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Geodesic motions in SO(2,1)

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Abstract: In this study, we have considered the rotational motions of a particle around the origin of the unit 2-sphere S_2^2 with constant angular velocity in semi-Euclidean 3-space with index two E_2^3, namely geodesic motions of $SO(2,1)$. Then we have obtained the vector and the matrix representations of the spherical rotations around the origin of a particle on S_2^2. Furthermore, we consider some relations between semi-Riemann spaces $SO(2,1)$ and $T_1S_2^2$ such as diffeomorphism and isometry. We have obtained the system of differential equations giving geodesics of Sasaki semi-Riemann manifold $(T_1S_2^2, g^S)$. Moreover, we consider the stationary motion of a particle on S_2^2 corresponding to one parameter curve of $SO(2,1)$, which determines a geodesic of $SO(2,1)$. Finally, we obtain the system of differential equations giving geodesics of the semi-Riemann space $(SO(2,1), h)$ and we show that the system of differential equations giving geodesics of Riemann space $(SO(2,1), h)$ is equal to that of $(T_1S_2^2, g^S)$.

Key words: Tangent sphere bundle, rotational group in semi-Euclidean 3-space, geodesics

1. Introduction

The particle kinematics on the unit 2- sphere S_2^2 in semi-Euclidean 3-space E_2^3 is a new research field, which has attracted the attention of researchers. The rotational motion of a particle around the origin of S_2^2 corresponds to a one-parameter curve of special orthogonal group $SO(2,1)$ in E_2^3. In this paper, we study the rotational motion of a particle with constant angular velocity around the origin of S_2^2, which defines a geodesic of $SO(2,1)$.

The spherical rotation of a vector around a fixed point was considered by Euler in 1765. He defined the vector representation of the spherical rotation of a vector about a fixed point in Euclidean 3-space. The matrix and quaternion representations corresponding to this rotation were obtained by Rodrigues and Hamilton, respectively [4].

Rotation motion is used for many different aims, such as describing the equations of the hydrodynamics of ideal fluids [1], generating the equations of motion for a robot manipulator [12], or the optimization of the rotation averaging problem [5].

The reason we deal with the geodesics of the special orthogonal group $SO(2,1)$ is to find a geometrical or dynamical interpretation to geodesics of the tangent sphere bundle $T_1S_2^2$.

Klingenberg and Sasaki defined an isomorphism from the tangent sphere bundle T_1S^2 to the special orthogonal group $SO(3)$. Moreover, they showed that this isomorphism is an isometry between the Sasaki

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Riemann manifold T_1S^2 with metric g^S and Riemann space $SO(3)$ with metric structure h derived by the Killing form. Then they considered the geodesics of T_1S^2 in detail [7].

Ayhan studied the geodesics of the tangent sphere bundle $T_1S_1^2$. He found the Sasaki semi-Riemann metric on $T_1S_1^2$ and then he obtained the system of differential equations giving geodesics on $T_1S_1^2$ [2].

Ayhan considered the geodesics of the special orthogonal group $SO(1,2)$ in E_1^3. He showed that the systems of differential equations giving geodesics of $SO(1,2)$ and $T_1H_1^2$ are equal [3].

Arnold defined the geodesics of the special orthogonal group in 3-dimensional Euclidean space by stationary motions on $SO(3)$. Moreover, he showed that the stationary motions are motions of particles with constant angular velocity in E^3 [1].

Novelia and O'Reilly indicated that a rotating particle with constant angular velocity corresponds to a one-parameter curve and this curve is a geodesic of the special orthogonal group $SO(3)$ in Euclidean 3-space E^3. Then they showed that this geodesic corresponds to a great circle on the unit 3-sphere. Moreover, they described the kinetic energy of a rotating particle in terms of the unit quaternion. They showed that kinetic energy of the rotating particle is constant along the geodesics of the special orthogonal group [9].

Jaferi and Yaylı studied the generalized quaternions and they have indicated how unit generalized quaternions can be used to describe rotation in 3-dimensional space $E_{\alpha\beta}^3$ [6].

Korolko and Leite proved that the kinematic equations for rolling the Lorentzian sphere are solved completely when rolling along geodesics [8].

Now let us take a closer look at the topics in the sections of the article.

In the second section of this paper, we examine the vector representation of the spherical rotational around the origin of a particle on the unit 2-sphere S_2^2 in E_2^3. Then we consider the matrix representation of this rotation depending on a rotation angles and a rotation axis. Moreover, we consider the tangent vector space $T_ISO(2,1)$ at identity rotation I of $SO(2,1)$ denoted by $so(2,1)$. Then we see that $so(2,1)$ consists of skew symmetric matrices and we obtain the expression of a tangent vector of $so(2,1)$ with respect to basis vectors of $so(2,1)$. Moreover, we consider the semi-Riemann metric on $SO(2,1)$. Finally, we are interested in the relations between $T_1S_2^2$ and $so(2,1)$.

In the third section, we study the expression with respect to the local coordinate functions of a point on $T_1S_2^2$, the orthonormal frame on $T_1S_2^2$, the covariant derivations of basis vectors of this orthonormal frame, Sasaki semi-Riemann metric g^S on $T_1S_2^2$, the adapted basis and dual basis vectors on $T_1S_2^2$ with respect to g^S, and geodesics of $\left(T_1S_2^2, g^S\right)$ inspired by [2]

In the fourth section, we examine the relation between the stationary motion of a rotating particle around the origin of S_2^2 and a geodesic of $SO(2,1)$. Then we obtain the stationary motion of a particle on S_2^2 with constant angular velocity producing a geodesic of $SO(2,1)$.

In the last section, we consider a new representation of an orthonormal basis of $T_1S_2^2$ via the Euler rotation matrices. Then we define a differentiable map between Riemann spaces $\left(T_1S_2^2, g^S\right)$ and $(SO(2,1),h)$. We show that the line element of $\left(T_1S_2^2, g^S\right)$ is equal to the line element of $(SO(2,1),h)$. Moreover, we obtain the second-order derivative of a rotation matrix R of $SO(2,1)$ with respect to components of R. Finally, we obtain the system of differential equations giving geodesics of $(SO(2,1),h)$ and we prove the equality of the systems of differential equations giving the geodesics $(SO(2,1),h)$ and $\left(T_1S_2^2, g^S\right)$.

2. Spherical rotations in $SO(2,1)$ and $T_1S_2^2$

In this section, the vectorial and matrix representations of the spherical rotation of a particle around the origin of S_2^2 are obtained. Then the tangent vector space at identity rotation I of $SO(2,1)$ denoted by $so(2,1)$, the skew symmetric structure of $so(2,1)$, and the expression of a vector of $so(2,1)$ with respect to the basis vectors of $so(2,1)$ are considered. Moreover, the symmetric metric structure on $SO(2,1)$, geodesics of $SO(2,1)$, and the relations between $SO(2,1)$ and $T_1S_2^2$ are studied.

The vectorial representation of the spherical rotation of a point P of S_2^2 about fixed point O along the n rotation axis by the angle of rotation φ is given by

$$r' \quad = \quad r + (n \times r)\sinh\varphi + n \times (n \times r)(-1 + \cosh\varphi), \tag{2.1}$$

where r and r' are the initial and final position vector of a point P of S_2^2 [8].

The matrix representation of a spherical rotation was considered by Rodrigues in 3-dimensional Euclidean space [4]. Assuming that N is a skew-symmetric matrix in semi-Euclidean 3-space E_2^3 corresponding to a unit vector $n = (n_1 \quad n_2 \quad n_3)$ given by

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}, \tag{2.2}$$

then the cross product $n \times r$ can be defined as follows:

$$n \times r = \begin{vmatrix} i & -j & -k \\ n_1 & n_2 & n_3 \\ r_1 & r_2 & r_3 \end{vmatrix},$$

and shown in matrix form as

$$n \times r = Nr. \tag{2.3}$$

If we put (2.3) into (2.1), we get the matrix representation of the rotation as follows:

$$r' = Rr, \tag{2.4}$$

where R is defined as follows:

$$R = I + N\sinh\varphi + N^2(-1 + \cosh\varphi), \tag{2.5}$$

where I is the unit matrix and $R = R_n(\varphi)$ is the rotation matrix described by the direction cosines n_1, n_2, n_3 of the rotation axis n and the rotation angle φ [6]. By calculating (2.5), we obtain $R_n(\varphi)$ by

$$\begin{pmatrix} \left(n_2^2 + n_3^2\right)(\cosh\varphi - 1) + 1 & -n_3\sinh\varphi - n_1n_2(\cosh\varphi - 1) & n_2\sinh\varphi - n_1n_3(\cosh\varphi - 1) \\ n_1n_2(\cosh\varphi - 1) - n_3\sinh\varphi & 1 - (\cosh\varphi - 1)\left(n_1^2 - n_3^2\right) & n_1\sinh\varphi - n_2n_3(\cosh\varphi - 1) \\ n_2\sinh\varphi + n_1n_3(\cosh\varphi - 1) & -n_1\sinh\varphi - n_2n_3(\cosh\varphi - 1) & 1 - (\cosh\varphi - 1)\left(n_1^2 - n_2^2\right) \end{pmatrix}. \tag{2.6}$$

Definition 2.1 *The set of length-preserving linear transformation in three-dimensional semi-Euclidean space with index 2 under the composition's operation of transformations is a group. This group is called the special orthogonal group, denoted by $SO(2,1)$ (see [11]) or $SO_2(3)$ (see [10]), and described by the following set:*

$$SO(2,1) = \left\{R : R^T\chi R = \chi \ \textit{and} \ \det R = 1\right\},$$

where

$$\chi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Theorem 2.2 *The $so(2,1)$ tangent vector space of $SO(2,1)$ at point I consists of skew-symmetric matrices.*

Proof Let $R = R_n(\varphi)$ be the rotation matrix described by the direction cosines n_1, n_2, n_3 of the rotation axis $n = (n_1, n_2, n_3)$ and the rotation angle φ. This rotation matrix $R = R_n(\varphi)$ is given by (2.6). A tangent vector of $so(2,1)$ has been obtained by taking into $\varphi = 0$ in the derivative of (2.6) with respect to φ as follows:

$$\dot{R}_n(0) = \left.\frac{d}{d\varphi}\right|_{\varphi=0} \{R_n(\varphi)\},$$

where

$$\dot{R}_n(\varphi) = \begin{pmatrix} \left(n_2^2 + n_3^2\right)\sinh\varphi & -n_1 n_2 \sinh\varphi - n_3\cosh\varphi & n_1 n_3 \sinh\varphi + n_2 \cosh\varphi \\ -n_1 n_2 \sinh\varphi - n_3\cosh\varphi & -\left(n_1^2 - n_3^2\right)\sinh\varphi & -n_2 n_3 \sinh\varphi + n_1\cosh\varphi \\ n_1 n_3 \sinh\varphi + n_2\cosh\varphi & -n_2 n_3 \sinh\varphi - n_1\cosh\varphi & -\left(n_1^2 - n_2^2\right)\sinh\varphi \end{pmatrix}$$

and

$$N = \dot{R}_n(0) = \begin{pmatrix} 0 & -n_3 & n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix}.$$

$\dot{R}_n(0) \in T_I SO(2,1)$ is a skew-symmetric matrix defined by $N^T = -\chi N \chi$, where χ is defined by

$$\chi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

□

Definition 2.3 *The basis vectors of the nondegenerate subspace of $so(2,1)$ consisting of timelike and spacelike vectors is given by the following matrices:*

$$b_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$b_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, b_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, b_6 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The expression with respect to basis vectors of the tangent vector $N \in so(2,1)$ is given by the following equality:

$$N = n_1 b_1 + n_2 b_2 + n_3 b_3 + 0.b_4 + 0.b_5 + 0.b_6.$$

Definition 2.4 *The symmetric metric structure on the tangent vector space* $so(2,1)$ *at a point* $I = R(\varepsilon)|_{\varepsilon=0}$ *of* $SO(2,1)$ *is defined as follows:*

$$\begin{array}{rccc} h: & so(2,1)\times so(2,1) & \to & R \\ & (X_I, Y_I) & \to & h(X_I,Y_I) = -\frac{1}{2}Trace\left\{X_I.Y_I\right\}, \end{array}$$

where the $(X_I,Y_I) \to Trace\left\{X_I.Y_I\right\}$ *map is called the Killing form of* $SO(3)$ *[9]. Since* h *has nondegenerate, symmetric, bilinear form,* h *will be a semi-Riemann metric on* $SO(2,1)$*. Thus,* $(SO(2,1),h)$ *is called a semi-Riemann space (see [3]).*

Now we show that the map between the rotation matrices of $SO(2,1)$ and the elements of the tangent sphere bundle $T_1S_2^2$ of S_2^2 is a diffeomorphism.

Theorem 2.5 $T_1S_2^2$ *is diffeomorphic to the special orthogonal group* $SO(2,1)$*.*

Proof Let ψ be a map from $T_1S_2^2$ to $SO(2,1)$ and y be an element of $T_1S_2^2$. The unit spacelike vector $e_1(y)$ issues from the center of S_2^2 and ends at the point $\pi(y)$ where $\pi: T_1S_2^2 \to S_2^2$. $e_2(y)$ is identical to y, i.e. $e_2(y) \equiv y$ is unit timelike vector. $e_1(y)\times e_2(y)$ is also a unit timelike vector, where × means cross product in E_2^3 and $e_2(y)$, $e_1(y)\times e_2(y)$ have the same Kozsul character. Thus, the map $\psi: T_1S_2^2 \to SO(2,1)$ defined by $y \to (e_1(y), e_2(y), e_1(y)\times e_2(y))$ is a diffeomorphism. □

Theorem 2.6 *Geodesics of* $T_1S_2^2$ *are either one-parameter subgroups of* $SO(2,1)$ *or their left cosets. These subgroups describe the geodesics of* $SO(2,1)$*.*

Proof Let H be a one-parameter subgroup of $SO(2,1)$. Then H is a group of rotations around a fixed axis l through the origin O. We denote I with (i,j,k) and elements of H by $f_\sigma, \sigma \in R \ mod 2\pi$. If we put $i(\sigma) = f_\sigma(i), j(\sigma) = f_\sigma(j)$, then $(i(\sigma), j(\sigma), i(\sigma)\times j(\sigma))$ draws a geodesic on $(SO(2,1),h)$ as σ varies. Thus, $j(\sigma)$ draws a geodesic of $(T_1S_2^2, g^S)$. When l does not have the direction i, the initial point of j, i.e. end point of $i(\sigma)$, draws a circle C on S_2^2 and $j(\sigma)$ makes a constant angle with C as σ varies. When l has the same direction as i, $i(\sigma)$ coincides with the fixed vector i. We denote the end point of i by x_0. Then $j(\sigma)$ draws a fiber $\pi^{-1}\left\{x_0\right\}$. Any geodesic of $(SO(2,1),h)$ that does not pass through I is given by a left coset of a one-parameter subgroup H, i.e. as a family of a frames $\widetilde{f}\left(i(\sigma), j(\sigma), i(\sigma)\times j(\sigma)\right)$, where $\widetilde{f} \in SO(2,1)$. This corresponds to a vector field $\widetilde{f}\left(j(\sigma)\right)$ on $\mathrm{T}_1S_2^2$. Therefore, the geodesic of $\mathrm{T}_1S_2^2$ that corresponds to a left coset of a one-parameter subgroup H of $SO(2,1)$ is either a unit vector field along a geodesic curve $\widetilde{f}(C)$ of S_2^2, which makes a constant angle with $\widetilde{f}\ (C)$, or a fiber $\pi^{-1}\left\{\widetilde{f}(x_0)\right\}$. □

Let us now show that the map ψ of $\left(T_1S_2^2, g^S\right)$ with $(SO(2,1),h)$ is an isometry.

Theorem 2.7 *The map* $\psi: T_1S_2^2 \to SO(2,1)$ *is an isometry of* $(T_1S_2^2, g^S)$ *with* $(SO(2,1),h)$*.*

Proof In order to show the isometry of the map ψ, it is sufficient to show the isometry of the differential of the map ψ, where ψ_* is a map from the tangent space $T_yT_1S_2^2$ at the point $y = \psi^{-1}(I)$ to the tangent space

$T_ISO(2,1)$ at the unit element I of $SO(2,1)$. We see that y is a tangent vector equal to $j=(0,1,0)$ at the point $i=(1,0,0)$. Now take an element $X_I=\eta_1 b_1+\eta_2 b_2+\eta_3 b_3$. Then it corresponds by ψ^{-1} to

$$\begin{aligned} e_1' &= -\eta_3 j+\eta_2 k, \qquad e_2^{'}=\eta_3 i-\eta_1 k,\\ e_3' &= e_1'\times j+i\times e_2', \end{aligned}$$

where i is spacelike and j and k are timelike vectors. Thus, we have

$$g^S(\left(\psi^{-1}\right)' X_I,\left(\psi^{-1}\right)' X_I)=<e_1',e_1'>+<e_2',k>^2=\eta_1^2-\eta_2^2-\eta_3^2=h(X_I,X_I).$$

Therefore, the correctness of the claim of the theorem is seen. □

3. Geodesics on $T_1S_2^2$

This section covers some issues such as the expression with respect to the local coordinate functions of any point on $T_1S_2^2$, the orthonormal frame of $T_1S_2^2$, the covariant derivations of basis vectors of this orthonormal frame, Sasaki semi-Riemann metric g^S on $T_1S_2^2$, and the adapted basis and dual basis vectors on $T_1S_2^2$ with respect to g^S. This section is inspired by [2].

Definition 3.1 *Let $e_1(a,\theta)$ be any point on S_2^2 given by*

$$e_1(a,\theta)=(\cosh a\cosh\theta,\cosh a\sinh\theta,\sinh a) \tag{3.1}$$

with respect to the geodesic polar coordinates a,θ of S_2^2. Then the unit vectors for the a-curve and θ-curve at point $e_1(a,\theta)$ are given by

$$f_2=\frac{\partial e_1}{\partial a}\quad \text{and}\quad f_3=\frac{1}{\sinh a}\frac{\partial e_1}{\partial\theta}. \tag{3.2}$$

In addition, the unit tangent vectors f_2 and f_3 have the following local expression:

$$\begin{aligned} f_2(a,\theta) &= (\sinh a\cosh\theta,\sinh a\sinh\theta,\cosh a),\\ f_3(a,\theta) &= (\sinh\theta,\cosh\theta,0), \end{aligned} \tag{3.3}$$

with respect to standard orthonormal basis of E_2^3. Thus f_2, f_3 are the base vectors, which span to tangent vector space at the point $e_1(a,\theta)$ of S_2^2, and e_1 is a unit spacelike and f_2 and f_3 are unit timelike vectors.

Theorem 3.2 *Let S_2^2 be the unit 2-sphere and $\{e_1,f_2,f_3\}$ be another orthonormal basis in semi-Euclidean space E_2^3. The covariant derivations of basis vectors are given by*

$$\begin{pmatrix} de_1\\ df_2\\ df_3\end{pmatrix}=\begin{pmatrix} 0 & da & \cosh a d\theta\\ da & 0 & \sinh a d\theta\\ \cosh a d\theta & -\sinh a d\theta & 0\end{pmatrix}\begin{pmatrix} e_1\\ f_2\\ f_3\end{pmatrix}.$$

Proof We use the covariant derivations of basis vectors e_1,f_2,f_3 in order to examine the change of the frames on two different points with infinitesimal distance on S_2^2 (i.e. (e_1,f_2,f_3) and $(e_1+de_1,f_2+df_2,f_3+df_3)$).

The covariant derivatives of these vectors are calculated by using partial derivation as follows:

$$\begin{aligned} de_1 &= \frac{\partial e_1}{\partial a}da + \frac{\partial e_1}{\partial \theta}d\theta = daf_2 + \cosh ad\theta f_3, \\ df_2 &= \frac{\partial f_2}{\partial a}da + \frac{\partial f_2}{\partial \theta}d\theta = dae_1 + \sinh ad\theta f_3, \\ df_3 &= \frac{\partial f_3}{\partial a}da + \frac{\partial f_3}{\partial \theta}d\theta = \cosh ad\theta e_1 - \sinh ad\theta f_2. \end{aligned}$$

□

Definition 3.3 *The disjoint union of the tangent vector spaces including all unit tangent vectors at each point of* S_2^2 *is called the tangent sphere bundle of* S_2^2 *and represented by* $T_1S_2^2 = \bigcup_{\forall e_1(a,\theta)\in S_2^2} T_{e_1}S_2^2$. *Let* $\pi : T_1S_2^2 \to S_2^2$ *be a canonical projection map and* e_2 *be an element of* $T_1S_2^2$ *at any point* $e_1(a,\theta)$ *of* S_2^2. *If we denote the angle between* f_2 *and* e_2 *by* ω, *then* (a,θ,ω) *can be considered as the local coordinates for* e_2. e_2 *and* e_3 *have the following local expression:*

$$\begin{aligned} e_2(a,\theta,\omega) &= \cos\omega f_2 + \sin\omega f_3, \\ e_3(a,\theta,\omega) &= -\sin\omega f_2 + \cos\omega f_3. \end{aligned} \tag{3.4}$$

Therefore, $\{e_1, e_2, e_3\}$ *is a new orthonormal system, which characterizes all points in* $T_1S_2^2$, *and* e_1 *is spacelike and* e_2 *and* e_3 *are timelike unit vectors.*

Theorem 3.4 *Let* $T_1S_2^2$ *be the tangent sphere bundle of* S_2^2 *and* e_1, e_2, e_3 *be unit orthogonal elements of* $T_1S_2^2$. *The covariant derivations of these elements are obtained by the following equations:*

$$\begin{aligned} de_1 &= (\cos\omega da + \sin\omega\cosh ad\theta)\, e_2 + (-\sin\omega da + \cos\omega\cosh ad\theta)\, e_3, \\ de_2 &= (\cos\omega da + \sin\omega\cosh ad\theta)\, e_1 + (d\omega + \sinh ad\theta)\, e_3, \\ de_3 &= (-\sin\omega da + \cos\omega\cosh ad\theta)\, e_1 - (d\omega + \sinh ad\theta)\, e_2. \end{aligned}$$

Proof We can use the covariant derivations of e_1, e_2, e_3 in order to examine the change of the frames on two different points with infinitesimal distance on $T_1S_2^2$ (i.e. (e_1, e_2, e_3) and $(e_1 + de_1, e_2 + de_2, e_3 + de_3)$). The covariant derivatives of e_1, e_2, e_3 are obtained by helping the partial derivation easily. □

Definition 3.5 *The 1-forms providing the equation* $\eta_k = w_{ij} =< de_i, e_j >$, *for* $i, j, k \in \{1, 2, 3\}$, *are called the connection 1-forms of* $T_1S_2^2$ *where* $\eta_k = w_{ij}$ *is given by*

$$\begin{aligned} \eta_1 &= w_{23} = -w_{32} = d\omega + \sinh ad\theta, \\ \eta_2 &= -w_{13} = -w_{31} = \sin\omega da - \cos\omega\cosh ad\theta, \\ \eta_3 &= w_{12} = w_{21} = \cos\omega da + \sin\omega\cosh ad\theta. \end{aligned} \tag{3.5}$$

Theorem 3.6 *The line element between two infinitely close points in* $T_1S_2^2$ *is equal to:*

$$d\sigma^2 = < de_1, de_1 > + < de_2, e_3 >^2 \tag{3.6}$$

$$= \eta_1 \wedge \eta_1 - \eta_2 \wedge \eta_2 - \eta_3 \wedge \eta_3 \tag{3.7}$$

$$= -(da)^2 - (d\theta)^2 + 2\sinh ad\theta d\omega + (d\omega)^2 . \tag{3.8}$$

Proof In semi-Euclidean space E_2^3, let $\{e_1, e_2, e_3\}$ be an orthonormal frame at any point $e_2 \in \pi^{-1}(\{e_1\})$ of $T_1S_2^2$ and $\{e_1 + de_1, e_2 + de_2, e_3 + de_3\}$ be the orthonormal frame at another point to be an infinitely close point to e_2. The infinitesimal length between these two points is obtained as follows:

$$\begin{aligned} d\sigma^2 &= < de_1, de_1 > + < de_2, e_3 >^2 \\ &= \eta_1 \wedge \eta_1 - \eta_2 \wedge \eta_2 - \eta_3 \wedge \eta_3 \\ &= -(da)^2 - (d\theta)^2 + 2\sinh a d\theta d\omega + (d\omega)^2 . \end{aligned}$$

□

Definition 3.7 *$d\sigma^2$ determines a metric structure denoted by g^S on the manifold $T_1S_2^2$. Moreover, $\{\eta_1, \eta_2, \eta_3\}$ is called an adapted basis 1-form for the cotangent space $T^*_{(e_1,e_2)}T_1S_2^2$ with respect to g^S. The tangent vectors $\xi_i; i \in \{1, 2, 3\}$ providing the following equation are called adapted basis vectors of the tangent space $T_{(e_1,e_2)}T_1S_2^2$ with respect to the metric structure g^S:*

$$\eta_i(\xi_i) = g^S(\xi_i, \xi_i) = \varepsilon_i, \varepsilon_i = \begin{cases} 1 & for \quad i = 1 \\ -1 & for \quad i = 2, 3, \end{cases} \tag{3.9}$$

where ξ_i is defined by

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial \omega}, \\ \xi_2 &= -\sin\omega \frac{\partial}{\partial a} + \frac{\cos\omega}{\cosh a}\frac{\partial}{\partial \theta} - \cos\omega \tanh a \frac{\partial}{\partial \omega}, \\ \xi_3 &= \cos\omega \frac{\partial}{\partial a} + \frac{\sin\omega}{\cosh a}\frac{\partial}{\partial \theta} - \sin\omega \tanh a \frac{\partial}{\partial \omega}. \end{aligned} \tag{3.10}$$

Definition 3.8 *Let $T_1S_2^2$ be the tangent sphere bundle of 2-sphere S_2^2 in 3-dimensional semi-Euclidean space E_2^3. If $T_{(e_1,e_2)}T_1S_2^2$ is a tangent vector space at any point (e_1, e_2) of $T_1S_2^2$, g^S is a semi-Riemann metric on $T_1S_2^2$, where g^S is defined by*

$$\begin{array}{lccc} g^S: & T_{(e_1,e_2)}T_1S_2^2 \times T_{(e_1,e_2)}T_1S_2^2 & \to & IR \\ & (X, Y) & \to & g^S(X, Y). \end{array} \tag{3.11}$$

Since g^S has a nondegenerate, symmetric, bilinear form, g^S must be a semi-Riemann metric on the tangent sphere bundle. g^S is called the Sasaki semi-Riemann metric and $\left(T_1S_2^2, g^S\right)$ is also called the Sasaki semi-Riemann manifold.

The induced semi-Riemann metric structure g^S on $T_1S_2^2$ has the matrix representation

$$g_{\alpha\beta} : \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & \sinh a \\ 0 & \sinh a & 1 \end{pmatrix} for\ \alpha, \beta \in \{1, 2, 3\}. \tag{3.12}$$

The inverse matrix of $g_{\alpha\beta}$ is given by

$$g^{\beta\alpha} : \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sec h^2 a & \sec ha \tanh a \\ 0 & \sec ha \tanh a & \sec h^2 a \end{pmatrix}. \tag{3.13}$$

Theorem 3.9 *Let* $\left(T_1S_2^2, g^S\right)$ *be a semi-Riemann manifold. Let* ∇ *be Levi-Civita connection of* $\left(T_1S_2^2, g^S\right)$ *and* $\Gamma_{\alpha\beta}^{\gamma}; \alpha, \beta, \gamma \in \{1,2,3\}$ *be coefficients of the Christoffel symbols related to* ∇. *At the same time,* ∇ *is symmetric. Then the nonzero Christoffel symbols of* $\left(T_1S_2^2, g^S\right)$ *are given as follows:*

$$\begin{array}{ll} \Gamma_{\theta\omega}^{a} = \frac{1}{2}\cosh a, & \\ \Gamma_{a\theta}^{\theta} = \frac{1}{2}\tanh a, & \Gamma_{a\omega}^{\theta} = -\frac{1}{2}\sec ha, \\ \Gamma_{a\theta}^{\omega} = \frac{1}{2}\sec ha, & \Gamma_{a\omega}^{\omega} = \frac{1}{2}\tanh a, \end{array} \tag{3.14}$$

where $\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma}$ *for all* $\alpha, \beta, \gamma \in \{a, \theta, \omega\}$.

Proof On the Sasaki semi-Riemann manifold $\left(T_1S_2^2, g^S\right)$, there is a unique connection ∇ such that ∇ is torsion-free and compatible with semi-Riemann metric g^S. This connection is called the Levi-Civita connection and characterized by the following Kozsul formula:

$$\begin{aligned} 2g^S\left(\nabla_{\partial_a}\partial_\theta, \partial_\omega\right) &= \partial_a g^S\left(\partial_\theta, \partial_\omega\right) + \partial_\theta g^S\left(\partial_\omega, \partial_a\right) - \partial_\omega g^S\left(\partial_a, \partial_\theta\right) + \\ &\quad -g^S\left(\left[\partial_a, \partial_\theta\right], \partial_\omega\right) + g^S\left(\left[\partial_\theta, \partial_\omega\right], \partial_a\right) + g^S\left(\left[\partial_\omega, \partial_a\right], \partial_\theta\right), \end{aligned}$$

where $\partial_a = \frac{\partial}{\partial a}, \partial_\theta = \frac{\partial}{\partial \theta}$ and $\partial_\omega = \frac{\partial}{\partial \omega}$. Since ∇ is symmetric, $\left[\partial_a, \partial_\theta\right], \left[\partial_\theta, \partial_\omega\right], \left[\partial_\omega, \partial_a\right]$ must be zero. If we get $\nabla_{\partial_a}\partial_\theta = \Gamma_{a\theta}^{a}\partial_a + \Gamma_{a\theta}^{\theta}\partial_\theta + \Gamma_{a\theta}^{\omega}\partial_\omega$, from the Kozsul formula, we obtain the following Christoffel symbols:

$$\begin{aligned} \Gamma_{a\theta}^{a} &= \frac{1}{2}g^{ak}\left(\partial_a g_{k\theta} + \partial_\theta g_{ak} - \partial_k g_{a\theta}\right) = 0, \\ \Gamma_{a\theta}^{\theta} &= \frac{1}{2}g^{\theta k}\left(\partial_a g_{k\theta} + \partial_\theta g_{ak} - \partial_k g_{a\theta}\right) = \frac{1}{2}\tanh a, \\ \Gamma_{a\theta}^{\omega} &= \frac{1}{2}g^{3k}\left(\partial_a g_{k\theta} + \partial_\theta g_{ak} - \partial_k g_{a\theta}\right) = \frac{1}{2}\sec ha, \end{aligned}$$

where $k \in \{a, \theta, \omega\}$. Other Christoffel symbols can be obtained by using a similar method. □

Theorem 3.10 *Let* $\left(T_1S_2^2, g^S\right)$ *be a semi-Riemann manifold and* $c : t \in R \to c(t) = (a(t), \theta(t), \omega(t)) \in T_1S_2^2$ *be a curve on* $T_1S_2^2$. *c is geodesic if and only if c provides the following system of differential equations:*

$$\begin{aligned} \ddot{a} + \cosh a\dot{\theta}\dot{\omega} &= 0, \\ \ddot{\theta} + \tanh a\dot{a}\dot{\theta} - \sec ha\dot{a}\dot{\omega} &= 0, \\ \ddot{\omega} + \sec ha\dot{a}\dot{\theta} + \tanh a\dot{a}\dot{\omega} &= 0. \end{aligned} \tag{3.15}$$

Proof $c(t) = (a(t), \theta(t), \omega(t))$ is geodesic if and only if $\nabla_{\dot{c}}\dot{c}$ must be zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega$, $\nabla_{\dot{c}}\dot{c}$ is equal to

$$\nabla_{\dot{a}\partial_a}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right) + \nabla_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right) + \nabla_{\dot{\omega}\partial_\omega}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right).$$

Therefore, we get

$$\begin{aligned}\nabla_{\dot{c}}\dot{c} &= \ddot{a}\partial_a + \dot{a}\dot{\theta}\left\{\frac{1}{2}\tanh a\partial_\theta + \left(\frac{1}{2}\sec ha\right)\partial_\omega\right\} \\ &\quad +\dot{a}\dot{\omega}\left(-\frac{1}{2}\sec ha\partial_\theta + \frac{1}{2}\tanh a\partial_\omega\right) + \ddot{\theta}\partial_\theta + \\ &\quad +\frac{1}{2}\cosh a\dot{\theta}\dot{\omega}\partial_a + \dot{a}\dot{\theta}\left\{\frac{1}{2}\tanh a\partial_\theta + \left(\frac{1}{2}\sec ha\right)\partial_\omega\right\} \\ &\quad +\dot{a}\dot{\omega}\left(-\frac{1}{2}\sec ha\partial_\theta + \frac{1}{2}\tanh a\partial_\omega\right) + \frac{1}{2}\cosh a\dot{\theta}\dot{\omega}\partial_a + \ddot{\omega}\partial_\omega.\end{aligned}$$

If we organize $\nabla_{\dot{c}}\dot{c}$,

$$\begin{aligned}\nabla_{\dot{c}}\dot{c} &= \left(\ddot{a} + \cosh a\dot{\theta}\dot{\omega}\right)\partial_a \\ &\quad + \left(\ddot{\theta} + \tanh a\dot{a}\dot{\theta} - \sec ha\dot{a}\dot{\omega}\right)\partial_\theta \\ &\quad + \left(\ddot{\omega} + \sec ha\dot{a}\dot{\theta} + \tanh a\dot{a}\dot{\omega}\right)\partial_\omega,\end{aligned}$$

it can be seen that the claim of the theorem is true. □

4. Rotations in SO(2,1)

In this section, the rotational motion of a particle around the origin of S_2^2 is studied. Then the kinetic energy of a rotating particle on S_2^2 is defined in terms of the semi-Riemann structure h on $SO(2,1)$ and the angular velocity vector of this particle. Then the fact that the rotational motion of a particle with constant angular velocity around the origin of the sphere produces a geodesic of $SO(2,1)$ is obtained.

Let $SO(2,1)$ be a group of rotations of semi-Euclidean 3-space, i.e. the configuration space of the rotational motions of particles around the origin of the unit 2-sphere S_2^2. The rotational motion of a particle on S_2^2 is described by a curve $\gamma = \gamma(t)$ in $SO(2,1)$. Let $so(2,1)$ be the space of angular velocities of all possible rotations. The value of $\gamma(t)$ at the initial instant, i.e. $t = 0$, corresponds to identity rotation I and the value of angular velocity of the rotating particle at the initial instant corresponds to angular velocity denoted by $\dot{\gamma}(0) = \dot{R}$.

Let us define the motion $\gamma : IR \to SO(2,1)$ such that $\gamma(0) = I$ and $\dot{\gamma}(0) = \dot{R}$. This motion is defined by the curve $\gamma(t) = \exp(\dot{R}t)$, which is a one-parameter curve of $SO(2,1)$ with angular velocity $\dot{R}$. $\dot{R}$ is the tangent vector to $SO(2,1)$ at the identity rotation I.

The rotational motion of a particle under inertia (with no external forces) around the origin of the unit sphere S_2^2 corresponds to the one-parameter curve on $SO(2,1)$, which is a geodesic of $(SO(2,1), h)$.

The geodesics of semi-Riemann space $(SO(2,1), h)$ are extremizers of kinetic energy T of a rotating particle under inertia around the origin of S_2^2. The kinetic energy of the rotating particle is determined by

$$T = h\left(\dot{R}, \dot{R}\right).$$

To every motion $t \to \gamma(t)$ of a rotating particle, we can associate following curves:

$$t \to \dot{\gamma}(t) \in so(2,1),$$

which are called the motion of the vectors of angular velocity.

Theorem 4.1 *The evolution of the vector $\dot{\gamma}$ in $so(2,1)$ is determined by the following differential equation:*

$$\frac{d\dot{\gamma}}{dt} = B(\dot{\gamma}, \dot{\gamma}),$$

where B defines an operator

$$B : so(2,1) \times so(2,1) \to so(2,1)$$

by the identity

$$h\left([a,b], c\right) = h\left(B(c,a), b\right),$$

for all $b \in so(2,1)$ (see, [1]).

Definition 4.2 *$v \in so(2,1)$ is called a stationary point if*

$$B(v,v) = 0.$$

Then the geodesic $\gamma(t) = \exp(vt)$, originating from the point $\gamma(0) = I$ with initial velocity $\dot{\gamma}(0) = v$, is called stationary motion [1].

Now we examine the relation between the stationary motion and angular velocity under the inertia of a rotating particle on S_2^2.

Theorem 4.3 *The rotational motion of $\gamma(t)$ in $SO(2,1)$ is a geodesic if $\gamma(t)$ is a motion with constant angular velocity.*

Proof Let the curve $\gamma(t)$ be a stationary motion, i.e. a geodesic of $SO(2,1)$. Then $\gamma(t)$ is a motion with acceleration free, i.e. $\ddot{\gamma} = 0$. Namely, $B(v,v) = 0$ for $\dot{\gamma}(0) = v$. Let $T = h\left(\dot{R}, \dot{R}\right)$ be the kinetic energy of a rotating particle on S_2^2. If we take the derivation of T with respect to the variable t, we get

$$2\dot{T} = h\left(\dot{\gamma}, \ddot{\gamma}\right) = h\left(\dot{\gamma}, B(\dot{\gamma}, \dot{\gamma})\right) = h\left([\dot{\gamma}, \dot{\gamma}], \dot{\gamma}\right) = 0.$$

Thus, the stationary motions on $SO(2,1)$ are motions with constant kinetic energy. Since the kinetic energy of the rotating particles with constant angular velocity is constant, the motions of these particles produce geodesics of $SO(2,1)$. □

Corollary 4.4 *The kinetic energy along the geodesic curves $\gamma(t)$ in the configuration space $SO(2,1)$ is conserved, i.e. constant.*

Now let us examine how the rotational motion of a rotating particle with constant angular velocity along a geodesic circle of the unit 2-sphere S_2^2 corresponds to a geodesic in $SO(2,1)$ and that the kinetic energy of this particle is constant at each stage of the movement.

Example 4.5 *The rotational motion of a rotating particle with constant angular velocity along the timelike geodesic circle lying on the* $z = 0$ *plane of* S_2^2 *corresponds to a geodesic of* $SO(2,1)$. *Furthermore, the kinetic energy of this particle is constant at every stage of its motion.*

The vector product of the position vector $(\cosh t, \sinh t, 0)$ *with the velocity vector* $(\sinh t, \cosh t, 0)$ *of this particle moving on the sphere* S_2^2 *gives the angular velocity vector* $n = (0,0,-1)$ *of this particle. Since the differential of* n *with respect to the variable* t *is equal to zero, the angular velocity of this particle is constant at each stage of the movement. The motion of this particle is the spherical rotation of a point* p *of* S_2^2 *about fixed point* O *along the rotation axis* $n = (0,0,-1)$ *and corresponds to the following skew-symmetric matrix:*

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, N *can also be seen as a tangent vector in the tangent vector space* $so(2,1)$ *of* $SO(2,1)$ *at point* I. *Since the exponential map carries the tangent vectors passing through the origin of* $so(2,1)$ *to the geodesics of* $SO(2,1)$ *through* I, $R(t) = \exp(Nt)$ *is a geodesic in* $SO(2,1)$. *By using the definition of the exponential map, we get*

$$R(t) = \exp(Nt) = I + \frac{Nt}{1!} + \frac{(Nt)^2}{2!} + \frac{(Nt)^3}{3!} + ...$$

If we calculate the different powers of N, *we can see that odd powers of* N *are equal to* N *and even powers of* N *are equal to the matrix* N^2 *given by*

$$N^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we edit the above $R(t)$ *equation, we get*

$$R(t) = I + (t + \frac{t^3}{3!} + \frac{t^5}{5!} + ...)N + (\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + ...)N^2$$

and

$$R(t) = I + (\sinh t)\, N + (-1 + \cosh t)N^2.$$

Thus, $R(t)$ *has the following matrix representation:*

$$R(t) = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and it is a geodesic in $SO(2,1)$. *This rotation matrix is used to determine the coordinates of the final position vector of a point* p *of* S_2^2 *given the initial position vector. Since the geodesic* $R(t) = \exp(Nt)$ *provides the equations* $R(0) = I$ *and* $\dot{R}(0) = N$, $R(t)$ *can be a stationary motion.*

Let us consider the kinetic energy of the rotating particle without any external force on the timelike geodesic circle given by $\gamma(t) = (\cosh t, \sinh t, 0)$ *of the unit 2-sphere* S_2^2 *in semi-Euclidean space* E_2^3. *The kinetic energy of this particle is equal to*

$$T = \frac{1}{2} < n, n >,$$

where n *is the angular velocity vector of this particle and* $<,>$ *is the standard inner product in* E_2^3*. If we calculate* T*, we can see that the kinetic energy of this particle is constant and equal to* -1 *regardless of* t*. Now we have shown that the kinetic energy of the stationary motion* $R(t)$ *in semi-Riemann space* $(SO(2,1),h)$ *is constant. The kinetic energy of this particle moving along the curve* $R(t)$ *is equal to*

$$T = h(N,N) = -\frac{1}{2}Trace\{N.N\} = -1,$$

where N *is equal to* $\dot{R}(0)$*. Thus, the kinetic energy of this particle moving along the curve* $R(t)$ *in* $SO(2,1)$ *is constant and equal to* -1*. Namely, the kinetic energy of this particle at every stage of its motion does not change.*

5. Geodesics of SO(2,1)

In this section, the expression of the orthonormal basis of $T_1S_2^2$ with respect to the Euler rotation matrices is obtained. Then a differentiable map between semi-Riemann spaces $(T_1S_2^2, g^S)$ and $(SO(2,1),h)$ is defined and it is shown that the line element of $(T_1S_2^2, g^S)$ is equal to the line element of $(SO(2,1),h)$. Furthermore, the second-order derivative of a rotation matrix R of $SO(2,1)$ is obtained. Finally, the system of differential equations giving geodesics of $(SO(2,1),h)$ is obtained and the equality of the systems of differential equations giving the geodesics $(SO(2,1),h)$ and $(T_1S_2^2, g^S)$ is found.

Let us take any point e_1 on the unit 2-sphere S_2^2 and the unit tangent vectors f_2 and f_3 passing from the point e_1. The local coordinate expressions of e_1, f_2, f_3 in 3-dimensional semi-Euclidean space are given by

$$\begin{aligned}
e_1(a,\theta) &= (\cosh a \cosh\theta, \cosh a \sinh\theta, \sinh a),\\
f_2(a,\theta) &= (\sinh a \cosh\theta, \sinh a \sinh\theta, \cosh a),\\
f_3(a,\theta) &= (\sinh\theta, \cosh\theta, 0),
\end{aligned}$$

with respect to geodesic polar coordinates a, θ. The unit tangent vectors f_2, f_3 belong to the tangent vector space of the unit 2-sphere at point e_1. This tangent vector space is denoted by $T_{e_1}S_2^2$. Let e_2 be both any tangent vector of $T_{e_1}S_2^2$ and an element of the tangent sphere bundle $\pi^{-1}\{e_1\} = T_{e_1}S_2^2 \subset T_1S_2^2$. To determine the position of e_2, we use the new coordinate denoted by ω. Let ω be any angle between f_2 and e_2. Thus, new basis vectors e_2 and e_3 of $T_{e_1}S_2^2$ are the following local coordinate expressions:

$$\begin{gathered}
e_2(a,\theta,\omega) = \cos\omega f_2 + \sin\omega f_3,\\
e_3(a,\theta,\omega) = -\sin\omega f_2 + \cos\omega f_3,
\end{gathered}$$

with respect to basis $\{f_2, f_3\}$ of $T_{e_1}S_2^2$ and

$$\begin{aligned}
e_2 &= (\cos\omega \sinh a \cosh\theta + \sin\omega \sinh\theta, \cos\omega \sinh a \sinh\theta - \sin\omega \cosh\theta, \cos\omega \cosh a),\\
e_3 &= (-\sin\omega \sinh a \cosh\theta + \cos\omega \sinh\theta, -\sin\omega \sinh a \sinh\theta - \cos\omega \cosh\theta, -\sin\omega \cosh a),
\end{aligned}$$

with respect to the standard orthonormal basis of E_2^3. Thus, e_1, e_2, e_3 are orthonormal basis elements of $T_1S_2^2$.

Theorem 5.1 *The matrix* $R = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}^T$ *is an element of* $SO(2,1)$ *where* R *is given by*

$$\begin{pmatrix} \cosh\theta\cosh a & \sinh\theta\cosh a & \sinh a \\ \sinh\theta\sin\omega + \cosh\theta\cos\omega\sinh a & \cosh\theta\sin\omega + \sinh\theta\cos\omega\sinh a & \cos\omega\cosh a \\ \sinh\theta\cos\omega - \cosh\theta\sin\omega\sinh a & \cosh\theta\cos\omega - \sinh\theta\sin\omega\sinh a & -\sin\omega\cosh a \end{pmatrix}.$$

Proof It has been straightforwardly seen that the rotation matrix R provides the equality $R^T\chi R = \chi$, where χ is defined as in Definition 2.2. Therefore, R is an element of $SO(2,1)$. □

Theorem 5.2 *The representation via the Euler rotation matrices of* $R = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}^T$ *is equal to the multiplication of the following rotation matrices:*

$$R = R_x(-\omega)R_z(-a)R_y(\theta)Q,$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and $R_x(-\omega), R_y(\theta)$ $R_z(-a)$ *and* Q *are elements of* $SO(2,1)$ *and* $R_z(-a)$ *describes the rotation matrix with respect to the* z *axis by the hyperbolic angle* $-a$ *.*

Proof $R_x(-\omega), R_y(\theta)$, $R_z(-a)$ and Q will be elements of $SO(2,1)$ since these matrices provide $R^T\chi R = \chi$. If we multiply the following matrices by Q:

$$R_x(-\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix},$$

$$R_z(-a) = \begin{pmatrix} \cosh a & \sinh a & 0 \\ \sinh a & \cosh a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_y(\theta) = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix},$$

we can see that the theorem is correct easily. □

Theorem 5.3 *The tangent vector* $\dot{R}$ *at point* I *of* $SO(2,1)$ *is a skew-symmetric matrix of* $so(2,1)$ *as follows:*

$$dR = \begin{pmatrix} de_1 \\ de_2 \\ de_3 \end{pmatrix} = \dot{R}R = \begin{pmatrix} 0 & \eta_3 & -\eta_2 \\ -\eta_3 & 0 & \eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

where η_1, η_2, η_3 *are given by*

$$\begin{aligned} \eta_1 &= d\omega + \sinh a d\theta, \\ \eta_2 &= \sin\omega da - \cos\omega\cosh a d\theta, \\ \eta_3 &= \cos\omega da + \sin\omega\cosh a d\theta. \end{aligned} \tag{5.1}$$

Proof If we use the covariant derivative of basis vectors of $T_1S_2^2$ in Theorem 3.4, the correctness of the claim of this theorem is seen easily. □

Theorem 5.4 *The derivative of the map* ψ *from* $T_1S_2^2$ *to* $SO(2,1)$ *is given by*

$$\psi_* = \,\widehat{}\circ f_*,$$

where $f_* : T_yT_1S_2^2 \to E_2^3$ *has the following matrix representation:*

$$f_* = \begin{pmatrix} \cos\omega & \cosh a\sin\omega & 0 \\ -\sin\omega & \cosh a\cos\omega & 0 \\ 0 & \cosh a\tanh a & 1 \end{pmatrix},$$

and the map $\widehat{} : E_2^3 \to so(2,1)$ *is defined by*

$$\widehat{r} = \begin{pmatrix} 0 & -r_3 & r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix},$$

for a vector $r = (r_1, r_2, r_3)$ *in* E_2^3.

Proof Since ψ is the map from $T_1S_2^2$ to $SO(2,1)$, ψ_* can be the map from $T_yT_1S_2^2$ to $T_{\psi(y)=I}SO(2,1)$ defined by

$$\psi_*(\xi_1) = b_1 \ \psi_*(\xi_2) = b_2, \ \psi_*(\xi_3) = b_3.$$

If we calculate $f_*(\xi_3)$, $f_*(\xi_2)$, $f_*(\xi_1)$, we should find the unit vector $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ of E_2^3, respectively. Then it can be easily seen that $\widehat{i} = b_1$, $\widehat{j} = b_2$, $\widehat{k} = b_3$. □

Theorem 5.5 *The line element* $d\rho$ *between two infinitely close points in* $(SO(2,1),h)$ *is equal to the line element* $d\sigma$ *between two infinitely close points in* $\left(T_1S_2^2, g^S\right)$.

Proof Since the other point that is infinitesimal close to $R = (e_1 \ \ e_2 \ \ e_3)^T$ is obtained by the matrix product $\dot{R}R$, the line element of $SO(2,1)$ determines the image of $\dot{R}$ under h:

$$\begin{aligned} d\rho^2 &= h\left(\dot{R},\dot{R}\right) = -\frac{1}{2}Trace(\dot{R}.\dot{R}) \\ &= \eta_1^2 - \eta_2^2 - \eta_3^2 \\ &= -(da)^2 - (d\theta)^2 + 2\sinh a d\theta d\omega + (d\omega)^2. \end{aligned}$$

The value of $d\rho^2$ is equal to the value of $d\sigma^2$ obtained by (3.8) Therefore, it is seen that the claim of theorem is correct. □

To find geodesic equations of $SO(2,1)$, let us calculate $d^2R = (\ d^2e_1 \quad d^2e_2 \quad d^2e_3\ )^T$.

Theorem 5.6 *The second-order derivation of the element* $R = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}^T$ *of* $SO(2,1)$ *is given by*

$$d^2R = \begin{pmatrix} d^2e_1 \\ d^2e_2 \\ d^2e_3 \end{pmatrix} = \ddot{R}R,$$

where

$$\begin{aligned} d^2e_1 &= \left((da)^2 + \cosh^2 a\,(d\theta)^2\right) e_1 + \\ &+ \left(\begin{array}{c} \cos\omega d^2a + 2\sinh a\sin\omega da d\theta + \\ + \cosh a\sin\omega d^2\theta - \sinh a\cosh a\cos\omega\,(d\theta)^2 \end{array}\right) e_2 + \\ &+ \left(\begin{array}{c} -\sin\omega d^2a + 2\sinh a\cos\omega da d\theta + \\ \cosh a\cos\omega d^2\theta + \sinh a\cosh a\sin\omega\,(d\theta)^2 \end{array}\right) e_3, \end{aligned}$$

$$\begin{aligned} d^2e_2 &= \left(\begin{array}{c} \cos\omega d^2a - 2\sin\omega da d\omega + \cosh a\sin\omega d^2\theta + \\ +2\cosh a\cos\omega d\theta d\omega + \sinh a\cosh a\cos\omega\,(d\theta)^2 \end{array}\right) e_1 + \\ &+ \left(\begin{array}{c} \cos^2\omega\,(da)^2 + 2\cosh a\sin\omega\cos\omega da d\theta - \sinh^2 a(d\theta)^2 + \\ + \cosh^2 a\sin^2\omega\,(d\theta)^2 - 2\sinh a d\theta d\omega - (d\omega)^2 \end{array}\right) e_2 + \\ &+ \left(\begin{array}{c} -\sin\omega\cos\omega\,(da)^2 + \sinh a d^2\theta + 2\cosh a\cos^2\omega da d\theta + \\ + \cosh^2 a\sin\omega\cos\omega\,(d\theta)^2 + d^2\omega \end{array}\right) e_3, \end{aligned}$$

$$\begin{aligned} d^2e_3 &= \left(\begin{array}{c} -\sin\omega d^2a - 2\cos\omega da d\omega + \cosh a\cos\omega d^2\theta - \\ -2\cosh a\sin\omega d\theta d\omega - \sinh a\cosh a\sin\omega\,(d\theta)^2 \end{array}\right) e_1 + \\ &+ \left(\begin{array}{c} -\sin\omega\cos\omega\,(da)^2 - \sinh a d^2\theta - 2\cosh a\sin^2\omega da d\theta + \\ + \cosh^2 a\sin\omega\cos\omega\,(d\theta)^2 - d^2\omega \end{array}\right) e_2 + \\ &+ \left(\begin{array}{c} \sin^2\omega\,(da)^2 - 2\cosh a\sin\omega\cos\omega da d\theta + \cosh^2 a\cos^2\omega\,(d\theta)^2 \\ -2\sinh a d\theta d\omega - \sinh^2 a\,(d\theta)^2 - (d\omega)^2 \end{array}\right) e_3. \end{aligned}$$

Proof Taking the partial differentials of $de_1,\ de_2,\ de_3$ given by Theorem 3.6 with respect to the variables a, $\theta,\ \omega$, we can obtain the second-order derivation of R straightforwardly. □

Theorem 5.7 *The system of differential equations giving the geodesics of* $SO(2,1)$ *is equal to the system of differential equations giving geodesics of* $(T_1S_1^2, g^S)$.

Proof The curve $\gamma(t)$ in $SO(2,1)$ is geodesic if and only if $\dot{T} = 2h\left(\dot{R},\ddot{R}\right) = -Trace\left(\dot{R},\ddot{R}\right)$ is equal to zero. If we calculate $\dot{T}$, we can find the following equation:

$$\begin{aligned} &\left(d^2a + \cosh a d\theta d\omega\right) da + \left(d^2\theta + \sinh a d^2\omega - \cosh a da d\omega\right) d\theta + \\ &\qquad + \left(d^2\omega + \sinh a d^2\theta + \cosh a da d\theta\right) d\omega = 0, \end{aligned}$$

where all components are equal to zero. Namely,

$$\begin{aligned} d^2a + \cosh a d\theta d\omega &= 0, & (5.2)\\ d^2\theta - \sinh a d^2\omega - \cosh a d a d\omega &= 0, & (5.3)\\ d^2\omega + \sinh a d^2\theta + \cosh a d a d\theta &= 0. & (5.4) \end{aligned}$$

If we multiply the equation in (5.3) by $\sinh a$ and the value of $\sinh a d^2\theta$ is put into equation (5.4), we get

$$d^2\omega + \sec h a d a d\theta + \tanh a d a d\omega = 0. \tag{5.5}$$

If we multiply the equation in (5.4) by $\sinh a$ and the value of $\sinh a d^2\omega$ is put into equation (5.3), we get

$$d^2\theta - \sec h a d a d\omega + \tanh a d a d\theta = 0. \tag{5.6}$$

Thus, if we organize equations (5.2), (5.5), and (5.6), we get the following system of differential equations:

$$\begin{aligned} \ddot{a} + \cosh a\dot{\theta}\dot{\omega} &= 0,\\ \ddot{\theta} + \tanh a\dot{a}\dot{\theta} - \sec h a\dot{a}\dot{\omega} &= 0,\\ \ddot{\omega} + \sec h a\dot{a}\dot{\theta} + \tanh a\dot{a}\dot{\omega} &= 0, \end{aligned} \tag{5.7}$$

and this system of differential equations is equal to the system of differential equations of $(T_1S_2^2, g^S)$ given by (3.15). □

Let us consider some geodesics on the tangent sphere bundle $T_1S_2^2$ with respect to the particular solutions $(a = a(t), \theta = \theta(t), \omega = \omega(t))$ providing the above system of differential equations.

Example 5.8 *Some geodesics of $T_1S_2^2$ can be determined by the following particular solutions providing the system of differential equations given by (5.7):*
Case 1. $a = t, \theta = 0, \omega = \frac{\pi}{2}$*;*
Case 2. $a = 0$*,* $\theta = t, \omega = \frac{\pi}{4}$*;*
Case 3. $a = 0$*,* $\theta = 0, \omega = t$*.*
Let us examine the relationships among the geodesics of $SO(2,1)$, S_2^2, and $T_1S_2^2$ for these three different cases.

As e_1 given by the equation in (3.1) defines a point or a geodesic of S_2^2, the matrix whose column vector is equal to e_1, f_2, f_3 that are given by equations (3.1), (3.3) is a geodesic of $SO(2,1)$ and $(e_1; e_2)$ defines a geodesic of $T_1S_2^2$ where e_2 is given by equation (3.4).

Case 1. If we substitute $a = t$, $\theta = 0$ in equations (3.1), (3.3), we will obtain the values of e_1, f_3, and f_2 as follows:

$$\begin{array}{rccc} e_1 = & (\cosh t, & 0, & \sinh t),\\ f_3 = & (0, & 1, & 0),\\ f_2 = & (\sinh t & 0 & \cosh t), \end{array}$$

which correspond to the rotation matrix in $SO(2,1)$ around the y axis. This rotation matrix is a geodesic of $SO(2,1)$. Furthermore, e_2 is equal to f_3 for $\omega = \frac{\pi}{2}$. As $e_1 = (\cosh t, 0, \sinh t)$ draws the timelike geodesic circle of S^2_2 in E^3_2, $(e_1;e_2)=(\cosh t,0,\sinh t\ ;\ 0,1,0)$ draws a geodesic of $T_1S^2_2$ in $E^3_2\times E^3_2$. e_2 makes a constant angle $\omega=\frac{\pi}{2}$ with the unit tangent vector $f_2\in T_{e_1}S^2_2$ at point e_1 for the different values of $t\in[0,2\pi]$. Since the velocity vectors at each point of the curves e_1 and $(e_1;e_2)$ are perpendicular to the acceleration vectors at that point, these curves are geodesic curves in E^3_2 and $E^3_2\times E^3_2$, respectively. The curve $(e_1;e_2)$ obtained by the parallel translation of the vector e_2 along the timelike geodesic circle of S^2_2 is a horizontal geodesic curve on $T_1S^2_2$.

Case 2. If we substitute $a=0,\ \theta=t$ in equations (3.1), (3.3), we will obtain the values of e_1, f_3, and f_2 as follows:

$$\begin{array}{rccc} e_1= & (\cosh t, & \sinh t, & 0), \\ f_3= & (\sinh t, & \cosh t, & 0), \\ f_2= & (0 & 0 & 1), \end{array}$$

which correspond to the rotation matrix in $SO(2,1)$ around the z axis. This rotation matrix is a geodesic of $SO(2,1)$. Furthermore, e_2 is equal to $\frac{1}{\sqrt{2}}(f_2+f_3)$ for $\omega=\frac{\pi}{4}$. As $e_1=(\cosh t,\sinh t,0)$ draws the timelike geodesic circle of S^2_2 in E^3_2, $(e_1;e_2)=(\cosh t,0,\sinh t\ ;\ \frac{1}{\sqrt{2}}\sinh t,\frac{1}{\sqrt{2}}\cosh t,\frac{1}{\sqrt{2}})$ draws a geodesic of $T_1S^2_2$ in $E^3_2\times E^3_2$, and e_2 makes a constant angle $\omega=\frac{\pi}{4}$ with the unit tangent vector $f_2\in T_{e_1}S^2_2$ at point e_1 for the different values of $t\in[0,2\pi]$. Since the velocity vectors at each point of the curves e_1 and $(e_1;e_2)$ are perpendicular to the acceleration vectors at that point, these curves are geodesic curves in E^3_2 and $E^3_2\times E^3_2$, respectively. The curve $(e_1;e_2)$ obtained by the parallel translation of the vector e_2 along the timelike geodesic circle of S^2_2 is a horizontal geodesic curve on $T_1S^2_2$.

Case 3. If we substitute $a=0,\ \theta=0$ in equations (3.1), (3.3), we will obtain the values of e_1, f_3, and f_2 as follows:

$$\begin{array}{rccc} e_1= & (1, & 0, & 0), \\ f_3= & (0, & 1, & 0), \\ f_2= & (0 & 0 & 1), \end{array}$$

which correspond to the unit matrix in $SO(2,1)$. This unit matrix is an identity element of $SO(2,1)$. Furthermore, e_2 is equal to $((\cos t)\,f_2+(\sin t)\,f_3)$ for $\omega=t$. As $e_1=(1,0,0)$ is a point of S^2_2 in E^3_2, $(e_1;e_2)=(1,0,0\ ;\ 0,\sin t,\cos t)$ draws a geodesic of $T_1S^2_2$ in $E^3_2\times E^3_2$. e_2 makes an angle $\omega=t\in[0,2\pi]$ with the unit tangent vector $f_2\in T_{e_1}S^2_2$ at point e_1. Since the velocity vector at each point of the curve $(e_1;e_2)$ is perpendicular to the acceleration vector at that point, this curve is a geodesic curve in $E^3_2\times E^3_2$ and $(e_1;e_2)$ defines the unit circle lying on the tangent vector space $T_{e_1}S^2_2$ at point e_1 of S^2_2, and it is a vertical geodesic curve on $T_1S^2_2$.

6. Conclusion

In this research, we propose a new insight into geodesics on the tangent sphere bundle of unit 2-spheres via geodesics of special orthogonal groups in semi-Euclidean space. By helping the diffeomorphism and isometry properties between the tangent sphere bundle and special orthogonal group, we have examined the relations between the differential geometric objects of these semi-Riemann spaces.

We have considered the vector and the matrix representation of a spherical rotation in semi-Euclidean space, the special orthogonal group and its tangent vector space, and the semi-Riemann structure on the special orthogonal group.

Then we have considered the manifold structure and geodesics of the tangent sphere bundle of the unit 2-sphere in semi-Euclidean 3-space with index two, inspired by [2].

Furthermore, we were interested in the geometrical and dynamical interpretation of rotational motion of a particle around the origin of the unit 2-sphere. We have seen that the stationary motion of a particle under inertia on the unit 2-sphere produces a geodesic of a special orthogonal group in semi-Euclidean space.

Finally, we have proved the equality of line elements and the systems of differential equations giving geodesics of the tangent sphere bundle and special orthogonal group.

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