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ON FACTOR RELATIONS BETWEEN WEIGHTED AND NÖRLUND MEANS

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Abstract. By (X,Y), we denote the set of all sequences $\epsilon=(\epsilon_n)$ such that $\Sigma\epsilon_n a_n$ is summable Y whenever Σa_n is summable X, where X and Y are two summability methods. In this study, we get necessary and sufficient conditions for $\epsilon \in \left(\left|N,q_n,u_n\right|_k, \left|\bar{N},p_n\right|\right)$ and $\epsilon \in \left(\left|\bar{N},p_n\right|, \left|N,q_n,u_n\right|_k\right)$, $k\geq 1$, using functional analytic tecniques, where $\left|\bar{N},p_n\right|$ and $\left|N,q_n,u_n\right|_k$ are absolute weighted and Nörlund summability methods, respectively, [1], [5]. Thus, in the special case, some well known results are also deduced.

1. Introduction

Let $A=(a_{nv})$ be an infinite matrix of complex numbers, Σa_v be a given infinite series with nth partial sum s_n and (u_n) be a sequence of nonnegative terms. Then the series Σa_v is called summable $|A,u_n|_k$, $k\geq 1$, if (see [16])

$$\sum_{n=0}^{\infty} u_n^{k-1}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty,\ A_{-1}(s)=0, \tag{1.1}$$

where $A(s)=(A_n(s))$, the A-transform sequence of the sequence $s=(s_n)$, i.e.,

$$A_n(s)=\sum_{v=0}^{\infty} a_{nv}s_v$$

converges for $n\geq 0$. Note that if A is chosen as the Nörlund matrix ($resp. u_n=n$), then the summability $|A,u_n|_k$ reduces to the absolute Nörlund summability $\left|N,p_n,u_n\right|_k$ [5] ($resp.$ the summability $\left|N,p_n\right|_k$, Borwein and Cass [2]), and also $\left|N,p_n\right|_1=\left|N,p_n\right|$, Mears [9]. Further, if $p_n=\binom{\alpha+n-1}{n}$ and $u_n=n$, then the summability $\left|N,p_n,u_n\right|_k$ is the same as the summability $|C,\alpha|_k$ in Flett's notation [4]. By a Nörlund matrix we mean one that

$$a_{nv}=\begin{cases} p_{n-v}/P_n, & 0\leq v\leq n\\ 0, & v>n,\end{cases} \tag{1.2}$$

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where (p_n) is a sequence of complex numbers with $P_n = p_0 + p_1 + \cdots + p_n \neq 0,\ P_{-(n+1)} = 0$ for $n \geq 0$. Also, if $A = (a_{nv})$ is the weighted matrix $(resp. u_n = P_n/p_n)$, i.e.

$$a_{nv} = \begin{cases} p_v/P_n, & 0 \leq v \leq n \\ 0, & v > n \end{cases} \tag{1.3}$$

then the summability $|A, u_n|_k$ reduces to the summability $\left|\bar{N}, p_n, u_n\right|_k$ ($resp.$ the summability $\left|\bar{N}, p_n\right|_k$, Bor [1]), where (p_n) is a sequence of positive numbers such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, Sulaiman [22]. For example, for the summability $\left|\bar{N}, p_n\right|_k$, the condition (1.1) may be stated as

$$\sum_{n=1}^{\infty} \left| \frac{1}{P_{n-1}} \left(\frac{p_n}{P_n}\right)^{1/k^*} \sum_{v=1}^{\infty} P_{v-1} a_v \right|^k < \infty.$$

Throughout this paper, $k^*$is the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

For any real α and integers $n \geq 0$, we define

$$\Delta^{\alpha} \epsilon_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1} \epsilon_v$$

whenever the series on right side of equality is convergent.

Let ϵ be a sequence and X and Y be two methods of summability. If $\Sigma \epsilon_n a_n$ is summable Y whenever Σa_n is summable X, then ϵ is said to be a summability factor of type (X, Y) and we denote it by $\epsilon \in (X, Y)$ [3]. The problems of summability factors dealing with absolute Cesàro and absolute weighted mean summabilities were widely examined by many authors (see [1-4], [8-11], [13-21]) et al. For example, for $\alpha \geq 0$, $k > 1$, the summability factors of type $\left(|C,\alpha|, \left|\bar{N}, p_n\right|\right)$, $\left(|C,\alpha|_k, \left|\bar{N}, p_n\right|\right)$, $\left(|C,\alpha|_k, |C,1|\right)$ and $\left(|C,1|_k, \left|\bar{N}, p_n\right|\right)$ were characterized by Mohapatra [11], Mazhar [8], Mehdi [10], Sarıgöl and Bor [17] and Sarıgöl [18], respectively. In a more recent paper, Sarıgöl [13] has extended these classes to $\alpha > -1$ and arbitrary positive sequence (p_n) in the following form.

Theorem 1.1. *Let $\alpha > -1$ and (p_n) be arbitrary sequence of positive numbers. Then, necessary and sufficient condition for $\epsilon \in \left(|C,\alpha|_k, \left|\bar{N}, p_n\right|\right)$, $k > 1$, is*

$$\sum_{m=1}^{\infty} m^{\alpha k^* + k^* - 1} \left(\sum_{n=m}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{r=m}^{\infty} A_{r-m}^{-\alpha-1} \frac{\epsilon_r}{r} P_{r-1} \right| \right)^{k^*} < \infty. \tag{1.4}$$

2. Main results

The purpose of this study is to generalize Theorem 1.1 by using Nörlund mean in place of Cesàro mean. Hence we characterize both classes $\left(\left|N, q_n, u_n\right|_k, \left|\bar{N}, p_n\right|\right)$ and

$\left(\left|\bar{N},p_n\right|,\left|N,q_n,u_n\right|_k\right)$, which gives us more than we need. In the special cases, some well known results are also deduced.

Before stating the theorems we recall the following lemmas which plays important role for the proof our theorems.

Lemma 2.1. *Let $1<k<\infty$. Then, $A(x)\in\ell$ whenever $x\in\ell_k$ if and only if*

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}<\infty,$$

where $\ell_k=\left\{x=(x_v):\Sigma|x_v|^k<\infty\right\}$ [13].

Lemma 2.2. *Let $1\le k<\infty$. Then, $A(x)\in\ell_k$ whenever $x\in\ell$ if and only if*

$$\sup_v\sum_{n=0}^{\infty}|a_{nv}|^k<\infty,$$

[7].

Now we begin with the theorem characterizing the class $\left(\left|N,q_n,u_n\right|_k,\left|\bar{N},p_n\right|\right)$.

Theorem 2.3. *Let q_0 be a non-zero number, (u_n) be a sequence of positive terms and (C_n) be a sequence satisfying*

$$\sum_{v=0}^{n}Q_{n-v}C_v=\begin{cases}1, & n=0\\ 0, & n\ge 1.\end{cases} \tag{2.1}$$

Then necessary and sufficient condition for $\epsilon\in\left(\left|N,q_n,u_n\right|_k,\left|\bar{N},p_n\right|\right)$, $k>1$, is

$$\sum_{m=1}^{\infty}\frac{1}{u_m}\left(\sum_{n=m}^{\infty}\left|\frac{p_n}{P_nP_{n-1}}\sum_{r=m}^{n}P_{r-1}\epsilon_rG_{rm}\right|\right)^{k^*}<\infty \tag{2.2}$$

where

$$G_{nr}=\sum_{v=r}^{n}C_{n-v}Q_v. \tag{2.3}$$

Proof. Let (t_n) and (T_n) be the sequences of Nörlund mean $\left(N,q_n\right)$ and weighted mean $\left(\bar{N},p_n\right)$ of the series Σa_n and $\Sigma\epsilon_na_n$, respectively, i.e

$$t_n=\frac{1}{Q_n}\sum_{v=0}^{n}q_{n-v}s_v=\frac{1}{Q_n}\sum_{v=0}^{n}Q_{n-v}a_v$$

and

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})\epsilon_va_v.$$

Then we define sequences $y=\left(y_n\right)$ and $\tilde{y}=\left(\tilde{y}_n\right)$ by

$$y_n=u_n^{1/k^*}\left(t_n-t_{n-1}\right) \tag{2.4}$$

and

$$\tilde{y}_n = T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^{n} P_{r-1}\epsilon_r a_r,\ n \geq 1 \text{ and } \tilde{y}_0 = a_0\epsilon_0$$

Then, $\epsilon \in \left(\left|N, q_n, u_n\right|_k, \left|\bar{N}, p_n\right|\right)$ if and only if $\tilde{y} \in l$ whenever $y \in l_k$. On the other hand, since q_0 is a non-zero, there exists a sequence (C_n) satisfying (2.1) and therefore it follows that

$$t_n = \frac{1}{Q_n} \sum_{v=0}^{n} Q_{n-v} a_v \text{ if and only if } a_n = \sum_{v=0}^{n} C_{n-v} Q_v t_v.$$

Hence we get from (2.4),

$$\begin{aligned} a_n &= \sum_{v=0}^{n} C_{n-v} Q_v \sum_{r=0}^{v} u_r^{-1/k^*} y_r \\ &= \sum_{r=0}^{n} u_r^{-1/k^*} \sum_{v=r}^{n} C_{n-v} Q_v y_r = \sum_{r=0}^{n} u_r^{-1/k^*} G_{nr} y_r \end{aligned}$$

where G_{nr} is defined by (2.3), and so, for $n \geq 1$,

$$\begin{aligned} \tilde{y}_n &= \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^{n} P_{r-1}\epsilon_r a_r \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^{n} P_{r-1}\epsilon_r \sum_{m=1}^{r} u_m^{-1/k^*} G_{rm} y_m \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{m=1}^{n} \left(u_m^{-1/k^*} \sum_{r=m}^{n} P_{r-1}\epsilon_r G_{rm} \right) y_m \\ &= \sum_{m=1}^{n} b_{nm} y_m \end{aligned}$$

where

$$b_{nm} = \begin{cases} \dfrac{u_m^{-1/k^*} p_n}{P_n P_{n-1}} \sum_{r=m}^{n} P_{r-1}\epsilon_r G_{rm}, & m \leq n \\ 0, & m > n. \end{cases} \tag{2.5}$$

Then, $\tilde{y} \in l$ whenever $y \in l_k$ if and only if

$$\sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} |b_{nm}| \right)^{k^*} < \infty$$

by Lemma 2.1, which is same as the condition (2.2). This completes the proof. ☐

It may be remarked that in the special case $q_n = A_n^{\alpha-1}$ and $u_n = n$, Theorem 2.3 reduces to Theorem 1.1. In fact, in this case it is obvious that $\left|N, q_n, u_n\right|_k = |C, \alpha|_k$. Also, we recall the following well known equality of Bosanquet and Das [3], for $\alpha \neq -1, -2, \ldots, v \geq 1$,

$$\sum_{r=v}^{n} A_r^{\alpha} A_{n-r}^{-\alpha-2} = \frac{v A_v^{\alpha} A_{n-v}^{-\alpha-1}}{n}, \tag{2.6}$$

Now, it is easy to see that $C_0 = A_0^{-\alpha-2} = 1$, $C_n = A_n^{-\alpha-2}$ and $A_{-n}^{\alpha} = 0$ for $n \geq 1$, and so, since

$$G_{rm} = \sum_{v=m}^{r} A_v^{\alpha} A_{r-v}^{-\alpha-2} = \frac{m A_m^{\alpha} A_{r-m}^{-\alpha-1}}{r}, 1 \leq r \leq v, \text{ and } 0 \text{ for } r > v,$$

by using (2.6), we get the matrix $B = (b_{nm})$ as

$$b_{nm} = \begin{cases} \dfrac{m^{1/k} A_m^{\alpha} p_n}{P_n P_{n-1}} \displaystyle\sum_{r=m}^{n} A_{r-m}^{-\alpha-1} \frac{\epsilon_r}{r} P_{r-1}, & m \leq n \\ 0, & m > n. \end{cases}$$

So by $A_n^{\alpha} \sim n^{\alpha}/\Gamma(\alpha+1)$ for $\alpha > -1$ [4], it follows from applying Lemma 2.1 to the matrix B that (2.2) is the same as (1.4), as asserted.

Note that $1 \in (X, Y)$ leads us to a comparison of summability fields of methods X and Y, where $1 = (1, 1, \ldots)$. So taking $\epsilon_n = u_n = 1$ for all $n \geq 1$ in Theorem 2.3 we get the following result.

Corollary 2.4. *If $k > 1$, then, $1 \in \left(\left|N, q_n\right|_k, \left|\bar{N}, p_n\right|\right)$ if and only if*

$$\sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} \left| \frac{p_n}{P_n P_{n-1}} \sum_{r=m}^{n} P_{r-1} G_{rm} \right| \right)^{k^*} < \infty. \tag{2.7}$$

This result also extends the following result of Kayashima [6] to $k > 1$.

Corollary 2.5. *If (p_n) and (q_n) are positive and nonincreasing sequences and (q_{n+1}/q_n) is nondecreasing, then $1 \in \left(\left|N, q_n\right|, \left|\bar{N}, p_n\right|\right)$.*

Theorem 2.6. *Let $k \geq 1$ and (u_n) be a sequence of nonnegative terms. Then, $\epsilon \in \left(\left|\bar{N}, p_n\right|, \left|N, q_n, u_n\right|_k\right)$ if and only if*

$$\sup_{v} \left\{ u_v^{1/k^*} \left| \frac{\epsilon_v P_v}{Q_v p_v} \right| \right\} < \infty \tag{2.8}$$

and

$$\sup_{v} \sum_{n=v+1}^{\infty} \left| u_n^{1/k^*} \left(\Omega_{nv}^{q} \epsilon_v \frac{P_v}{p_v} - \Omega_{n,v+1}^{q} \epsilon_{v+1} \frac{P_{v-1}}{p_v} \right) \right|^{k} < \infty, \tag{2.9}$$

where

$$\Omega_{nv}^{q} = \frac{Q_{n-v}}{Q_n} - \frac{Q_{n-v-1}}{Q_{n-1}}, Q_{-1} = 0. \tag{2.10}$$

Proof. As in proof of Theorem 2.3, we define sequences $y = (y_n)$ and $\tilde{y} = (\tilde{y}_n)$ by $y_0 = \epsilon_0 a_0$,

$$y_n = u_n^{1/k^*} (t_n - t_{n-1}) = u_n^{1/k^*} \sum_{v=0}^{n} \left(\frac{Q_{n-v}}{Q_n} - \frac{Q_{n-v-1}}{Q_{n-1}} \right) \epsilon_v a_v, n \geq 1 \tag{2.11}$$

$$\tilde{y}_0 = a_0,\ \tilde{y}_n = T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{r=1}^{n} P_{r-1} a_r,\ n \geq 1 \tag{2.12}$$

Then, $\epsilon \in \left(\left|\bar{N}, p_n\right|, \left|N, q_n, u_n\right|_k\right)$ iff $y \in l_k$ whenever $\tilde{y} \in l$. On the other hand, from (2.12) we write

$$a_n = \frac{P_n}{p_n}\tilde{y}_n - \frac{P_{n-2}}{p_{n-1}}\tilde{y}_{n-1}, n \geq 1 \text{ and } a_0 = \tilde{y}_0$$

Hence, by (2.11) we get

$$\begin{aligned} y_n &= u_n^{1/k^*} \sum_{v=1}^{n} \Omega_{nv}^q \epsilon_v a_v = u_n^{1/k^*} \sum_{v=1}^{n} \Omega_{nv}^q \epsilon_v \left(\frac{P_v}{p_v}\tilde{y}_v - \frac{P_{v-2}}{p_{v-1}}\tilde{y}_{v-1}\right) \\ &= u_n^{1/k^*} \left\{\Omega_{nn}^q \epsilon_n \frac{P_n}{p_n}\tilde{y}_n + \sum_{v=1}^{n-1} \left(\Omega_{nv}^q \epsilon_v \frac{P_v}{p_v} - \Omega_{n,v+1}^q \epsilon_{v+1} \frac{P_{v-1}}{p_v}\right)\tilde{y}_v\right\} \\ &= \sum_{v=1}^{n} c_{nv}\tilde{y}_v \end{aligned}$$

where

$$c_{nv} = \begin{cases} u_n^{1/k^*}\left(\Omega_{nv}^q \epsilon_v \frac{P_v}{p_v} - \Omega_{n,v+1}^q \epsilon_{v+1} \frac{P_{v-1}}{p_v}\right), & 1 \leq v \leq n-1 \\ u_n^{1/k^*} \Omega_{nn}^q \epsilon_n \frac{P_n}{p_n}, & v = n \\ 0, & v > n. \end{cases}$$

So $y \in l_k$ whenever $\tilde{y} \in l$ if and only if

$$\sup_v \sum_{n=v}^{\infty} |c_{nv}|^k < \infty$$

by Lemma 2.2 or, equivalently,

$$\sup_v \left\{u_v^{1/k^*}\left|\frac{\epsilon_v P_v}{Q_v p_v}\right|\right\} < \infty,\ v \geq 1$$

and

$$\sup_v \sum_{n=v+1}^{\infty} \left|u_n^{1/k^*}\left(\Omega_{nv}^q \epsilon_v \frac{P_v}{p_v} - \Omega_{n,v+1}^q \epsilon_{v+1} \frac{P_{v-1}}{p_v}\right)\right|^k < \infty.$$

Thus the proof is completed. □

Corollary 2.7. *If $k \geq 1$, then, $1 \in \left(\left|\bar{N}, p_n\right|, \left|N, q_n\right|_k\right)$ if and only if*

$$\sup_v \left\{v^{1/k^*}\left|\frac{P_v}{Q_v p_v}\right|\right\} < \infty \tag{2.13}$$

and

$$\sup_v \sum_{n=v+1}^{\infty} \left|n^{1/k^*}\left(\Omega_{nv}^q \frac{P_v}{p_v} - \Omega_{n,v+1}^q \frac{P_{v-1}}{p_v}\right)\right|^k < \infty. \tag{2.14}$$

Proof. Put $\epsilon_n = 1$ and $u_n = n$ for all $n \geq 1$ in Theorem 2.6.

This result, for $k = 1$, reduces to the following theorem of Kayashima [6].

Corollary 2.8. *If (p_n) and (q_n) are positive and nondecreasing sequences and (q_{n+1}/q_n) is nonincreasing, then $1 \in (|\bar{N}, p_n|, |N, q_n|)$.*

Proof. By considering that (p_n) and (q_n) are positive and nondecreasing sequences, we have

$$\frac{P_n}{Q_n p_n} \leq \frac{n+1}{Q_n} \leq q_0^{-1} \text{ for all } n \geq 0.$$

Also, by hypotheses on the sequence (q_n), it converges to a number, $\lim_n q_{n+1}/q_n = \sigma$ say. So, there exists a nonincreasing null sequence (x_n) such that $q_{n+1} = (\sigma + x_n) q_n$ for all $n \geq 0$, where $\sigma \geq 1$. Then, it can be written that

$$Q_n = q_0 + \sigma Q_{n-1} + \sum_{v=1}^{n} q_{v-1} x_{v-1}$$

which gives

$$\frac{Q_n}{Q_{n-1}} = \frac{q_0}{Q_{n-1}} + \sigma + Z_n \to \sigma \text{ as } n \to \infty \tag{2.15}$$

where

$$Z_n = \frac{1}{Q_{n-1}} \sum_{v=1}^{n} q_{v-1} x_{v-1}$$

Since (x_n) is nonincreasing, it is easily seen that (Z_n) is nonincreasing, which implies that (Q_n/Q_{n-1}) is nonincreasing. So it follows that, for $0 \leq v \leq n$,

$$\Omega_{nv}^{q} = \frac{Q_{n-v-1}}{Q_n} \left(\frac{Q_{n-v}}{Q_{n-v-1}} - \frac{Q_n}{Q_{n-1}} \right) \geq 0$$

Further,

$$C_{nv} = \Omega_{nv}^{q} \frac{P_v}{p_v} - \Omega_{n,v+1}^{q} \frac{P_{v-1}}{p_v} \geq 0.$$

In fact, if $q_{n-v}/Q_n - q_{n-v-1}/Q_{n-1} \geq 0$, then it is clear that $C_{nv} \geq 0$, since

$$C_{nv} = \frac{P_v}{p_v} \left(\frac{q_{n-v}}{Q_n} - \frac{q_{n-v-1}}{Q_{n-1}} \right) + \Omega_{n,v+1}^{q}. \tag{2.16}$$

If $q_{n-v}/Q_n - q_{n-v-1}/Q_{n-1} < 0$, then, it can be deduced from the condition on (q_n) that

$$\begin{aligned} \frac{q_{n-v}}{Q_n} - \frac{q_{n-v-1}}{Q_{n-1}} &\geq \frac{q_{n-v-1}}{Q_n} \left(\frac{q_{n-m}}{q_{n-m-1}} - \frac{Q_n}{Q_{n-1}} \right) \\ &\geq \frac{q_{n-m}}{Q_n} - \frac{q_{n-m-1}}{Q_{n-1}}, \ 0 \leq m \leq v, \end{aligned}$$

which implies

$$v\left(\frac{q_{n-v}}{Q_n}-\frac{q_{n-v-1}}{Q_{n-1}}\right)\geq\sum_{m=0}^{v-1}\left(\frac{q_{n-m}}{Q_n}-\frac{q_{n-m-1}}{Q_{n-1}}\right)=-\Omega^q_{nv}.$$

Also, since (p_v) is a positive nondecreasing sequence, we can write $P_v \leq v p_v$ for all $v \geq 1$, which gives us, by (2.17),

$$\frac{P_v}{p_v}\left(\frac{q_{n-v}}{Q_n}-\frac{q_{n-v-1}}{Q_{n-1}}\right)\geq-\frac{P_v}{vp_v}\Omega^q_{nv}\geq-\Omega^q_{nv}.$$

This means that $C_{nv}\geq 0$ for $0\leq v\leq n$. Hence, by considering (2.15), we have

$$\begin{aligned}\sup_v\sum_{n=v+1}^{\infty}\left|\frac{P_v}{p_v}\Omega^q_{nv}-\frac{P_{v-1}}{p_v}\Omega^q_{n,v+1}\right| &= \sup_v\lim_m\sum_{n=v+1}^{m}\left(\frac{P_v}{p_v}\Omega^q_{nv}-\frac{P_{v-1}}{p_v}\Omega^q_{n,v+1}\right)\\ &\leq \sup_v\lim_m\left[\frac{P_v}{p_v}\left(\frac{Q_{m-v}}{Q_m}-\frac{q_0}{Q_v}\right)-\frac{P_{v-1}Q_{m-v-1}}{p_vQ_m}\right]\\ &= \sup_v\left[\frac{P_v}{p_v}\left(\frac{1}{\sigma^v}-\frac{q_0}{Q_v}\right)-\frac{P_{v-1}}{p_v\sigma^{v-1}}\right]<\infty,\end{aligned}$$

which completes the proof. □

Further the following result of [12] is obtained form Corollary 2.7 by choosing $q_n = 1$ for $n \geq 1$.

Corollary 2.9. *If $k\geq 1$, then, $1\in\left(\left|\bar{N},p_n\right|,|C,1|_k\right)$ if and only if*

$$\sup_v\frac{P_v}{v^{1/k}p_v}<\infty.$$

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