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ORIGINAL ARTICLE

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Exp-function method for the conformable space-time fractional STO, ZKBBM and coupled Boussinesq equations

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ABSTRACT
In the present paper, new analytical solutions for the conformable space-time fractional Sharma-Tasso-Olever (STO), Zakharov Kuznetsov Benjamin Bona Mahony (ZKBBM) and coupled Boussinesq equations are obtained by using the Exp-function method. The obtained traveling wave solutions are presented by exponential functions. Simulations of the obtained solutions are given at the end of the paper.

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Space-time fractional STO equation; space-time fractional ZKBBM equation; space-time fractional coupled Boussinesq equations; conformable derivative; exp-function method

1. Introduction

Conformable fractional derivative has been defined by Khalil et al. (Khalil, Horani, Yousef, & Sababheh, 2014). Whereas other derivatives such as Riemman-Liouville, Caputo, Grünwald-Letnikov are defined with complex formulas, conformable fractional derivative is defined with simple formula. The applicability of the conformable derivative model has been theoretically and practically verified by investigating the chloride ions transport in reinforced concrete (Khitab, Lorente, & Ollivier, 2005; Thomas & Bamforth, 1999). Conformable fractional partial differential equations have been also used in modeling electromagnetic fields of media, quantum mechanics (see, for example (Anderson & Ulness, 2015; Zhao, Pan, & Luo, 2018).

In the solitary wave theory, traveling waves are particularly interesting. They appear in many areas such as elastic media, plasmas, solid state physics, condensed matter physics, electrical circuits, optical fibers, chemical kinematics, fluids, bio-genetics, etc. Three types of traveling waves are given as (Kichenassamy & Olver, 1992; Whitham, 1999): the solitary waves, which are localized traveling waves, asymptotically zero at large distances, the periodic solutions, the kink waves which rise or descend from one asymptotic state to another. Solitonic solutions of nonlinear partial differential equations have been investigated in (Dai & Wang, 2008; Dai & Xu, 2015; Ding et al., 2017; Wang, Zhang, & Dai, 2016).

Exp-function method has been proposed to seek traveling wave solutions of nonlinear differential equations in (He & Wu, 2006). The method has been also applied to the following nonlinear evolution equations: Drinfel'd-Sokolov-Wilson system, Burgers-type equation, Schrodinger equation, Calogero-Bogoyavlenskii-Schiff equation, Zakharov equations, Cahn-Hilliard equation, Allen-Cahn equation and Steady-State equation (see, for example (Abdou, 2008; Abdou, Soliman, & Basyony, 2007; Ali, Iqbal, & Mohyud-Din, 2016; Ayub, Khan, & Mahmood-Ul-Hassan, 2017; El-Wakil, Madkour, & Abdou, 2007; Gurefe & Misirli, 2011; Mohyud-Din, Noor, & Noor, 2010; Parand & Rad, 2012; Wu & He, 2007). However, its applications to fractional nonlinear evolution equations have been studied. For example, nonlinear fractional Telegraph equation, Kolmogorov-Petrovskii-Piskunovequation (Guner & Bekir, 2017), fractional Fokas equation and the nonlinear fractional Sharma-Tasso-Olver equation (Zheng, 2013), fractional reaction-diffusion and nonlinear fractional wave equations (Bekir, Guner, Bhraw, & Biswas, 2015), nonlinear fractional Zoomeron equation (Guner, Bekir, & Bilgil, 2015), fractional Boussinesq-Like equations (Rahmatullah, Ellahi, Mohyud-Din, & Khan, 2018), fractional Kawahara equation and fractional advection-diffusion-reaction equation (Guner &

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Atik, 2016) have been solved by using Exp-function method. Here, fractional derivatives have been defined in Jumaries modified Riemann-Liouville sense. In (He, 2013; Jia, Hu, Chen, & Jai, 2015), the method has been applied to the nonlinear fractional evolution equations with local fractional.

In recent years, many techniques have been used to obtain analytical and numerical solutions of the fractional STO, ZKBBM and coupled Boussinesq equations. Time fractional STO equation with Jumarie's modified Riemann-Liouville derivative has been solved by using the iterative method, simplest equation method, Khater method and modified trial equation method in (Bibi, Mohyud-Din, Khan, & Ahmed, 2017; Bulut & Pandir, 2013; Sontakke & Shaikh, 2016; Taghizadeh, Mirzazadeh, Rahimian, & Akbari, 2013), respectively. Time fractional STO equations with Caputo derivative and conformable derivative have been studied in (Rezazadeh, Khodadad, & Manafian, 2017; Song et al., 2009), respectively. Improved tanh-coth method has been applied to the space-time fractional STO equation with Jumarie's modified Riemann-Liouville derivative in (Cesar & Gomez, 2015).(G'/G^2) -expansion method, sub-equation method, Jacobi elliptic function method and exponential rational function method have been applied to the space-time fractional ZKBBM equation with Jumarie's modified Riemann-Liouville derivative in (Aksoy, Kaplan, & Bekir, 2016; Alzaidy, 2013; Gepreel, 2014; Mohyud-Din & Bibi, 2018), respectively. Traveling wave solutions for the space-time fractional coupled Boussinesq equations with the Jumaries modified Riemann-Liouville derivative have been obtained by using the modified extended tanh method (Shallal, Jabbar, & Ali, 2018). Time fractional coupled Boussinesq equations with the conformable derivative have been solved by using $\exp(-\phi(\xi))$ method and modified Kudryashov method in (Hosseini, Bekir, & Ansari, 2017; Hosseini & Ansari, 2017).

In general, STO, ZKBBM and coupled Boussinesq equations have been studied for the case of time fractional in the literature. For the space-time fractional STO, ZKBBM and coupled Boussinesq equations, Jumarie's modified Riemann-Liouville derivatives have been used. In this paper, we consider space-time fractional STO, ZKBBM and coupled Boussinesq equations. Here, fractional derivatives are defined in conformable sense. Applying Exp-function method we have obtain analytic solutions including exponential functions for conformable space-time fractional STO, ZKBBM and coupled Boussinesq equations.

2. Description of conformable fractional derivative and its properties

For a function $f\colon (0,\infty)\to R$, the conformable fractional derivative of f of order $0<\alpha<1$ is defined as (see, for example (Khalil et al., 2014))

$$T_t^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}. \tag{1}$$

Some important properties of the the conformal fractional derivative are as follows:

$$\begin{aligned} T_t^\alpha(af+bg)(t) &= aT_t^\alpha f(t) + bT_t^\alpha g(t), \text{ for all } a,b\in R,\\ T_t^\alpha(t^\mu) &= \mu t^{\mu-\alpha},\\ T_t^\alpha(f(g(t))) &= t^{1-\alpha}g'(t)f'(g(t)). \end{aligned} \tag{2}$$

3. Analytic solutions to the conformable space-time fractional STO equation

Conformable space-time fractional STO equation is denoted by (Sontakke & Shaikh, 2016; Taghizadeh et al., 2013)

$$\begin{aligned} T_t^\alpha u + 3c\left(T_x^\beta u\right)^2 + 3cu^2T_x^\beta u + 3cuT_x^\beta T_x^\beta u + cT_x^\beta T_x^\beta T_x^\beta u = 0,\\ 0<\alpha\le 1,\ 0<\beta\le 1. \end{aligned} \tag{3}$$

Note that for $\alpha=\beta=1$, conformable space-time fractional STO equation is reduced to classical STO equation. Classical STO equation is a prominent double nonlinear dispersive model. Here $c\neq 0$ is a constant, $u=u(x,t)$ is a field variable, x is the spatial coordinate in the propagation direction and t is the temporal coordinates, which occur in different contexts in mathematical physics. The dissipative u_{xxx} term provides damping at small scales, and the nonlinear term u^2u_x stabilizes by transferring energy between large and small scales.

Using the following transformation for Eq. (3)

$$u(x,t) = U(\xi),\quad \xi = k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}, \tag{4}$$

where k and m are non zero arbitrary constants, and integrating resulting equation with zero constant we have

$$kU + 3cm^2UU' + cmU^3 + cm^3U'' = 0. \tag{5}$$

According to Exp-function method, the solution of Eq. (5) can be expressed in the following form

$$U(\xi) = \frac{\sum_{i=-r}^{s} a_i\exp[i\xi]}{\sum_{j=-h}^{l} b_j\exp[j\xi]} \tag{6}$$

where t, s, h and l are positive integers which are known to be further determined, a_i and b_j are unknown constants.

Substituting Eq. (6) into Eq. (5) and balancing in the obtained equation, we get $r=s=h=l=1$, so Eq. (6) reduces to

$$U(\xi) = \frac{a_1\exp[\xi] + a_0 + a_{-1}\exp[-\xi]}{b_1\exp[\xi] + b_0 + b_{-1}\exp[-\xi]}. \tag{7}$$

By substituting Eq. (7) into Eq. (5), and collecting all the terms with the same power of e^s ($s=$

$3, 2, 1, 0, -1, -2, -3)$, we can obtain a set of algebraic equations for the unknowns a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, k, m:

$$
\begin{aligned}
&cma_1^3 + ka_1b_1^2 = 0,\\
&3b_0ca_1^2m^2 + 3a_0ca_1^2m - b_0ca_1b_1m^3 - 3a_0ca_1b_1m^2\\
&+2b_0ka_1b_1 + a_0cb_1^2m^3 + a_0kb_1^2 = 0,\\
&3ca_0^2a_1m - 3ca_0^2b_1m^2 + 3ca_0a_1b_0m^2 - ca_0b_0b_1m^3\\
&+2ka_0b_0b_1 + 6b_{-1}ca_1^2m^2\\
&+3a_{-1}ca_1^2m + ca_1b_0^2m^3 + ka_1b_0^2 - 4b_{-1}ca_1b_1m^3\\
&-6a_{-1}ca_1b_1m^2 + 2b_{-1}ka_1b_1\\
&+4a_{-1}cb_1^2m^3 + a_{-1}kb_1^2 = 0,\\
&a_0b_0^2k + a_0^3cm + 2a_0b_1b_{-1}k + 2a_1b_0b_{-1}k\\
&+2a_{-1}b_0b_1k + 6a_0a_1a_{-1}cm\\
&+9a_0a_1b_{-1}cm^2 - 9a_0a_{-1}b_1cm^2 - 6a_0b_1b_{-1}cm^3\\
&+3a_1b_0b_{-1}cm^3 + 3a_{-1}b_0b_1cm^3 = 0,\\
&3ca_0^2a_{-1}m + 3ca_0^2b_{-1}m^2 - 3ca_0a_{-1}b_0m^2\\
&-ca_0b_0b_{-1}m^3 + 2ka_0b_0b_{-1} - 6b_1ca_{-1}^2m^2\\
&+3a_1ca_{-1}^2m + ca_{-1}b_0^2m^3 + ka_{-1}b_0^2 - 4b_1ca_{-1}b_{-1}m^3\\
&+6a_1ca_{-1}b_{-1}m^2 + 2b_1ka_{-1}b_{-1} + 4a_1cb_{-1}^2m^3\\
&+a_1kb_{-1}^2 = 0,\\
&-3b_0ca_{-1}^2m^2 + 3a_0ca_{-1}^2m - b_0ca_{-1}b_{-1}m^3\\
&+3a_0ca_{-1}b_{-1}m^2 + 2b_0ka_{-1}b_{-1}\\
&+a_0cb_{-1}^2m^3 + a_0kb_{-1}^2 = 0,\\
&cma_{-1}^3 + ka_{-1}b_{-1}^2 = 0.
\end{aligned}
$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions:

Case 1:

$$
a_1 = a_1, \quad a_0 = a_0, \quad a_{-1} = \frac{a_0^2}{4a_1},
$$
$$
b_1 = \frac{a_1}{m}, \quad b_0 = 0, \quad b_{-1} = -\frac{a_0^2}{4a_1m}, \quad k = -cm^3,
$$

where a_1 and a_0 are free parameters.

$$
u_1(x,t) = \frac{a_1 \exp\left[-cm^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right] + a_0 + \frac{a_0^2}{4a_1}\exp\left[-\left(-cm^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right]}{\frac{a_1}{m}\exp\left[\left(-cm^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right] - \frac{a_0^2}{4a_1m}\exp\left[-\left(-cm^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right]}. \tag{8}
$$

Case 2:

$$
a_1 = a_1, \quad a_0 = a_0, \quad a_{-1} = 0,
$$
$$
b_1 = \frac{a_1}{m}, \quad b_0 = -\frac{a_0}{m}, \quad b_{-1} = -\frac{2a_0^2}{a_1m}, \quad k = -cm^3,
$$

where a_1 and a_0 is free parameter.

$$
u_2(x,t) = \frac{a_1 \exp\left[\left(-cm^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right] + a_0}{\frac{a_1}{m}\exp\left[\left(-cm^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right] - \frac{a_0}{m} - \frac{2a_0^2}{a_1m}\exp\left[\left(cm^3\frac{t^\alpha}{\alpha} - m\frac{x^\beta}{\beta}\right)\right]}. \tag{9}
$$

4. Analytic solutions to the conformable space-time fractional ZKBBM equation

Conformable space-time fractional ZKBBM equation is given in the following form (Mohyud-Din & Bibi, 2018; Shakeel & Tauseef Mohyud-Din, 2015)

$$
T_t^\alpha u + T_x^\beta u - 2auT_x^\beta u - bT_t^\alpha\left(T_x^\beta T_x^\beta u\right) = 0, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1, \tag{10}
$$

where a, b are real-valued constants. It is well known that ZK (Zakharov Kuznetsov) equation models are weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasmas. ZK-BBM equation is the conjunction of ZK equation and BBM (Benjamin-Bona-Mahony) equation that models shallow water waves.

Substituting Eq. (4) into Eq. (10) and integrating resulting equation with zero constant we have

$$
(k+m)U - amU^2 - bm^2kU'' = 0. \tag{11}
$$

Substituting Eq. (6) into Eq. (11) and balancing in the obtained equation, we get $r = s = h = l = 1$, so Eq. (6) reduces to Eq. (7). By substituting Eq. (7) into Eq. (11), and collecting all the terms with the same power of e^s $(s = 3, 2, 1, 0, -1, -2, -3)$, we can obtain a set of algebraic equations for the unknowns a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, k, m:

$$
\begin{aligned}
&a_1b_1^2k + a_1b_1^2m - aa_1^2b_1m = 0,\\
&a_0b_1^2k + a_0b_1^2m - aa_1^2b_0m + 2a_1b_0b_1k + 2a_1b_0b_1m\\
&-a_0bb_1^2km^2 - 2aa_0a_1b_1.m + a_1bb_0b_1km^2 = 0,\\
&a_1b_0^2k + a_{-1}b_1^2k + a_1b_0^2m + a_{-1}b_1^2m - aa_0^2b_1m\\
&-aa_1^2b_{-1}m + 2a_0b_0b_1k\\
&+2a_1b_1b_{-1}k + 2a_0b_0b_1m + 2a_1b_1b_{-1}m - a_1bb_0^2km^2\\
&-4a_{-1}bb_1^2km^2 - 2aa_0a_1.b_0m - 2aa_1a_{-1}b_1m\\
&+a_0bb_0b_1km^2 + 4a_1bb_1b_{-1}km^2 = 0,\\
&a_0b_0^2k + a_0b_0^2m - aa_0^2b_0m + 2a_0b_1b_{-1}k + 2a_1b_0b_{-1}k\\
&+2a_{-1}b_0b_1k + 2a_0b_1.b_{-1}m + 2a_1b_0b_{-1}m\\
&+2a_{-1}b_0b_1m - 2aa_0a_1b_{-1}m - 2aa_0a_{-1}b_1m - 2aa_1a_{-1}\\
&.b_0m + 6a_0bb_1b_{-1}km^2 - 3a_1bb_0b_{-1}km^2\\
&-3a_{-1}bb_0b_1km^2 = 0,\\
&a_{-1}b_0^2k + a_1b_{-1}^2k + a_{-1}b_0^2m + a_1b_{-1}^2m - aa_0^2b_{-1}m\\
&-aa_{-1}^2b_1m + 2a_0b_0b_{-1}k\\
&+2a_{-1}b_1b_{-1}k + 2a_0b_0b_{-1}m + 2a_{-1}b_1b_{-1}m\\
&-a_{-1}bb_0^2km^2 - 4a_1bb_{-1}^2km^2\\
&-2aa_0a_{-1}b_0m - 2aa_1a_{-1}b_{-1}m + a_0bb_0b_{-1}km^2\\
&+4a_{-1}bb_1b_{-1}km^2 = 0,\\
&a_0b_{-1}^2k + a_0b_{-1}^2m - aa_{-1}^2b_0m + 2a_{-1}b_0b_{-1}k\\
&+2a_{-1}b_0b_{-1}m - a_0bb_{-1}^2km^2\\
&-2aa_0a_{-1}b_{-1}m + a_{-1}bb_0b_{-1}km^2 = 0,\\
&a_{-1}b_{-1}^2k + a_{-1}b_{-1}^2m - aa_{-1}^2b_{-1}m = 0.
\end{aligned}
$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solution:

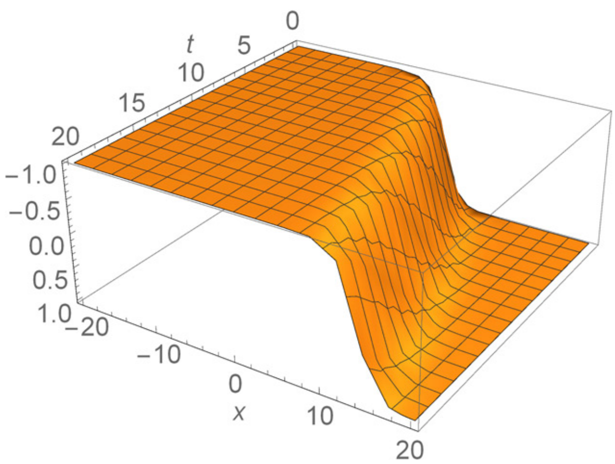

Figure 1. 3D plot of the obtained traveling wave solutions $u_1(x,t)$ of Eq. (3).

$$a_1 = \frac{b_1 bm^2}{a(1+m^2b)}, \quad a_0 = -\frac{2bb_0m^2}{a(1+m^2b)},$$
$$a_{-1} = \frac{bb_0m^2}{4b_1a(1+m^2b)},$$
$$b_1 = b_1, \quad b_0 = b_0, \quad b_{-1} = \frac{b_0^2}{4b_1}, \quad k = -\frac{m}{1+bm^2},$$

where b_1 and b_0 are free parameters.

$$u(x,t) = \frac{\frac{b_1bm^2}{a(1+m^2b)}\exp\left[\left(-\frac{m}{1+bm^2}\frac{t^\alpha}{\alpha}+m\frac{x^\beta}{\beta}\right)\right]-\frac{2bb_0m^2}{a(1+m^2b)}}{b_1\exp\left[\left(-\frac{m}{1+bm^2}\frac{t^\alpha}{\alpha}+m\frac{x^\beta}{\beta}\right)\right]+b_0+\frac{b_0^2}{4b_1}\exp\left[\left(\frac{m}{1+bm^2}\frac{t^\alpha}{\alpha}-m\frac{x^\beta}{\beta}\right)\right]}$$
$$+\frac{\frac{bb_0m^2}{4b_1a(1+m^2b)}\exp\left[\left(\frac{m}{1+bm^2}\frac{t^\alpha}{\alpha}-m\frac{x^\beta}{\beta}\right)\right]}{b_1\exp\left[\left(-\frac{m}{1+bm^2}\frac{t^\alpha}{\alpha}+m\frac{x^\beta}{\beta}\right)\right]+b_0+\frac{b_0^2}{4b_1}\exp\left[\left(\frac{m}{1+bm^2}\frac{t^\alpha}{\alpha}-m\frac{x^\beta}{\beta}\right)\right]}. \tag{12}$$

5. Analytic solutions to the conformable space-time fractional coupled Boussinesq equations

Finally, we consider the conformable space-time fractional coupled Boussinesq equations (Hosseini et al., 2017; Hosseini & Ansari, 2017)

$$T_t^\alpha u + T_x^\beta v = 0, \tag{13}$$

$$T_t^\alpha v + \lambda T_x^\beta(u^2) - \mu T_x^\beta T_x^\beta T_x^\beta u = 0, \quad 0<\alpha\le 1, \quad 0<\beta\le 1. \tag{14}$$

Boussinesq type equations can be considered as the first model for nonlinear, dispersive wave propagation and describe the surface water waves whose horizontal scale is much larger than the depth of the water (Madsen, Murray, & Sorensen, 1991).

Substituting Eq. (4) into Eqs. (13)–(14) we obtain the following differential equations

$$kU' + mV' = 0, \tag{15}$$

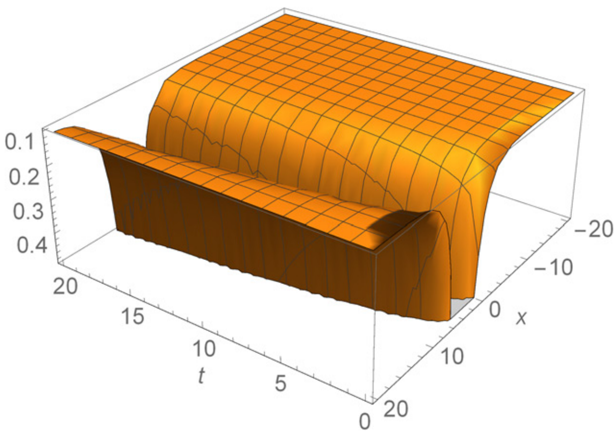

Figure 3. 3D plot of the obtained traveling wave solutions $u(x, t)$ of Eq. (10).

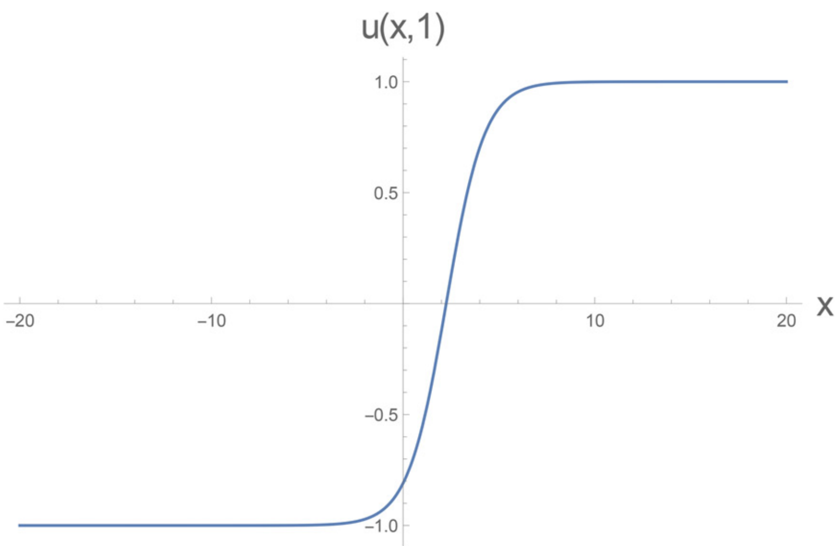

Figure 2. 2D plot of the obtained traveling wave solutions $u_1(x,1)$ of Eq. (3).

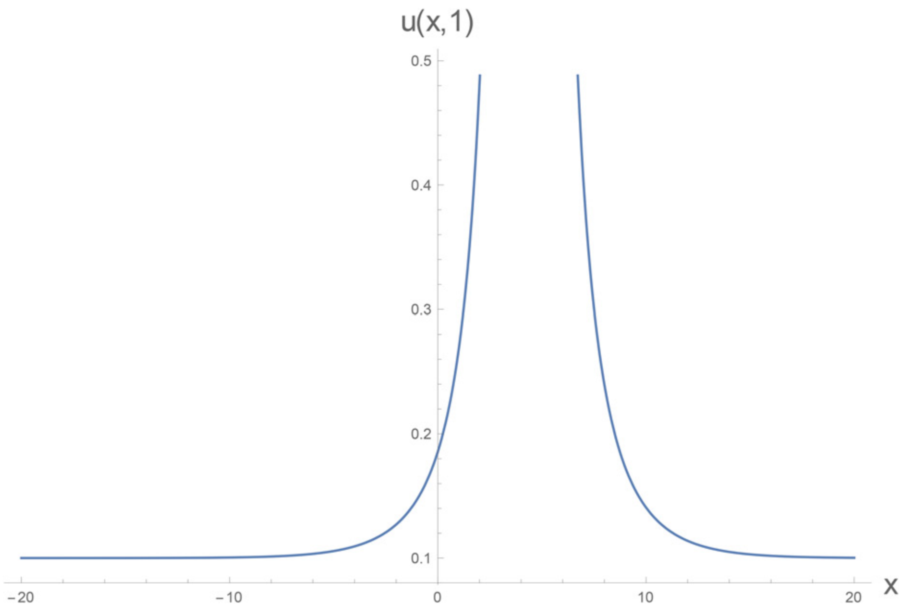

Figure 4. 2D plot of the obtained traveling wave solutions $u(x,1)$ of Eq. (10).

$$kV' + \lambda m(U^2)' - \mu m^3 U''' = 0. \tag{16}$$

Integrating Eqs. (15)–(16) and using $V = -\frac{k}{m}U$ we have

$$-\frac{k^2}{m}U + \lambda m U^2 - \mu m^3 U'' = 0. \tag{17}$$

By balancing in Eq. (17), we set $r = s = h = l = 1$, so Eq. (6) reduces to form of the Eq. (7). By substituting Eq. (7) into Eq. (17) and collecting all the terms with the same power of e^s $(s = 3, 2, 1, 0, -1, -2, -3)$, we can obtain a set of algebraic equations for the unknowns a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, k, m:

$$\lambda a_1^2 b_1 m^2 - a_1 b_1^2 k^2 = 0,$$
$$b_0\lambda a_1^2 m^2 - 2b_0 a_1 b_1 k^2 + b_0\mu a_1 b_1 m^4 + 2a_0\lambda a_1 b_1 m^2$$
$$-a_0 b_1^2 k^2 - a_0\mu b_1^2 m^4 = 0,$$
$$\lambda a_0^2 b_1 m^2 + 2\lambda a_0 a_1 b_0 m^2 - 2a_0 b_0 b_1 k^2 + \mu a_0 b_0 b_1 m^4$$
$$+b_{-1}\lambda a_1^2 m^2 - a_1 b_0^2 k^2$$
$$-\mu a_1 b_0^2 m^4 - 2b_{-1} a_1 b_1 k^2 + 4b_{-1}\mu a_1 b_1 m^4$$
$$+2a_{-1}\lambda a_1 b_1 m^2 - a_{-1} b_1^2 k^2 - 4a_{-1}\mu b_1^2 m^4 = 0,$$
$$a_0^2 b_0 \lambda m^2 - 2a_0 b_1 b_{-1} k^2 - 2a_1 b_0 b_{-1} k^2 - 2a_{-1} b_0 b_1 k^2$$
$$-a_0 b_0^2 k^2 + 2a_0 a_1 b_{-1}$$
$$.\lambda m^2 + 2a_0 a_{-1} b_1 \lambda m^2 + 2a_1 a_{-1} b_0 \lambda m^2 + 6a_0 b_1 b_{-1} m^4 \mu$$
$$-3a_1 b_0 b_{-1} m^4 \mu - 3a_{-1} b_0 b_1 m^4 \mu = 0,$$
$$\lambda a_0^2 b_{-1} m^2 + 2\lambda a_0 a_{-1} b_0 m^2 - 2a_0 b_0 b_{-1} k^2$$
$$+\mu a_0 b_0 b_{-1} m^4 + b_1 \lambda a_{-1}^2 m^2$$
$$-a_{-1} b_0^2 k^2 - \mu a_{-1} b_0^2 m^4 - 2b_1 a_{-1} b_{-1} k^2 + 4b_1 \mu a_{-1} b_{-1} m^4$$
$$+2a_1 \lambda a_{-1} b_{-1} m^2 - a_1 b_{-1}^2 k^2 - 4a_1 \mu b_{-1}^2 m^4 = 0,$$
$$b_0 \lambda a_{-1}^2 m^2 - 2b_0 a_{-1} b_{-1} k^2 + b_0 \mu a_{-1} b_{-1} m^4$$
$$+2a_0 \lambda a_{-1} b_{-1} m^2 - a_0 b_{-1}^2 k^2 - a_0 \mu b_{-1}^2 m^4 = 0,$$
$$\lambda a_{-1}^2 b_{-1} m^2 - a_{-1} b_{-1}^2 k^2 = 0.$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solution:

$$a_1 = \frac{b_1 m^2 \mu}{\lambda}, \quad a_0 = -\frac{2b_0 m^2 \mu}{\lambda}, \quad a_{-1} = \frac{b_0^2 m^2 \mu}{4b_1 \lambda},$$
$$b_1 = b_1, \quad b_0 = b_0, \quad b_{-1} = \frac{b_0^2}{4b_1}, \quad k = \mp m^2\sqrt{\mu},$$

where b_1 and b_0 are free parameters.

$$u(x,t) = \frac{\frac{b_1 m^2 \mu}{\lambda}\exp\left[\left(\mp m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right] - \frac{2b_0 m^2 \mu}{\lambda}}{b_1 \exp\left[\left(\mp m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right] + b_0 + \frac{b_0^2}{4b_1}\exp\left[\left(\pm m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} - m\frac{x^\beta}{\beta}\right)\right]}$$
$$+\frac{\frac{b_0^2 m^2 \mu}{4b_1 \lambda}\exp\left[\left(\pm m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} - m\frac{x^\beta}{\beta}\right)\right]}{b_1 \exp\left[\left(\mp m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right] + b_0 + \frac{b_0^2}{4b_1}\exp\left[\left(\pm m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} - m\frac{x^\beta}{\beta}\right)\right]}, \tag{18}$$
$$v(x,t) = \pm m\sqrt{\mu}.U\left(\mp m^2\sqrt{\mu}\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right).$$

6. Results and discussion

In this section, the solutions (8), (12) and (18) of fractional STO, ZKBBM and coupled Boussinesq equations are simulated as traveling wave solutions for various values of the physical parameters in Figures 1–6. Figure 1 and Figure 2 show kink wave solution $u_1(x,t)$ in Eq. (8). 3 D plot of the obtained solution $u_1(x,t)$ is given for $\alpha = 0.75,\ \beta = 1, a_1 = 1,\ a_0 = -5, c = 1,\ m = 1$. Figure 2 also illustrates the same solution with 2 D plot for $-20 < x < 20$ at $t = 1$. Figure 3 and Figure 4 show singular kink wave solution $u(x, t)$ in Eq. (12). Figure 3 is 3 D plot of the singular kink wave solution $u(x, t)$ for $\alpha = 0.5,\ \beta = 1, b_0 = 1,\ b_1 = -2, a = 2,\ b = 1,\ m = -0.5, -20 < x < 20,\ 0 < t < 20$. Figure 4 shows 2 D plot of the traveling wave solution $u(x, 1)$ for the same parameters. Figure 5 and Figure 6 show solitary wave solution $u(x, t)$ in Eq. (18). Figure 5 is 3 D plot of the traveling wave solution $u(x, t)$ in Eq. (18) for $\alpha = 0.75,\ \beta = 1, b_0 = 1,\ b_1 = 1, \lambda = 1,\ \mu = 1,\ m = 0.5, -20 < x < 20,\ 0 < t < 10$. Figure 6 also illustrates the same solution with 2 D plot for $-20 < x < 20$ at $t = 1$.

Note that the 3 D graphs describe the behavior of u in space x at time t, which represents the change of amplitude and shape for each obtained solitary wave solutions. 2 D graphs describe the behavior of u in space x at fixed time $t = 1$. All graphics are drawn by the aid of Mathematica 10.

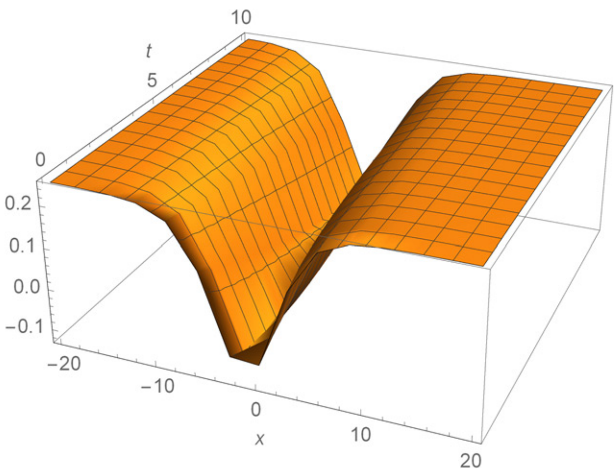

Figure 5. 3D plot of the obtained traveling wave solutions $u(x, t)$ of Eqs. (13)–(14).

7. Conclusion

In this paper, Exp-function method has been applied to the conformable space-time fractional STO, ZKBBM and coupled Boussinesq equations. The method can be used directly without requiring linearization, discretization or perturbation. New solitary wave solutions for conformable space-time fractional STO, ZKBBM and coupled Boussinesq equations have been obtained. It has been checked that all of the obtained solutions satisfy the corresponding equations.

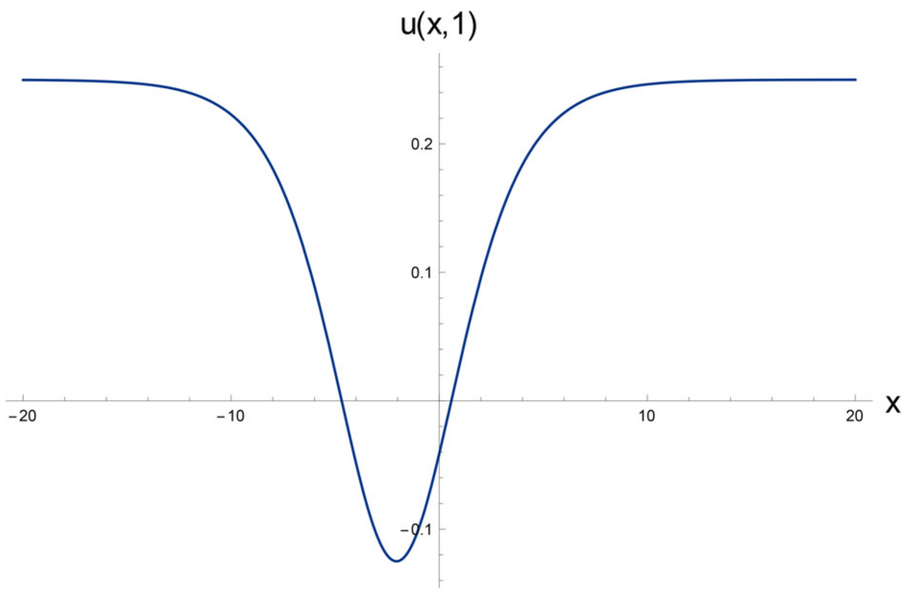

Figure 6. 2D plot of the obtained traveling wave solutions $u(x, 1)$ of Eqs. (13)–(14).

Disclosure statement

No potential conflict of interest was reported by the authors.

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