GENERALIZED FRACTIONAL MAXIMAL OPERATOR ON GENERALIZED LOCAL MORREY SPACES

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ABSTRACT. In this paper, we study the boundedness of generalized fractional maximal operator M_ρ on generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$ and generalized Morrey spaces $M_{p,\varphi}$, including weak estimates. Firstly, we prove the Spanne type boundedness of M_ρ from the space $LM_{p,\varphi_1}^{\{x_0\}}$ to another $LM_{q,\varphi_2}^{\{x_0\}}$, $1 < p < q < \infty$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to the weak space $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$ and $1 < q < \infty$. Secondly, we prove the Adams type boundedness of M_ρ from the space $M_{p,\varphi^{\frac{1}{p}}}$ to another $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$ and $1 < q < \infty$. In all cases the conditions for the boundedness of M_ρ are given in terms of supremal-type integral inequalities on $(\varphi_1, \varphi_2, \rho)$ and (φ, ρ), which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ and $\varphi(x,r)$ in r.

1. INTRODUCTION

The classical Morrey spaces $M_{p,\lambda}$ were first introduced by Morrey in [21] to study the local behavior of solutions to second order elliptic partial differential equations. The generalized Morrey spaces $M_{p,\varphi}$ are obtained by replacing r^λ in the definition of the Morrey space. During the last decades various classical operators, such as maximal, singular and potential operators were widely investigated in both in classical, generalized Morrey spaces and generalized local Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operators on these spaces, we refer the readers to [1, 9, 15, 16, 20, 22].

For a measurable function $\rho : (0, \infty) \to (0, \infty)$ the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ are defined by

$$M_\rho f(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)| dy,$$

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy$$

for any suitable function f on $\mathbb{R}^n$. If $\rho(t) \equiv t^\alpha$, then $M_\alpha \equiv M_{t^\alpha}$ is the fractional maximal operator and $I_\alpha \equiv I_{t^\alpha}$ is the Riesz potential.

Spanne [24] and Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results can be summarized as follows.

Theorem A. *(Spanne, but published by Peetre [24]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.*

Theorem B. *(Adams [1]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Nakai [22] proved the boundedness of the operators I_ρ and M_ρ from the generalized Morrey spaces M_{p,φ_1} to the spaces M_{q,φ_2} for suitable functions φ_1 and φ_2. The boundedness of M_ρ and I_ρ from the generalized Morrey spaces M_{p,φ_1} to the spaces M_{q,φ_2} is studied by Nakai [23], Eridani [10], Gunawan [18], Eridani, Gunawan and Nakai [12], Sawano, Sugano, Tanaka [25], Eridani, Gunawan, Nakai, Sawano [11], Guliyev, Ismayilova, Kucukaslan, Serbetci [17], Kucukaslan, Hasanov, Aykol [19].

In particular, the following statement containing both Theorem A and Theorem B was proved in [3, 4].

Theorem C. *([3, 4]) Let $1 \le p < q < \infty$, $0 < \lambda, \mu < n$ and*

$$0 < \alpha = \frac{n-\lambda}{p} - \frac{n-\mu}{q} < \frac{n}{p}.$$

Then, for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$, and, for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.

In [3, 4] it was also proved that, under the assumptions of Theorem C, the operator I_α, for $p > 1$, is bounded from the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ to $LM_{q,\mu}^{\{x_0\}}$, and, for $p = 1$ from $LM_{1,\lambda}^{\{x_0\}}$ to the weak local Morrey space $WLM_{q,\mu}^{\{x_0\}}$.

Since, for some $c > 0$, $\left(M_\alpha f\right)(x) \le c\left(I_\alpha(|f|)\right)(x)$, $x \in \mathbb{R}^n$, it follows that in Theorems A, B, C the operator I_α can be replaced by the operator M_α (including

also the case $p = q$). For the operator M_α Theorem C was, in fact, earlier proved in [5, 6].

Guliyev [14] proved the Spanne and Adams type boundedness of I_α from the spaces $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$ without any assumption on monotonicity of φ_1, φ_2. Paper [7] should be mentioned where for $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ necessary and sufficient conditions of φ_1 and φ_2 are obtained. In [17], by using the method given in [13] the Spanne and Adams type boundedness of the operator I_ρ from the generalized local Morrey space $LM_{p,\varphi_1}^{\{x_0\}}$ to another one $LM_{q,\varphi_2}^{\{x_0\}}$ were proved.

The main goal of this paper is to show that the boundedness of the generalized fractional maximal operator M_ρ in generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$ and generalized Morrey spaces $M_{p,\varphi}$ can be obtained under weaker assumptions on ρ, namely in terms of the so-called supremal operators. More precisely, we find sufficient conditions, in supremal terms, on the functions $(\varphi_1, \varphi_2, \rho)$ which ensure the boundedness of the operator M_ρ from one generalized local Morrey space $LM_{p,\varphi_1}^{\{x_0\}}$ to another $LM_{q,\varphi_2}^{\{x_0\}}$ for $1 < p < q < \infty$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to the weak space $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$ and $1 < q < \infty$. We also find conditions on the pair (φ, ρ) which ensure the Adams type boundedness of M_ρ from the spaces $M_{p,\varphi^{\frac{1}{p}}}$ to another $M_{q,\varphi^{\frac{1}{q}}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to the weak space $WM_{q,\varphi^{\frac{1}{q}}}$ for $p = 1$ and $1 < q < \infty$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. PRELIMINARIES

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. Therefore $|B(x, r)| = w_n r^n$, where w_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Definition 2.1. *Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$, the weak Morrey space $WM_{p,\lambda}$ respectively, under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \ WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local (central) Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} |B(0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized local (central) Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} |B(0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 2.3. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0 + \cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0 + \cdot)\|_{WLM_{p,\varphi}} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \ WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.4. *Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We define a cone $\mathbb{A}$ by the set of the functions $\varphi \in \mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and such that $\lim_{t\to 0+} \varphi(t) = 0$, briefly*

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Definition 2.5. [8] *Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by*

$$(\overline{S}_u g)(r) := \|u(t)g(t)\|_{L_\infty(r,\infty)}, \ r \in (0,\infty).$$

Let v be a non-negative measurable function on $(0, \infty)$. We denote by $L_{\infty,v}(0, \infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)g(t)$$

and $L_{\infty}(0, \infty) \equiv L_{\infty,1}(0, \infty)$. The following lemma is proved analogously to Lemma 5.2 in [8].

Lemma 2.1. [8] *Let v_1 and v_2 be weights and $0 < \|v_1\|_{L_{\infty}(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0, \infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0, \infty)$ to $L_{\infty,v_2}(0, \infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|_{L_{\infty}(\cdot,\infty)}^{-1} \right) \right\|_{L_{\infty}(0,\infty)} < \infty.$$

The following lemma was proved in [17].

Lemma 2.2. [17] *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_{\infty}(t,\infty)} < \infty$ for any $t > 0$. Then the identity operator I is bounded from $L_{\infty,v_1}(0, \infty)$ to $L_{\infty,v_2}(0, \infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \left(\|v_1\|_{L_{\infty}(\cdot,\infty)}^{-1} \right) \right\|_{L_{\infty}(0,\infty)} < \infty.$$

3. SPANNE TYPE RESULT FOR THE OPERATOR M_ρ IN THE SPACES $LM_{p,\varphi}^{\{x_0\}}$

We assume that

$$\sup_{1 \le t < \infty} \frac{\rho(t)}{t^n} < \infty, \tag{3.1}$$

so that the fractional maximal functions $M_\rho f$ are well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x) = \frac{\chi_{\mathbb{R}^n \setminus B(0,1)}(x)}{|x|^{2n}}.$$

In addition, we shall also assume that ρ satisfies the growth condition: there exist constants $C_1 > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r < s \le 2r} \frac{\rho(s)}{s^n} \le C_1 \sup_{k_1 r < t < k_2 r} \frac{\rho(t)}{t^n}, \quad r > 0. \tag{3.2}$$

This condition is weaker than the usual doubling condition for the function $\frac{\rho(t)}{t^n}$: there exists a constant $C_2 > 0$ such that

$$\frac{1}{C_2} \frac{\rho(t)}{t^n} \le \frac{\rho(r)}{r^n} \le C_2 \frac{\rho(t)}{t^n},$$

whenever r and t satisfy $r, t > 0$ and $\frac{1}{2} \le \frac{r}{t} \le 2$.

Remark 3.1. *Typical examples of $\rho(t)$ that we envisage are, for $0 < \alpha < n$*

$$\rho(t) \equiv \begin{cases} t^\alpha \log(e/t), & 0 < t \le 1 \\ \frac{t^\alpha}{\log(et)}, & 1 \le t < \infty \end{cases}$$

and, for $c > 0$

$$\rho(t) \equiv \begin{cases} t^\alpha, & 0 < t \le 1 \\ e^c e^{-ct^2}, & 1 \le t < \infty. \end{cases}$$

The second one is used to control the Bessel potential (see also [26]).

The boundedness of the operator I_ρ in the spaces $L_p(\mathbb{R}^n)$ can be found in [11]. Let $\frac{\rho(t)}{t^n}$ be almost decreasing, that is, there exists a constant C such that $\frac{\rho(t)}{t^n} \le C \frac{\rho(s)}{s^n}$ for $s < t$. In this case we get

$$\begin{aligned} M_\rho f(x) &= \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(y)| dy \\ &\lesssim \sup_{t>0} \int_{B(x,t)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \\ &= \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy = I_\rho(|f|)(x). \end{aligned}$$

For proving our main results, we need the following estimate.

Lemma 3.3. *If $B_0 := B(x_0, r_0) \subset B(x, r)$ and ρ satisfies the doubling condition. Then $\rho(r_0) \lesssim M_\rho \chi_{B_0}(x)$ for every $x \in B_0$.*

Proof. Let ρ satisfy the doubling condition, then

$$\mathcal{M}_\rho f(x) \lesssim M_\rho f(x), \tag{3.3}$$

where $\mathcal{M}_\rho(f)(x) = \sup_{B \ni x} \frac{\rho(r_B)}{|B|} \int_B |f(y)| dy$ and r_B is the center of the ball B.

Now let $x \in B_0$. By using (3.3), we get

$$\begin{aligned} M_\rho \chi_{B_0}(x) &\gtrsim \mathcal{M}_\rho \chi_{B_0}(x) = \sup_{B \ni x} \frac{\rho(r_B)}{|B|} |B \cap B_0| \\ &\gtrsim \frac{\rho(r_0)}{|B_0|} |B_0 \cap B_0| = \rho(r_0). \end{aligned}$$

$\square$

The following lemma is valid.

Lemma 3.4. *Let $1 \le p < q < \infty$.*
   (1) The condition

$$\rho(r) \le C r^{\frac{n}{p} - \frac{n}{q}} \tag{3.4}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$. Moreover, if $p > 1$, then the condition (3.4) is sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

(2) *If ρ satisfies the doubling condition, then the condition (3.4) is necessary for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ and from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$.*

(3) *If ρ satisfies the doubling condition and the supremal regularity condition*

$$\sup_{r < t < \infty} \rho(t)\, t^{-\frac{n}{p}} \leq C\rho(r) r^{-\frac{n}{p}}$$

holds for all $r > 0$, where $C > 0$ does not depend on r, then the condition (3.4) is necessary and sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$. Moreover, if $p > 1$, then the condition (3.4) is necessary and sufficient for the boundedness of M_ρ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

Proof. (1) Suppose ρ satisfies the condition (3.4). Then

$$M_\rho f(x) \lesssim M_{\frac{n}{p}-\frac{n}{q}} f(x). \tag{3.5}$$

Since the operator $M_{\frac{n}{p}-\frac{n}{q}}$ is bounded from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ and for $p > 1$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, then from (3.5) we get the statement (1).

(2) Now we shall prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.3, we have $\rho(r_0) \lesssim M_\rho \chi_{B_0}(x)$. Therefore, we have

$$\rho(r_0) \lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{WL_q(B_0)} \lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{WL_q(\mathbb{R}^n)}$$
$$\lesssim r_0^{-\frac{n}{q}} \|\chi_{B_0}\|_{L_p(\mathbb{R}^n)} \lesssim r_0^{\frac{n}{p}-\frac{n}{q}}$$

and for $p > 1$

$$\rho(r_0) \lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{L_q(B_0)} \lesssim r_0^{-\frac{n}{q}} \|M_\rho \chi_{B_0}\|_{L_q(\mathbb{R}^n)}$$
$$\lesssim r_0^{-\frac{n}{q}} \|\chi_{B_0}\|_{L_p(\mathbb{R}^n)} \lesssim r_0^{\frac{n}{p}-\frac{n}{q}}$$

holds for every $r_0 > 0$, hence the proof of statement (2) is completed.

(3) From the first and second statements the third statement of the lemma follows. $\qquad\square$

The following lemma is valid.

Lemma 3.5. *Let $1 \leq p < q < \infty$ and let $\rho(t)$ satisfy the conditions (3.1), (3.2) and (3.4). Then the inequality*

$$\|M_\rho f\|_{WL_q(B(x_0,r))} \lesssim \|f\|_{L_p(B(x_0,2r))} + r^{\frac{n}{q}} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}$$

holds for any ball $B(x_0, r)$ and for all $f_p^{\mathrm{loc}}(\mathbb{R}^n)$.

If $p > 1$, then the inequality

$$\|M_\rho f\|_{L_q(B(x_0,r))} \lesssim \|f\|_{L_p(B(x_0,2r))} + r^{\frac{n}{q}} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}} \tag{3.6}$$

holds for any ball $B(x_0, r)$ and for all $f_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 \leq p < q < \infty$ and let $\rho(t)$ satisfy the conditions (3.1), (3.2) and (3.4). For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\|M_\rho f\|_{WL_q(B)} \leq \|M_\rho f_1\|_{WL_q(B)} + \|M_\rho f_2\|_{WL_q(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $M_\rho f_1 \in WL_q(\mathbb{R}^n)$ and by Lemma 3.4 M_ρ is bounded from $L_p(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$. Thus it follows that

$$\|M_\rho f_1\|_{WL_q(B)} \leq \|M_\rho f_1\|_{WL_q(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

Let x be an arbitrary point from B. If $B(x,t) \cap {}^\complement(2B) \neq \emptyset$, then $t > r$. Indeed, if $y \in B(x,t) \cap {}^\complement(2B)$, then $t > |x - y| \geq |x_0 - y| - |x_0 - x| > 2r - r = r$.

On the other hand, $B(x,t) \cap {}^\complement(2B) \subset B(x_0, 2t)$. Indeed, $y \in B(x,t) \cap {}^\complement(2B)$, then we get $|x_0 - y| \leq |x - y| + |x_0 - x| < t + r < 2t$.

Hence

$$M_\rho f_2(x) = \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(x,t)\cap{}^\complement(2B)} |f(y)|dy$$

$$\lesssim \sup_{t>r} \frac{\rho(2t)}{(2t)^n} \int_{B(x_0,2t)} |f(y)|dy$$

$$= \sup_{t>2r} \frac{\rho(t)}{t^n} \int_{B(x_0,t)} |f(y)|dy.$$

Therefore, for all $x \in B$ we have

$$M_\rho f_2(x) \lesssim \sup_{t>2r} \frac{\rho(t)}{t^n} \int_{B(x_0,t)} |f(y)|dy. \tag{3.7}$$

Thus

$$\|M_\rho f\|_{WL_q(B)} \leq \|M_\rho f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{q}} \sup_{t>2r} \|f\|_{L_1(B(x_0,t))} \frac{\rho(t)}{t^n}$$

$$\lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{q}} \sup_{t>2r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}. \tag{3.8}$$

Let $p > 1$. From the (p,q) boundedness of M_ρ and (3.4) it follows that:

$$\|M_\rho f_1\|_{L_q(B)} \leq \|M_\rho f_1\|_{L_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_p(\mathbb{R}^n)} = \|f\|_{L_p(2B)}. \tag{3.9}$$

Then by (3.8) and (3.9) we get the inequality (3.6). $\qquad\square$

The following theorem is one of the main results of the paper in which we get the Spanne type boundedness of the generalized fractional maximal operator M_ρ in the generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$.

Theorem 3.1. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < q < \infty$, and let the function ρ satisfy the conditions (3.1), (3.2) and (3.4). Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)s^{\frac{n}{p}} \le C\,\varphi_2\big(x_0, \frac{t}{2}\big)\,t^{\frac{n}{q}}, \tag{3.10}$$

$$\sup_{t>r}\Big(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)s^{\frac{n}{p}}\Big)\frac{\rho(t)}{t^{\frac{n}{p}}} \le C\,\varphi_2(x_0, r),$$

where C does not depend on x_0 and r. Then the operator M_ρ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ and for $p > 1$ from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$. Moreover,

$$\|M_\rho f\|_{WLM_{q,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}},$$

and for $p > 1$

$$\|M_\rho f\|_{LM_{q,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}.$$

Proof. Let the function ρ satisfy the conditions (3.1), (3.2), (3.4), and also (φ_1, φ_2) satisfy the conditions (3.10) and (3.11). By Lemmas 2.1, 2.2 and 3.5 we have

$$\|M_\rho f\|_{WLM_{q,\varphi_2}^{\{x_0\}}} \lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-\frac{n}{q}} \|f\|_{L_p(B(x_0, 2r))}$$

$$+ \sup_{r>0} \varphi_2(x_0, r)^{-1} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}$$

$$\approx \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0, r))} = \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}$$

and for $p > 1$

$$\|M_\rho f\|_{LM_{q,\varphi_2}^{\{x_0\}}} \lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-\frac{n}{q}} \|f\|_{L_p(B(x_0, 2r))}$$

$$+ \sup_{r>0} \varphi_2(x_0, r)^{-1} \sup_{t>r} \|f\|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}$$

$$\approx \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0, r))} = \|f\|_{LM_{p,\varphi_1}^{\{x_0\}}}.$$

$\square$

In the following corollary we get the boundedness of the generalized fractional maximal operator M_ρ on generalized Morrey spaces $M_{p,\varphi}$.

Corollary 3.1. *Let $1 \le p < q < \infty$, the function ρ satisfy the conditions (3.1), (3.2) and (3.4). Let also (φ_1, φ_2) satisfy the following conditions*

$$\operatorname*{ess\,inf}_{r<t<\infty} \varphi_1(x, t)t^{\frac{n}{p}} \le C\,\varphi_2\big(x, \frac{r}{2}\big)\,r^{\frac{n}{q}},$$

$$\sup_{t>r}\Big(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s)s^{\frac{n}{p}}\Big)\frac{\rho(t)}{t^{\frac{n}{p}}} \le C\,\varphi_2(x, r),$$

where C does not depend on x and r. Then the operator M_ρ is bounded from M_{p,φ_1} to WM_{q,φ_2} and for $p > 1$ from M_{p,φ_1} to M_{q,φ_2}.

In the case $\rho(t) = t^\alpha$ from Theorem 3.1 we get new Spanne type result for fractional maximal operator M_α on generalized local Morrey spaces.

Corollary 3.2. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 \le p < q < \infty$ and $1/p - 1/q = \alpha/n$. Let also (φ_1, φ_2) satisfy the condition*

$$\sup_{t>r} \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}} \right) t^{-\frac{n}{q}} \le C \varphi_2(x_0, r), \tag{3.11}$$

where C does not depend on r. Then the operator M_α is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$.

Also in the case $\rho(t) = t^\alpha$ and $\varphi(x,t) = t^{\frac{\lambda-n}{p}}$, $0 < \lambda < n$ from Theorem 3.1 we get local Morrey space variant of Theorem A.

Corollary 3.3. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\alpha = \frac{n}{p} - \frac{n}{q}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$, the operator M_α is bounded from $LM_{p,\lambda}^{\{x_0\}}$ to $LM_{q,\mu}^{\{x_0\}}$ and for $p = 1$, M_α is bounded from $LM_{1,\lambda}^{\{x_0\}}$ to $WLM_{q,\mu}^{\{x_0\}}$.*

Remark 3.2. *For this case $\alpha = \frac{n}{p} - \frac{n}{q}$ necessary and sufficient conditions for the boundedness of I_α from M_{p,φ_1} to M_{q,φ_2} are obtained in [4].*

4. ADAMS TYPE RESULT FOR THE OPERATOR M_ρ IN THE SPACES $M_{p,\varphi}$

The following theorem was proved in [2].

Theorem D. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} t^{-\frac{n}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s) \, s^{\frac{n}{p}} \le C \varphi_2(x, r),$$

where C does not depend on x and r. Then the operator M is bounded from M_{p,φ_1} to WM_{p,φ_2} and for $p > 1$, the operator M is bounded from M_{p,φ_1} to M_{p,φ_2}.

The following theorem is another main result of the paper, in which we get the Adams type boundedness of the generalized fractional maximal operator M_ρ in the generalized Morrey spaces $M_{p,\varphi}$.

Theorem 4.2. *Let $1 \le p < q < \infty$, $\frac{\rho(t)}{t^n}$ be almost decreasing, and let $\rho(t)$ satisfy the condition (3.2) and the inequality*

$$\int_0^{k_2 r} \frac{\rho(s)}{s} ds \le C\rho(r),$$

where k_2 is given by the condition (3.2) and C does not depend on $r > 0$. Let also $\varphi(x,t)$ satisfy the conditions

$$\sup_{r<t<\infty} t^{-n} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x, s) \, s^n \le C \varphi(x, r), \tag{4.1}$$

and

$$\rho(r)\varphi(x,r) + \sup_{t>r} \rho(t)\varphi(x,t) \leq C\varphi(x,r)^{\frac{p}{q}}, \tag{4.2}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator M_ρ is bounded from $M_{p,\varphi^{\frac{1}{p}}}$ to $WM_{q,\varphi^{\frac{1}{q}}}$ and for $p > 1$ from $M_{p,\varphi}$ to $M_{q,\varphi^{\frac{1}{q}}}$.

Proof. Let $x_0 \in \mathbb{R}^n$, $1 \leq p < q < \infty$ and $f \in M_{p,\varphi^{\frac{1}{p}}}$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Then we have

$$M_\rho f(x) \leq M_\rho f_1(x) + M_\rho f_2(x).$$

For $M_\rho f_1(y)$, $y \in B(x,r)$, following Hedberg's trick (see for instance [27], p. 354), we obtain

$$
\begin{aligned}
M_\rho f_1(y) &= \sup_{t>0} \frac{\rho(t)}{t^n} \int_{B(y,t)\cap B(x,2r)} |f(z)|dz \\
&\lesssim \sup_{t>0} \int_{B(y,t)\cap B(x,2r)} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)|dz \\
&\approx \sup_{t>0} \sum_{k=-\infty}^{0} \int_{B(y,t)\cap\left(B(x,2^{k+1}r)\setminus B(x,2^k r)\right)} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)|dz \\
&\lesssim \sup_{t>0} \sum_{k=-\infty}^{0} \int_{2^k k_1 r}^{2^k k_2 r} \frac{\rho(s)}{s^{n+1}} ds \int_{B(y,t)\cap B(x,2^{k+1}r)} |f(z)|dz \\
&\approx Mf(x) \sup_{t>0} \sum_{k=-\infty}^{0} \int_{2^k k_1 r}^{2^k k_2 r} \frac{\rho(s)}{s} ds \\
&= Mf(x) \int_0^{k_2 r} \frac{\rho(s)}{s} ds \lesssim Mf(x)\rho(r). \tag{4.3}
\end{aligned}
$$

For $M_\rho f_2(y)$, $y \in B(x,r)$ from (3.7) we have

$$
\begin{aligned}
M_\rho f_2(y) &\lesssim \sup_{t>2r} \frac{\rho(t)}{t^n} \int_{B(x,t)} |f(z)|dz \\
&\lesssim \sup_{t>2r} \|f\|_{L_p(B(x,t))} \frac{\rho(t)}{t^{\frac{n}{p}}}. \tag{4.4}
\end{aligned}
$$

Then from condition (4.2) and inequalities (4.3), (4.4) for all $y \in B(x,r)$ we get

$$
\begin{aligned}
M_\rho f(y) &\lesssim \rho(r)\, Mf(x) + \sup_{t>r} \|f\|_{L_p(B(x,t))} \frac{\rho(t)}{t^{\frac{n}{p}}} \\
&\leq \rho(r)\, Mf(x) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{t>r} \varphi(x,t)\rho(t). \tag{4.5}
\end{aligned}
$$

Thus, by (4.2) and (4.5) we obtain

$$M_\rho f(y) \lesssim \min\left\{\varphi(x,t)^{\frac{p}{q}-1} M f(x), \varphi(x,t)^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}\right\}$$

$$\lesssim \sup_{s>0} \min\left\{s^{\frac{p}{q}-1} M f(x), s^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}\right\} = (Mf(x))^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}},$$

where we have used that the supremum is achieved when the minimum parts are balanced. Hence for all $y \in B(x,r)$, we have

$$M_\rho f(y) \lesssim (Mf(x))^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Consequently the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}$ provided by Theorem D in virtue of condition (4.1).

$$\|M_\rho f\|_{WM_{q,\varphi^{\frac{1}{q}}}} = \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|M_\rho f\|_{WL_q(B(x,t))}$$

$$\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{WL_p(B(x,t))}^{\frac{p}{q}}$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{WL_p(B(x,t))}\right)^{\frac{p}{q}}$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|Mf\|_{WM_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}}$$

$$\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}},$$

and

$$\|M_\rho f\|_{M_{q,\varphi^{\frac{1}{q}}}} = \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|M_\rho f\|_{L_q(B(x,t))}$$

$$\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{n}{q}} \|Mf\|_{L_p(B(x,t))}^{\frac{p}{q}}$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{n}{p}} \|Mf\|_{L_p(B(x,t))}\right)^{\frac{p}{q}}$$

$$= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}}$$

$$\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}},$$

if $1 < p < q < \infty$. $\square$

In the case $\rho(t) = t^\alpha$ from Theorem 4.2 we get the Adams type result on generalized Morrey spaces (see [16, Theorem 5.7, p. 182]).

In the case $\rho(t) = t^\alpha$, $\varphi(x,t) = t^{\lambda-n}$, $0 < \lambda < n$ from Theorem 4.2 we get the following Adams's result for the fractional maximal operator.

Corollary 4.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$, the operator M_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$, M_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Remark 4.3. *Note that, the condition* (3.1) *is weaker than the following condition which was given in* [17] *for I_ρ:*

$$\int_1^\infty \frac{\rho(t)}{t^n} \frac{dt}{t} < \infty. \tag{4.6}$$

For example, the function

$$\rho(t) = \frac{t^n}{\log(e + t)}, \ \ t > 0$$

satisfies (3.1), *but not* (4.6). *This example shows that the function ρ satisfies Theorems 3.1 and 4.2, but does not satisfy the assumptions of Theorems 16 and 22 in* [17]. *In other words, the condition* (3.1) *which satisfies our main theorems, is better (more general and comprehensive) than the condition* (4.8) *which satisfies the main theorems were given in* [17].

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