear dispersive waves (Whitham, 1965) and developed a theory for slowly varying wave trains (Whitham and Lighthill, 1967).

In this paper, we use the auxiliary equation method to find a solution set for the system given in Eq. (10) by means of conformable fractional derivative. By using the conformable fractional derivative Eq. (10) is generalized to non-integer order partial differential equation as follows:

$$D_t^{\alpha} v + D_x(uv) + D_{xxx}u = 0;$$
$$D_t^{\alpha} u + D_x v + uD_x u = 0. \tag{11}$$

3 A brief description of the auxiliary equation method

Auxiliary equation method has been used to get exact solutions of nonlinear partial differential equations (Sirendaoreji and Jiong, 2003). This method is applicable to all nonlinear partial differential equations if the equations consist of only even-order partial derivative terms or only odd-order partial derivative terms. Using this method, Sirendaoreji and Jiong (2003) provided new exact travelling wave solutions with the aid of symbolic computation. Zhang and Xia defined a generalized auxiliary equation method (Zhang and Xia, 2007) inspired by Tasbozan et al. (2018), and applied their method to the combined KdV-mKdV equation and the (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations. Yomba (2008) applied the auxiliary equation method to solve the nonlinear Klein–Gordon equation and generalized Camassa–Holm equations.

Auxiliary equation method which depends on the differential equation was firstly mentioned by Sirendaoreji and Jiong (2003):

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\xi}\right)^2 = az^2(\xi) + bz^3(\xi) + cz^4(\xi) \tag{12}$$

By using Eq. (12), they obtained the analytical solutions of some nonlinear partial differential equations (Sirendaoreji and Jiong, 2003). To explain the method clearly, we will illustrate the steps as follows.

Step 1. The general form of nonlinear conformable fractional differential equation can be written as:

$$P\left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0, \tag{13}$$

where the arguments, subscripts of Polynomial P shows partial derivatives and $\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}$ means two times conformable derivative of the function $u(x,t)$.

Step 2. Using the wave transformation

$$u(x,t) = u(\xi),\ \xi = kx + w\frac{t^{\alpha}}{\alpha}, \tag{14}$$

in which k shows the number of wave and w denotes the velocity of the wave. With the aid of this transformation, Eq. (13) fractional derivatives can be rewritten as:

$$\frac{\partial^{\alpha}(.)}{\partial t^{\alpha}} = k\frac{\mathrm{d}(.)}{\mathrm{d}\xi},\ \frac{\partial(.)}{\partial x} = w\frac{\mathrm{d}(.)}{\mathrm{d}\xi},\dots \tag{15}$$

Using the transformation given in Eq. (14) inside Eq. (13), we obtain the following ordinary differential equation

$$G(U, U', U'', U''', \dots) = 0, \tag{16}$$

where the derivatives are respect to ξ.

Step 3. Now, consider $U(\xi)$ is a sum of serial such as:

$$U(\xi) = \sum_{i=0}^{n} a_i z^i(\xi), \tag{17}$$

where $z(\xi)$ is the solution of the nonlinear differential Eq. (12), a, b, c, k, w, a_i are the real constants and n is the positive integer to be determined by a balancing procedure (Malfliet, 1992).

Step 4. Balancing the linear terms of highest order in the ordinary differential equation (ODE) Eq. (16) with the highest order nonlinear terms gives us the parameter n. Then we place Eq. (17) into the ODE Eq. (16). After this procedure we get an equation consisting of the powers of $z(\xi)$. All coefficients of $z(\xi)$ are equal to zero in the final equation. This procedure arouses the system of algebraic equations including a, b, c, k, w, a_i. Solving this system with respect to these parameters and using the following table which expresses the exact solutions of Eq. (12) give the analytical solutions.

4 Implementation of the method

Consider the time fractional coupled Boussinesq–Whitham–Broer–Kaup equation

$$D_t^{\alpha}v+D_x(uv)+D_{xxx}u=0;$$
$$D_t^{\alpha}u+D_xv+uD_xu=0, \tag{18}$$

where the fractional derivatives are in conformable sense. With the wave transform Eq. (14) and integrating both equations once, the system becomes:

$$wv+kuv+k^3u''=0;$$
$$wu+kv+k\frac{u^2}{2}=0, \tag{19}$$

where the prime denotes the derivative of the functions with respect to ξ. From the second equation, $v=-\frac{w}{k}u-\frac{u^2}{2}$ is obtained. Using this equality in the first equation of Eq. (10) yields

$$2w^2u+3kwu^2+k^2u^3-2k^4u''=0. \tag{20}$$

Now using the balancing procedure for the highest order nonlinear term and highest order linear term in Eq. (20) yields $n=1$. Thus, the unknown function $u(\xi)$ can be considered as:

$$u(\xi)=a_0+a_1z(\xi). \tag{21}$$

Placing Eq. (21) into Eq. (20) and using Eq. (12) led to an algebraic equation with respect to $z(\xi)$. Equating all the coefficients of the same powers of $z(\xi)$ to zero arouses an algebraic equation system. Solving this system gives following solution sets.

Set 1:

$$a_0=0,\ a_1=1,\ a=\frac{w^2}{k^4},\ b=\frac{w}{k^3},\ c=\frac{1}{4k^2}.$$

Set 2:

$$a_0=-\frac{2w}{k},\ a_1=1,\ a=\frac{w^2}{k^4},\ b=-\frac{w}{k^3},\ c=\frac{1}{4k^2}.$$

Set 3:

$$a_0=-\frac{w}{k},\ a_1=1,\ a=-\frac{w^2}{2k^4},\ b=0,\ c=\frac{1}{4k^2}.$$

Using the solution of Set 1, we obtain $\Delta=0$ and by looking Table 1, the new wave solutions of time fractional coupled Boussinesq–Whitham–Broer–Kaup Eq. (18) can be given as follows:

$$u_1(x,t)=\frac{w(1-\coth A)}{k};$$

$$v_1(x,t)=\frac{w^2(1-\coth A)}{k^2}-\frac{w^2(1-\coth A)^2}{2k^2};$$

$$u_2(x,t)=-\frac{w(1-\tanh A)}{k};$$

$$v_2(x,t)=\frac{w^2(1-\tanh A)}{k^2}-\frac{w^2(1-\tanh A)^2}{2k^2};$$

$$u_3(x,t)=-\frac{w^3\operatorname{sech}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\tanh A+1)^2}{4k^6}\right]^{-1};$$

$$v_3(x,t)=\frac{w^4\operatorname{sech}^2A}{k^8\left[\frac{w^2}{k^6}-\frac{w^2(\tanh A+1)^2}{4k^6}\right]}-\frac{w^6\operatorname{sech}^4A}{2k^{14}\left[\frac{w^2}{k^6}-\frac{w^2(\tanh A+1)^2}{4k^6}\right]^2};$$

$$u_4(x,t)=\frac{w^3\operatorname{csch}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\coth A+1)^2}{4k^6}\right]^{-1};$$

$$v_4(x,t)=-\frac{w^6\operatorname{csch}^4A}{2k^{14}\left[\frac{w^2}{k^6}-\frac{w^2(\coth A+1)^2}{4k^6}\right]^2}-\frac{w^4\operatorname{csch}^2A}{k^8\left[\frac{w^2}{k^6}-\frac{w^2(\coth A+1)^2}{4k^6}\right]},$$

$$u_5(x,t)=\frac{4w^2e^{2A}}{k^4}\left[\left(e^{2A}-\frac{w}{k^3}\right)^2-\frac{w^2}{k^6}\right]^{-1};$$

$$v_5(x,t)=-\frac{8w^4e^{4A}}{k^8\left[\left(e^{2A}-\frac{w}{k^3}\right)^2-\frac{w^2}{k^6}\right]^2}-\frac{4w^3e^{2A}}{k^5\left[\left(e^{2A}-\frac{w}{k^3}\right)^2-\frac{w^2}{k^6}\right]},$$

$$u_6(x,t)=-\frac{w^2\operatorname{sech}^2A}{k^4}\left(\frac{w}{k^3}+\sqrt{\frac{w^2}{k^6}}\tanh A\right)^{-1},$$

$$v_6(x,t)=\frac{w^3\operatorname{sech}^2A}{k^5\left(\frac{w}{k^3}+\sqrt{\frac{w^2}{k^6}}\tanh A\right)}-\frac{w^4\operatorname{sech}^4A}{2k^8\left(\frac{w}{k^3}+\sqrt{\frac{w^2}{k^6}}\tanh A\right)^2},$$

$$u_7(x,t)=\frac{w^2\operatorname{csch}^2A}{k^4}\left(\frac{w}{k^3}+\sqrt{\frac{w^2}{k^6}}\coth A\right)^{-1},$$

$$v_7(x,t)=-\frac{w^4\operatorname{csch}^4A}{2k^8\left(\frac{w}{k^3}+\sqrt{\frac{w^2}{k^6}}\coth A\right)^2}-\frac{w^3\operatorname{csch}^2A}{k^5\left(\frac{w}{k^3}+\sqrt{\frac{w^2}{k^6}}\coth A\right)}.$$

where, $A=\frac{1}{2}\sqrt{\frac{w^2}{k^4}}\left(kx+\frac{wt^{\alpha}}{\alpha}\right)$.

Considering the solution Set 2, we obtain $\Delta=0$. With the aid of Table 1, wave solutions of the coupled system (18) are given below

$$u_8(x,t)=\frac{w(1-\tanh A)}{k}-\frac{2w}{k};$$

$$v_8(x,t)=-\frac{1}{2}\left[\frac{w(1-\tanh A)}{k}-\frac{2w}{k}\right]^2-\frac{w}{k}\left[\frac{w(1-\tanh A)}{k}-\frac{2w}{k}\right];$$

$$u_9(x,t)=\frac{w(1-\coth A)}{k}-\frac{2w}{k};$$

$$v_9(x,t)=-\frac{1}{2}\left[\frac{w(1-\coth A)}{k}-\frac{2w}{k}\right]^2-\frac{w}{k}\left[\frac{w(1-\coth A)}{k}-\frac{2w}{k}\right];$$

$$u_{10}(x,t)=\frac{4w^2e^{2A}}{k^4}\left[\left(e^{2A}+\frac{w}{k^3}\right)^2-\frac{w^2}{k^6}\right]^{-1}-\frac{2w}{k};$$

$$v_{10}(x,t)=-\frac{1}{2}\left\{\frac{4w^2e^{2A}}{k^4}\left[\left(e^{2A}+\frac{w}{k^3}\right)^2-\frac{w^2}{k^6}\right]^{-1}-\frac{2w}{k}\right\}^2-\frac{w}{k}\left\{\frac{4w^2e^{2A}}{k^4}\left[\left(e^{2A}+\frac{w}{k^3}\right)^2-\frac{w^2}{k^6}\right]^{-1}-\frac{2w}{k}\right\};$$

$$u_{11}(x,t)=\frac{w^2\mathrm{csch}^2A}{k^4}\left(\sqrt{\frac{w^2}{k^6}}\coth A-\frac{w}{k^3}\right)^{-1}-\frac{2w}{k};$$

$$v_{11}(x,t)=-\frac{1}{2}\left[\frac{w^2\mathrm{csch}^2A}{k^4}\left(\sqrt{\frac{w^2}{k^6}}\coth A-\frac{w}{k^3}\right)^{-1}-\frac{2w}{k}\right]^2-\frac{w}{k}\left[\frac{w^2\mathrm{csch}^2A}{k^4}\left(\sqrt{\frac{w^2}{k^6}}\coth A-\frac{w}{k^3}\right)^{-1}-\frac{2w}{k}\right],$$

$$u_{12}(x,t)=-\frac{w^2\mathrm{sech}^2A}{k^4}\left(\sqrt{\frac{w^2}{k^6}}\tanh A-\frac{w}{k^3}\right)^{-1}-\frac{2w}{k},$$

$$v_{12}(x,t)=-\frac{1}{2}\left[-\frac{w^2\mathrm{sech}^2A}{k^4}\left(\sqrt{\frac{w^2}{k^6}}\tanh A-\frac{w}{k^3}\right)^{-1}-\frac{2w}{k}\right]^2-\frac{w}{k}\left[-\frac{w^2\mathrm{sech}^2A}{k^4}\left(\sqrt{\frac{w^2}{k^6}}\tanh A-\frac{w}{k^3}\right)^{-1}-\frac{2w}{k}\right];$$

$$u_{13}(x,t)=\frac{w^3\mathrm{sech}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\tanh A+1)^2}{4k^6}\right]^{-1}-\frac{2w}{k},$$

$$v_{13}(x,t)=-\frac{1}{2}\left\{\frac{w^3\mathrm{sech}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\tanh A+1)^2}{4k^6}\right]^{-1}-\frac{2w}{k}\right\}^2-\frac{w}{k}\left\{\frac{w^3\mathrm{sech}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\tanh A+1)^2}{4k^6}\right]^{-1}-\frac{2w}{k}\right\},$$

$$u_{14}=-\frac{w^3\mathrm{csch}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\coth A+1)^2}{4k^6}\right]^{-1}-\frac{2w}{k},$$

$$v_{14}=-\frac{1}{2}\left\{-\frac{w^3\mathrm{csch}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\coth A+1)^2}{4k^6}\right]^{-1}-\frac{2w}{k}\right\}^2-\frac{w}{k}\left\{-\frac{w^3\mathrm{csch}^2A}{k^7}\left[\frac{w^2}{k^6}-\frac{w^2(\coth A+1)^2}{4k^6}\right]^{-1}-\frac{2w}{k}\right\}.$$

Finally regarding solutions Set 3, we obtain $\Delta=\frac{w^2}{2k^6}$. With the help of Table 1, wave solutions of Eq. (18) are obtained as:

$$u_{15}(x,t)=-\frac{\sqrt{2}w^2\sec\left(\sqrt{2}A\right)}{k^4\sqrt{w^2/k^6}}-\frac{w}{k};$$

$$v_{15}(x,t)=-\frac{1}{2}\left[-\frac{\sqrt{2}w^2\sec\left(\sqrt{2}A\right)}{k^4\sqrt{w^2/k^6}}-\frac{w}{k}\right]^2-\frac{w}{k}\left[-\frac{\sqrt{2}w^2\sec\left(\sqrt{2}A\right)}{k^4\sqrt{w^2/k^6}}-\frac{w}{k}\right];$$

$$u_{16}(x,t)=-\frac{\sqrt{2}w^2\csc\left(\sqrt{2}A\right)}{k^4\sqrt{w^2/k^6}}-\frac{w}{k},$$

$$v_{16}(x,t)=-\frac{1}{2}\left[-\frac{\sqrt{2}w^2\csc\left(\sqrt{2}A\right)}{k^4\sqrt{w^2/k^6}}-\frac{w}{k}\right]^2-\frac{w}{k}\left[-\frac{\sqrt{2}w^2\csc\left(\sqrt{2}A\right)}{k^4\sqrt{w^2/k^6}}-\frac{w}{k}\right],$$

$$u_{17}(x,t)=\frac{w^2\csc\left(A/\sqrt{2}\right)\sec\left(A/\sqrt{2}\right)}{\sqrt{2}k^4\sqrt{w^2/k^6}}-\frac{w}{k},$$

$$v_{17}(x,t)=-\frac{1}{2}\left[\frac{w^2\csc\left(A/\sqrt{2}\right)\sec\left(A/\sqrt{2}\right)}{\sqrt{2}k^4\sqrt{w^2/k^6}}-\frac{w}{k}\right]^2-\frac{w}{k}\left[\frac{w^2\csc\left(A/\sqrt{2}\right)\sec\left(A/\sqrt{2}\right)}{\sqrt{2}k^4\sqrt{w^2/k^6}}-\frac{w}{k}\right].$$

Table 1 Solutions of Eq. (12) with $\Delta=b^2-4ac$ and $\varepsilon=\pm1$

No	$z(\xi)$	
1	$\dfrac{-ab\mathrm{sech}^2\left(\sqrt{a}\xi/2\right)}{b^2-ac\left[1+\varepsilon\tanh\left(\sqrt{a}\xi/2\right)\right]^2}$	$a>0$
2	$\dfrac{ab\mathrm{csch}^2\left(\sqrt{a}\xi/2\right)}{b^2-ac\left[1+\varepsilon\coth\left(\sqrt{a}\xi/2\right)\right]^2}$	$a>0$
3	$\dfrac{2a\mathrm{sech}(\sqrt{a}\xi)}{\varepsilon\sqrt{\Delta}-b\mathrm{sech}(\sqrt{a}\xi)}$	$a>0,\ \Delta>0$
4	$\dfrac{2a\sec(\sqrt{-a}\xi)}{\varepsilon\sqrt{\Delta}-b\sec(\sqrt{-a}\xi)}$	$a<0,\ \Delta>0$
5	$\dfrac{2a\mathrm{csch}(\sqrt{a}\xi)}{\varepsilon\sqrt{-\Delta}-b\mathrm{csch}(\sqrt{a}\xi)}$	$a>0,\ \Delta<0$
6	$\dfrac{2a\csc(\sqrt{-a}\xi)}{\varepsilon\sqrt{\Delta}-b\csc(\sqrt{-a}\xi)}$	$a<0,\ \Delta>0$
7	$\dfrac{-a\mathrm{sech}^2\left(\sqrt{a}\xi/2\right)}{b+2\varepsilon\sqrt{ac}\tanh\left(\sqrt{a}\xi/2\right)}$	$a>0,\ c>0$
8	$\dfrac{-a\sec^2\left(\sqrt{a}\xi/2\right)}{b+2\varepsilon\sqrt{-ac}\tan\left(\sqrt{a}\xi/2\right)}$	$a<0,\ c>0$
9	$\dfrac{a\mathrm{csch}^2\left(\sqrt{a}\xi/2\right)}{b+2\varepsilon\sqrt{ac}\coth\left(\sqrt{a}\xi/2\right)}$	$a>0,\ c>0$
10	$\dfrac{-a\csc^2\left(\sqrt{a}\xi/2\right)}{b+2\varepsilon\sqrt{-ac}\cot\left(\sqrt{a}\xi/2\right)}$	$a<0,\ c>0$
11	$-\dfrac{a}{b}\left[1+\varepsilon\tanh\left(\sqrt{a}\xi/2\right)\right]$	$a>0,\ \Delta=0$
12	$-\dfrac{a}{b}\left[1+\varepsilon\coth\left(\sqrt{a}\xi/2\right)\right]$	$a>0,\ \Delta=0$
13	$\dfrac{4ae^{\varepsilon\sqrt{a}\xi}}{\left(e^{\varepsilon\sqrt{a}\xi}-b\right)^2-4ac}$	$a>0$
14	$\dfrac{\pm4ae^{\varepsilon\sqrt{a}\xi}}{1-4ace^{2\varepsilon\sqrt{a}\xi}}$	$a>0,\ b=0$

Remark 4.1 We take $\varepsilon = 1$ in the solutions given above. One can easily find any other solutions for $\varepsilon = -1$ using Table 1.

5 Graphical simulations

In this chapter we give some graphical illustrations of chosen solutions for different values of k, w, α in different ranges of x, t. Figs. 1–6 show that the obtained solutions are wave solutions of Eq. (18).

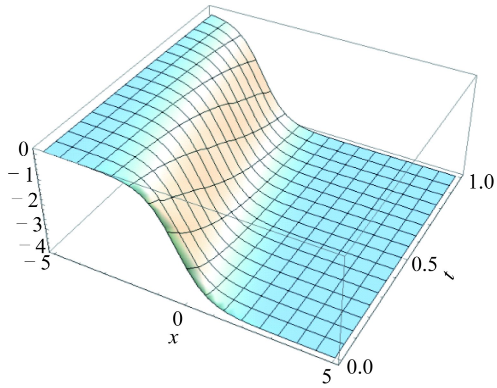

Fig. 1. Graph of the exact solution $u_3(x,t)$ in Eq. (18) where $w = 2$, $k = 1$, $\alpha = 0.8$.

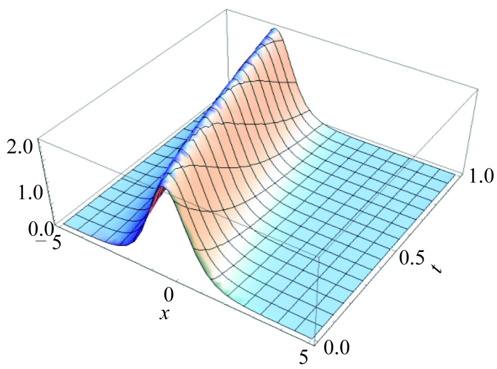

Fig. 2. Graph of the exact solution $v_3(x,t)$ in Eq. (18) where $w = 2$, $k = 1$, $\alpha = 0.8$.

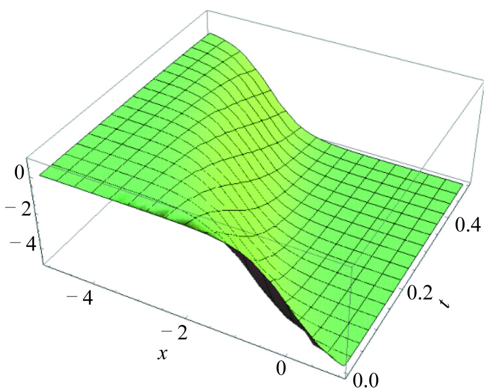

Fig. 3. Graph of the exact solution $u_8(x,t)$ in Eq. (18) where $w = 2$, $k = 0.8$, $\alpha = 0.5$.

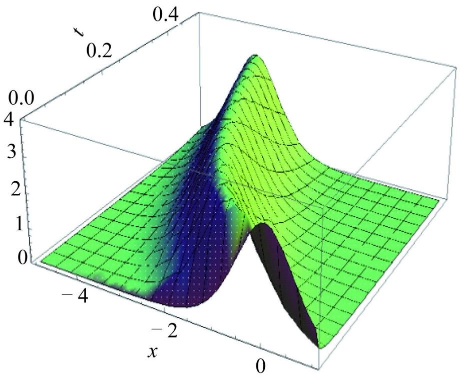

Fig. 4. Graph of the exact solution $v_8(x,t)$ in Eq. (18) where $w = 2$, $k = 0.8$, $\alpha = 0.5$.

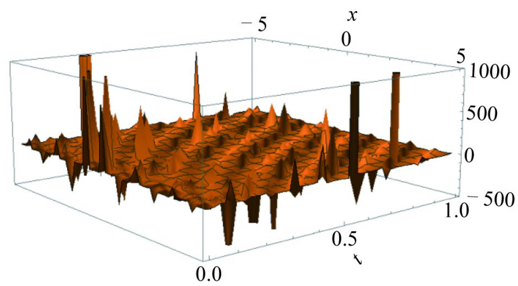

Fig. 5. Graph of the exact solution $u_{17}(x,t)$ in Eq. (18) where $w = 5$, $k = 1$, $\alpha = 0.5$.

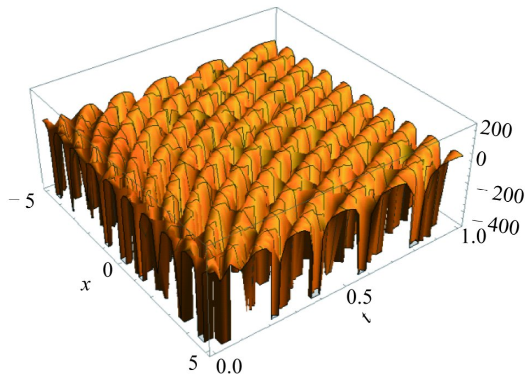

Fig. 6. Graph of the exact solution $v_{17}(x,t)$ in Eq. (18) where $w = 4$, $k = 1$, $\alpha = 0.5$.

6 Conclusion

We have successfully found many new types of exact traveling wave solutions of time fractional coupled Boussinesq–Whitham–Broer–Kaup equation by using the auxiliary equation method. The procedure shows that using conformable fractional derivative and auxiliary equation method gives a reliable and effective way to obtain the non-linear fractional partial differential equations. This method is based on an auxiliary differential equation so the solutions procedure becomes simple and understandable. By using conformable fractional derivative one can obtain analytical solutions of the nonlinear partial differential equations which cannot be solved in Caputo and Riemann–Liouville definitions.

References

Abdeljawad, T., 2015. On conformable fractional calculus, *Journal of Computational and Applied Mathematics*, 279, 57–66.

Aminikhah, H., Sheikhani, A.H.R. and Rezazadeh, H., 2016. Sub-equation method for the fractional regularized long-wave equations with conformable fractional derivatives, *Scientia Iranica*, 23(3), 1048–1054.

Atangana, A., Baleanu, D. and Alsaedi, A., 2015. New properties of conformable derivative, *Open Mathematics*, 13(1), 2391–5455.

Batarfi, H., Losada, J., Nieto, J.J. and Shammakh, W., 2015. Three-point boundary value problems for conformable fractional differential equations, *Journal of Function Spaces*, 2015, 706383.

Biswas, A., Al-Amr, M.O., Rezazadeh, H., Mirzazadeh, M., Eslami, M., Zhou, Q., Moshokoa, S.P. and Belic, M., 2018a. Resonant optical solitons with dual-power law nonlinearity and fractional temporal evolution, *Optik*, 165, 233–239.

Biswas, A., Rezazadeh, H., Mirzazadeh, M., Eslami, M., Ekici, M., Zhou, Q., Moshokoa, S.P. and Belic, M., 2018b. Optical soliton perturbation with Fokas-Lenells equation using three exotic and efficient integration schemes, *Optik*, 165, 288–294.

Biswas, A., Rezazadeh, H., Mirzazadeh, M., Eslami, M., Zhou, Q., Moshokoa, S.P. and Belic, M., 2018c. Optical solitons having weak non-local nonlinearity by two integration schemes, *Optik*, 164, 380–384.

Boussinesq, J., 1872. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, *Journal de Mathématiques Pures et Appliquées*, 17, 55–108. (in French)

Eslami, M. and Mirzazadeh, M., 2013. Topological 1-soliton solution of nonlinear Schrödinger equation with dual-power law nonlinearity in nonlinear optical fibers, *The European Physical Journal Plus*, 128(11), 140.

Eslami, M., 2016. Trial solution technique to chiral nonlinear Schrodinger's equation in (1+2)-dimensions, *Nonlinear Dynamics*, 85(2), 813–816.

Eslami, M. and Mirzazadeh, M., 2016a. Optical solitons with Biswas-Milovic equation for power law and dual-power law nonlinearities, *Nonlinear Dynamics*, 83(1–2), 731–738.

Eslami, M. and Rezazadeh, H., 2016b. The first integral method for Wu-Zhang system with conformable time-fractional derivative, *Calcolo*, 53(3), 475–485.

Eslami, M., Rezazadeh, H., Rezazadeh, M. and Mosavi, S.S., 2017. Exact solutions to the space-time fractional Schrödinger-Hirota equation and the space-time modified KDV-Zakharov-Kuznetsov equation, *Optical and Quantum Electronics*, 49(8), 279.

Iyiola, O.S. and Nwaeze, E.R., 2016. Some new results on the new conformable fractional calculus with application using D'Alambert approach, *Progress in Fractional Differentiation and Applications*, 2(2), 115–122.

Khalil, R., Al Horani, M., Yousef, A. and Sababheh, M., 2014. A new definition of fractional derivative, *Journal of Computational and Applied Mathematics*, 264, 65–70.

Khodadad, F.S., Nazari, F., Eslami, M. and Rezazadeh, H., 2017. Soliton solutions of the conformable fractional Zakharov-Kuznetsov equation with dual-power law nonlinearity, *Optical and Quantum Electronics*, 49(11), 384.

Malfliet, W., 1992. Solitary wave solutions of nonlinear wave equations, *American Journal of Physics*, 60(7), 650–654.

Osman, M.S., Korkmaz, A., Rezazadeh, H., Mirzazadeh, M., Eslami, M. and Zhou, Q., 2018. The unified method for conformable time fractional Schrödinger equation with perturbation terms, *Chinese Journal of Physics*, 56(5), 2500–2506.

Rezazadeh, H., 2018. New solitons solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity, *Optik*, 167, 218–227.

Rezazadeh, H., Mirhosseini-Alizamini, S.M., Eslami, M., Rezazadeh, M., Mirzazadeh, M. and Abbagari, S., 2018a. New optical solitons of nonlinear conformable fractional Schrödinger-Hirota equation, *Optik*, 172, 545–553.

Rezazadeh, H., Mirzazadeh, M., Mirhosseini-Alizamini, S.M., Neirameh, A., Eslami, M. and Zhou, Q., 2018b. Optical solitons of Lakshmanan-Porsezian-Daniel model with a couple of nonlinearities, *Optik*, 164, 414–423.

Rezazadeh, H., Osman, M.S., Eslami, M., Ekici, M., Sonmezoglu, A., Asma, M., Othman, W.A.M., Wong, B.R., Mirzazadeh, M., Zhou, Q., Biswas, A. and Belic, M., 2018c. Mitigating Internet bottleneck with fractional temporal evolution of optical solitons having quadratic-cubic nonlinearity, *Optik*, 164, 84–92.

Sabatier, J., Agrawal, O.P. and Tenreiro MacHado, J.A., 2007. *Advances in Fractional Calculus*: *Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht.

Sachs, R.L., 1988. On the integrable variant of the boussinesq system: Painlevé property rational solutions, a related many-body system, and equivalence with the AKNS hierarchy, *Physica D*: *Nonlinear Phenomena*, 30(1–2), 1–27.

Sirendaoreji, and Jiong, S., 2003. Auxiliary equation method for solving nonlinear partial differential equations, *Physics Letters A*, 309(5–6), 387–396.

Tariq, K.U., Younis, M., Rezazadeh, H., Rizvi, S.T.R. and Osman, M.S., 2018. Optical solitons with quadratic-cubic nonlinearity and fractional temporal evolution, *Modern Physics Letters B*, 32(26), 1850317.

Tasbozan, O., Şenol, M., Kurt, A. and Özkan, O., 2018. New solutions of fractional Drinfeld-Sokolov-Wilson system in shallow water waves, *Ocean Engineering*, 161, 62–68.

Whitham, G.B., 1965. A general approach to linear and non-linear dispersive waves using a Lagrangian, *Journal of Fluid Mechanics*, 22(2), 273–283.

Whitham, G.B. and Lighthill, M.J., 1967. Variational methods and applications to water waves, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 299(1456), 6–25.

Yomba, E., 2008. A generalized auxiliary equation method and its application to nonlinear Klein-Gordon and generalized nonlinear Camassa-Holm equations, *Physics Letters A*, 372(7), 1048–1060.

Zhang, S. and Xia, T.C., 2007. A generalized new auxiliary equation method and its applications to nonlinear partial differential equations, *Physics Letters A*, 363(5–6), 356–360.