ELECTRONIC RESEARCH ARCHIVE
Volume **28**, Number **3**, September **2020**

doi:10.3934/era.2020067
pp. **1227–1238**

COLIMITS OF CROSSED MODULES IN MODIFIED CATEGORIES OF INTEREST

Ali Aytekin

Department of Mathematics,
Pamukkale University, Denizli, Turkey

Kadir Emir*

Department of Mathematics and Statistics
Masaryk University, Brno, Czech Republic

Abstract. In this paper, we give the constructions of the coequalizer and coproduct objects for the category of crossed modules, in a modified category of interest (MCI). In other words, we prove that the corresponding category is finitely cocomplete.

1. **Introduction.** The notions of category of interest [18] and groups with operations [19] are date back to Higgins [15]. They both aim to unify various algebraic structures and their properties. Precisely the notion of groups with operations is given as a relaxed version of category of interest. Therefore, groups with operations do not capture some algebraic structures which categories of interest do – clearly, every category of interest is a group with operation as well. Although many well-known algebraic categories (such as groups, vector spaces, associative algebras, Lie algebras, etc.) are the essential examples of categories of interest, there are some others which are not. For instance, the categories of cat^1-Lie (associative, Leibniz, etc.) algebras are not categories of interest.

At this point, a new and more general type of this notion is introduced in [4] which is called a modified category of interest. It satisfies all axioms of the former notion except one, which is replaced by a new and modified one. According to this definition, every category of interest becomes a modified category of interest. Further examples of modified categories of interest are those, which are equivalent to the categories of crossed modules in the categories of groups, associative algebras, commutative algebras, dialgebras, Lie algebras, Leibniz algebras, etc.

A crossed module of groups [21] $\partial\colon E \to G$ is given by a group homomorphism together with a group action $\triangleright$ of G on E satisfying the following relations (for all $e, f \in E$ and $g \in G$):

$$\partial(g \triangleright e) = g + \partial(e) - g, \quad \partial(e) \triangleright f = e + f - e.$$

Crossed modules are used for modeling homotopy systems of connected CW-complexes, and also for the classification of algebraic 2-types [17]. On the other hand, the category of crossed modules is also equivalent to the category of cat^1-groups [16] as well as to the categories of interest in the sense of [10, 11]. The

2020 *Mathematics Subject Classification.* Primary: 17A30; Secondary: 18G45, 18A30.
Key words and phrases. Colimit, crossed module, modified category of interest.
* Corresponding author: emir@math.muni.cz.

definition of crossed module is adapted to modified categories of interest in [4] that unifies all crossed module structures of the algebraic structures we mentioned above. It is strongly recommended to see [6, 7] for a very detailed survey of crossed modules and related structures. Some categorical properties of crossed modules are examined in [1, 2, 3, 5, 8, 12, 13, 14, 20] for various algebraic structures. In fact, some of them are examples of modified categories of interest.

As the modified category of interest is the unification of many well-known algebraic structures and their properties, it is natural to ask whether it is possible to unify some categorical properties of crossed modules via modified categories of interest. In this context, constructions of limits (of crossed modules) in a modified category of interest are given in [13] that yields the completeness of the corresponding category. Following that study, in this paper, we prove that a category of crossed modules in modified categories of interest is (finitely) cocomplete, namely, it has all finite colimits.

2. **Preliminaries.**

We recall some notions from [4] that will be used in the sequel.

Definition 2.1. Let $\mathbb{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) the group operations (written additively : $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \setminus \{+\}$, $\Omega_1' = \Omega_1 \setminus \{-\}$. Assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$ and assume $\Omega_0 = \{0\}$;

(c) for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;

(d) for each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and either the identity $\omega(x*y) = \omega(x)*\omega(y)$ or the identity $\omega(x*y) = \omega(x)*y$.

Denote by Ω_{1S}' the subset of those elements in Ω_1', which satisfy the identity $\omega(x * y) = \omega(x) * y$, and by Ω_1'' all other unary operations, i.e. those which satisfy the first identity from (d).

Let C be an object of $\mathbb{C}$ and $x_1, x_2, x_3 \in C$:

(e) $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$,

(f) For each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that

$$\begin{aligned}(x_1 * x_2)\ \overline{*}\ x_3 = W\big(&x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,\\ &x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2\big),\end{aligned}$$

where each juxtaposition represents an operation in Ω_2'.

A category of groups with operations $\mathbb{C}$ satisfying conditions (a)-(f) is called a "modified category of interest", or "MCI" for short.

Remark 1. Let us fix an arbitrary modified category of interest $\mathbb{C}$ throughout this section.

Definition 2.2. Let A, B be two objects of $\mathbb{C}$. A morphism in $\mathbb{C}$ is a map $f\colon A \to B$ commutes with all possible $w \in \Omega_1'$, such that

$$f(a + a') = f(a) + f(a'),$$
$$f(a * a') = f(a) * f(a'),$$

for all $a, a' \in A$, $* \in \Omega_2'$.

Example 2.3. The categories of groups, (commutative) algebras, modules over a ring, vector spaces, Lie algebras, Leibniz algebras, dialgebras are well-known examples of modified categories of interest.

However, there exist other well-known algebraic categories that are not modified categories of interest. For instance, the categories of Leibniz-Rinehart algebras, Hopf algebras, racks (or quandles), etc.

As we underlined in the introduction, the following are the essential examples of modified categories of interest (which are not categories of interest), and they were the main motivation to define modified categories of interest.

Example 2.4. The categories of cat^1-(commutative) algebras, cat^1-Lie algebras and cat^1-Leibniz algebras are also modified categories of interest.

Definition 2.5. Let B be an object of $\mathbb{C}$. A subobject of B is called an ideal if it is the kernel of some morphism. In other words, A is an ideal of B if A is a normal subgroup of B, and $a * b \in A$, for all $a \in A$, $b \in B$ and $* \in \Omega_2'$.

Definition 2.6. Let A, B be two objects of $\mathbb{C}$. An extension of B by A is a sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B \longrightarrow 0$$

where p is surjective and i is the kernel of p. We say that an extension is split if there exists a morphism $s\colon B \to E$ such that $ps = 1_B$.

Definition 2.7. The split extension induces a set of actions of B on A corresponding to the operations in $\mathbb{C}$ with being

$$\begin{aligned} b \cdot a &= s(b) + a - s(b), \\ b * a &= s(b) * a, \end{aligned}$$

for all $b \in B$, $a \in A$ and $* \in \Omega_2'$.

Actions defined by the previous equations are called *derived actions* of B on A. Remark that we use the notation "$*$" to denote both the star operation and the star action.

Definition 2.8. Given an action of B on A, the semi-direct product $A \rtimes B$ is a universal algebra, whose underlying set is $A \times B$, and the operations are defined by

$$\begin{aligned} \omega(a,b) &= (\omega\,(a)\,, \omega\,(b)), \\ (a', b') + (a, b) &= (a' + b' \cdot a, b' + b), \\ (a', b') * (a, b) &= (a' * a + a' * b + b' * a, b' * b), \end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$, $* \in \Omega_2'$.

Remark that, an action of B on A is a derived action, if and only if, $A \rtimes B$ is an object of $\mathbb{C}$.

Theorem 2.9. *Denote a general category of groups with operations of a modified category of interest $\mathbb{C}$ by $\mathbb{C}_G$. A set of actions of B on A in $\mathbb{C}_G$ is a set of derived actions, if and only if, it satisfies the following conditions:*

1. $0 \cdot a = a$,
2. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
3. $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
4. $b * (a_1 + a_2) = b * a_1 + b * a_2$,
5. $(b_1 + b_2) * a = b_1 * a + b_2 * a$,

6. $(b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2$,
7. $(b_1 * b_2) \cdot (a * b) = a * b$,
8. $a_1 * (b \cdot a_2) = a_1 * a_2$,
9. $b * (b_1 \cdot a) = b * a$,
10. $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,
11. $\omega(a * b) = \omega(a) * b = a * \omega(b)$ *for any* $\omega \in \Omega'_{1S}$, *and* $\omega(a * b) = \omega(a) * \omega(b)$ *for any* $\omega \in \Omega''_1$,
12. $x * y + z * t = z * t + x * y$,

for each $\omega \in \Omega'_1$, $* \in \Omega'_2$, b, b_1, $b_2 \in B$, $a, a_1, a_2 \in A$ *and for* $x, y, z, t \in A \cup B$ *whenever each side of 12 makes sense.*

Definition 2.10. A "crossed module" (C_1, C_0, ∂) in $\mathbb{C}$ is given by a morphism $\partial\colon C_1 \to C_0$ with a derived action of C_0 on C_1 such that

X1) $$\begin{aligned} \partial(c_0 \cdot c_1) &= c_0 + \partial(c_1) - c_0\,, \\ \partial(c_0 * c_1) &= c_0 * \partial(c_1)\,, \end{aligned}$$

X2) $$\begin{aligned} \partial(c_1) \cdot c'_1 &= c_1 + c'_1 - c_1\,, \\ \partial(c_1) * c'_1 &= c_1 * c'_1\,, \end{aligned}$$

for all $c_0 \in C_0$, $c_1, c'_1 \in C_1$, $* \in \Omega'_2$. Without the second condition, we call it a precrossed module.

A morphism between (pre)crossed modules $(C_1, C_0, \partial) \to (C'_1, C'_0, \partial')$ is a pair (μ_1, μ_0) of morphisms $\mu_0\colon C_0 \to C'_0$ and $\mu_1\colon C_1 \to C'_1$, such that the diagram

$$\begin{tikzcd} C_1 \arrow[r, "\partial"] \arrow[d, "\mu_1"'] & C_0 \arrow[d, "\mu_0"] \\ C'_1 \arrow[r, "\partial'"'] & C'_0 \end{tikzcd}$$

commutes and also

$$\begin{aligned} \mu_1(c_0 \cdot c_1) &= \mu_0(c_0) \cdot \mu_1(c_1)\,, \\ \mu_1(c_0 * c_1) &= \mu_0(c_0) * \mu_1(c_1)\,, \end{aligned}$$

for all $c_0 \in C_0$, $c_1 \in C_1$ and $* \in \Omega'_2$.

We denote the category of crossed modules by XMod, and similarly, of precrossed modules by PXMod.

The following two are the characteristic examples of crossed modules in any modified category of interest $\mathbb{C}$.

Example 2.11. Let B be an object of $\mathbb{C}$ and A is an ideal of B. Then, the inclusion map $A \hookrightarrow B$ becomes a crossed module where the action is defined via conjugation, namely

$$\begin{aligned} b \cdot a &= b + a - b\,, \\ b * a &= b * a\,, \end{aligned}$$

for all $a \in A$ and $b \in B$.

Example 2.12. Let B be an object of $\mathbb{C}$. Then, we have a natural crossed module $0 \to B$ with the trivial action. More generally, if A is an abelian object (i.e. $x + y = y + x$ and $x * y = 0$, for all $x, y \in A$ and $* \in \Omega'_2$), then the zero map $A \to B$ defines a crossed module with any derived action, for all B.

Considering Example 2.3, the following well-known crossed module definitions are particular examples of crossed modules in a modified category of interest.

Example 2.13. A crossed module of groups [5] is a group homomorphism $\partial\colon E \to G$, together with a group action $\triangleright$ of G on E such that

$$\begin{aligned} &\text{X1)} \quad \partial(g \triangleright e) = g + \partial(e) - g, \\ &\text{X2)} \quad \partial(e) \triangleright f = e + f - e, \end{aligned}$$

for all $e, f \in E$ and $g \in G$.

Example 2.14. A dialgebra crossed module [9] is a dialgebra homomorphism $\partial : D_1 \to D_0$ with a dialgebra action (via four bilinear maps) of D_0 on D_1, such that

$$\begin{aligned} \text{X1)} \quad & \partial(d_0 \triangleright_{\dashv} d_1) = d_0 \dashv \partial(d_1), \\ & \partial(d_0 \triangleright_{\vdash} d_1) = d_0 \vdash \partial(d_1), \\ & \partial(d_1 \triangleleft_{\dashv} d_0) = \partial(d_1) \dashv d_0, \\ & \partial(d_1 \triangleleft_{\vdash} d_0) = \partial(d_1) \vdash d_0, \end{aligned}$$

$$\begin{aligned} \text{X2)} \quad & \partial(d_1) \triangleright_{\dashv} d_1' = d_1 \dashv d_1' = d_1 \triangleleft_{\dashv} \partial(d_1'), \\ & \partial(d_1) \triangleright_{\vdash} d_1' = d_1 \vdash d_1' = d_1 \triangleleft_{\vdash} \partial(d_1'), \end{aligned}$$

for all $d_1, d_1' \in D_1$, $d_0 \in D_0$.

Example 2.15. A Lie algebra crossed module [9] is a Lie algebra homomorphism $\partial\colon \mathfrak{e} \to \mathfrak{g}$, together with a Lie algebra action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{e}$ such that

$$\begin{aligned} &\text{X1)} \quad \partial(g \triangleright e) = [g, \partial(e)], \\ &\text{X2)} \quad \partial(e) \triangleright f = [e, f], \end{aligned}$$

for all $e, f \in \mathfrak{e}$ and $g \in \mathfrak{g}$.

3. **Colimits in** $\mathrm{XMod/C_0}$**.** From now on, $\mathbb{C}$ will be a fixed MCI where $\mathbb{E}$ includes the identity $x + y = y + x$. Remark that Lie algebras, (commutative) associative algebras, dialgebras are all examples of $\mathbb{C}$.

Definition 3.1. Consider the subcategory $\mathrm{XMod/C_0}$ of crossed modules with a fixed codomain C_0. Its objects will be called crossed C_0-modules, and the morphism between (E, ∂_E) and (D, ∂_D) is defined by a morphism[1] $\mu\colon E \to D$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\partial_E"] \arrow[dr, "\mu"'] & & C_0 \\ & D \arrow[ur, "\partial_D"'] & \end{tikzcd}$$

and also

$$\begin{aligned} \mu(c_0 \cdot e) &= c_0 \cdot \mu(e), \\ \mu(c_0 * e) &= c_0 * \mu(e), \end{aligned}$$

for all $c_0 \in C_0$, $e \in E$, $* \in \Omega_2'$.

A crossed C_0-module $\partial\colon E \to C_0$ will be denoted by (E, ∂_E) for short.

[1]In full, it is a tuple (μ, id_{C_0}).

Proposition 1. *Let $\mu, \mu' : (E, \partial_E) \to (D, \partial_D)$ be two crossed C_0-module morphisms. Then,*

$$I = \{\mu(e) - \mu'(e) \mid e \in E\}$$

is an ideal of D.

Proof. For all $d \in D$, we have

$$\begin{aligned} d + \mu(e) - \mu'(e) - d &= d + \mu(e) - d + d - \mu'(e) - d \\ &= \partial_D(d) \cdot \mu(e) + \partial_D(d) \cdot \mu'(-e) \\ &= \mu(d \cdot e) + \mu'(d \cdot (-e)) \\ &= \mu(d \cdot e) - \mu'(d \cdot e) \in I\,, \end{aligned}$$

and

$$\begin{aligned} d * (\mu(e) - \mu'(e)) &= d * \mu(e) - d * \mu'(e) \\ &= \partial_D(d) * \mu(e) - \partial_D(d) * \mu'(e) \\ &= \mu(d * e) - \mu'(d * e) \in I\,, \end{aligned}$$

from which I becomes an ideal of D. □

Proposition 2. *Let I be the ideal given in Proposition 1. Then (I, ∂_D) is a crossed C_0-module with the action induced from that of C_0 on D.*

Proof. We only prove that the action of C_0 on I is well-defined. Let $c \in C_0$ and $\mu(e) - \mu'(e) \in I$. Then we have

$$\begin{aligned} c \cdot (\mu(e) - \mu'(e)) &= c \cdot \mu(e) - c \cdot \mu(e) \\ &= \mu(c \cdot e) - \mu'(c \cdot e) \in I\,, \end{aligned}$$

and, similarly,

$$c * (\mu(e) - \mu'(e)) = \mu(c * e) - \mu'(c * e),$$

that completes the proof. □

Theorem 3.2. *Any pair of parallel morphisms $\mu, \mu' \colon (E, \partial_E) \to (D, \partial_D)$ has a coequalizer.*

Proof. Consider the diagram

$$(E, \partial_E) \overset{\mu}{\underset{\mu'}{\rightrightarrows}} (D, \partial_D) \xrightarrow{p} (D/I, \overline{\partial}_D)\,.$$

We obviously have $p\,\mu = p\,\mu'$ since

$$\begin{aligned} p(\mu(e)) &= \mu(e) + I \\ &= \mu'(e) - \underbrace{\mu'(e) + \mu(e)}_{\in I} + I \\ &= \mu'(e) + I \\ &= p(\mu'(e))\,, \end{aligned}$$

for all $e \in E$.

Let $q \colon (D, \partial_D) \to (F, \partial_F)$ be a morphism such that $q\,\mu = q\,\mu'$. Define

$$\begin{aligned} q' : (D/I, \overline{\partial}_D) &\longrightarrow (F, \partial_F) \\ d + I &\longmapsto q(d). \end{aligned}$$

Then we have

$$\begin{aligned}\partial_F(q'(d+I)) &= \partial_F(q(d))\\ &= \partial_D(d)\\ &= \overline{\partial}_D(p(d))\\ &= \overline{\partial}_D(d+I),\end{aligned}$$

and

$$\begin{aligned}q'(c\cdot(d+I)) &= q'(c\cdot d+I)\\ &= q(c\cdot d)\\ &= c\cdot q(d)\\ &= c\cdot(q'(d+I)),\end{aligned}$$

and, similarly,

$$q'(c*(d+I)) = c*q'((d+I)),$$

for all $c\in C_0$ and $d+I\in D/I$. Hence, we obtain that

$$q' : (D/I,\overline{\partial}_D) \longrightarrow (F,\partial_F)$$

is a crossed C_0-module morphism.

Morever, let $h : (D/I,\overline{\partial}_D) \to (F,\partial_F)$ be a crossed C_0-module morphism such that $h\,p = q$. We have

$$\begin{aligned}h(d+I) &= hp(d)\\ &= q(d)\\ &= q'p(d)\\ &= q'(d+I),\end{aligned}$$

for all $d+I\in D/I$, that proves q' is unique and completes the proof. □

Proposition 3. *Let (D,∂_D) and (E,∂_E) be two crossed C_0-modules. Then, the set of actions of D on E defined via ∂_D is a set of derived actions.*

Proof. We show that the set of actions defined by

$$\begin{aligned}d\cdot e &= \partial_D(d)\cdot e\,,\\ d*e &= \partial_D(d)*e\,,\end{aligned}$$

satisfies the conditions in Theorem 2.9, as follows:

2) For all $d\in D$, $e,e'\in E$, we have

$$\begin{aligned}d\cdot(e+e') &= \partial_D(d)\cdot(e+e')\\ &= \partial_D(d)\cdot e+\partial_D(d)\cdot e'\\ &= d\cdot e+d\cdot e',\end{aligned}$$

11) For all $d\in D$, $e\in E$ and $w\in\Omega_1'$, we have

$$\begin{aligned}w(d*e) &= w(\partial_D(d)*e)\\ &= w(\partial_D(d))*e\\ &= \partial_D(w(d))*e\\ &= w(d)*e,\end{aligned}$$

and the other conditions follow immediately. □

Proposition 4. *With the assumptions in Theorem 3.2, C_0 acts on $E \rtimes D$ componentwise, i.e.*

$$c \cdot (e,d) = (c \cdot e, c \cdot d)\,,$$
$$c * (e,d) = (c * e, c * d)\,,$$

for all $c \in C_0$ and $(e,d) \in E \rtimes D$.

Proof. Since C_0 acts on D and E, we have the split extensions

$$0 \longrightarrow D \longrightarrow X \underset{s_D}{\leftrightarrows} C_0 \longrightarrow 0$$

and

$$0 \longrightarrow E \longrightarrow X' \underset{s_E}{\leftrightarrows} C_0 \longrightarrow 0$$

Consequently, we have the following split extension

$$0 \longrightarrow E \rtimes D \longrightarrow X' \times X \underset{(s_E,s_D)}{\leftrightarrows} C_0 \longrightarrow 0$$

with $(s_E, s_D)(c) = (s_E(c), s_D(c))$, from which we get the derived actions

$$c \cdot (e,d) = (c \cdot e, c \cdot d),$$
$$c * (e,d) = (c * e, c * d),$$

for all $c \in C_0$ and $(e,d) \in E \rtimes D$. □

Proposition 5. *Let (D, ∂_D), (E, ∂_E) be two crossed C_0-modules. Then*

$$\begin{array}{rcl} \partial : E \rtimes D & \longrightarrow & C_0 \\ (e,d) & \longmapsto & \partial_E(e) + \partial_D(d) \end{array}$$

is a precrossed C_0-module.

Proof. First of all, ∂ is a morphism in $\mathbb{C}$ since

$$\begin{aligned} \partial((e,d)+(e',d')) &= \partial(e + d \cdot e', d + d') \\ &= \partial_E(e + d \cdot e') + \partial_D(d + d') \\ &= \partial_E(e) + \partial_E(d \cdot e') + \partial_D(d + d') \\ &= \partial_E(e) + \partial_E(\partial_D(d) \cdot e') + \partial_D(d + d') \\ &= \partial_E(e) + \partial_D(d) + \partial_E(e') - \partial_D(d) + \partial_D(d) + \partial_D(d') \\ &= \partial_E(e) + \partial_D(d) + \partial_E(e') + \partial_D(d') \\ &= \partial(e,d) + \partial(e',d'), \end{aligned}$$

and

$$\begin{aligned} &\partial((e,d) * (e',d')) \\ &\quad = \partial(e * e' + d * e' + e * d', d * d') \\ &\quad = \partial_E(e * e' + d * e' + e * d') + \partial_D(d * d') \\ &\quad = \partial_E(e * e') + \partial_E(d * e') + \partial_E(e * d') + \partial_D(d * d') \end{aligned}$$

$$
\begin{aligned}
&= \partial_E(e * e') + \partial_E(\partial_D(d) * e') + \partial_E(e * \partial_D(d')) + \partial_D(d * d') \\
&= \partial_E(e) * \partial_E(e') + \partial_D(d) * \partial_E(e') + \partial_E(e) * \partial_D(d') + \partial_D(d) * \partial_D(d') \\
&= (\partial_E(e) + \partial_D(d)) * \partial_E(e') + (\partial_E(e) + \partial_D(d)) * \partial_D(d') \\
&= (\partial_E(e) + \partial_D(d)) * (\partial_E(e') + \partial_D(d')) \\
&= \partial(e, d) * \partial(e', d'),
\end{aligned}
$$

for all $(e, d), (e', d') \in E \rtimes D$.

On the other hand, since ∂_E and ∂_D are crossed modules, we have

$$
\begin{aligned}
\partial(c \cdot (e, d)) &= \partial(c \cdot e, c \cdot d) \\
&= \partial_E(c \cdot e) + \partial_D(c \cdot d) \\
&= c + \partial_E(e) - c + c + \partial_D(d) - c \\
&= c + \partial_E(e) + \partial_D(d) - c \\
&= c + \partial(e, d) - c\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial(c * (e, d)) &= \partial(c * e, c * d) \\
&= \partial_E(c * e) + \partial_D(c * d) \\
&= c * \partial_E(e) + c * \partial_D(d) \\
&= c * (\partial_E(e) + \partial_D(d)) \\
&= c * \partial(e, d)\,,
\end{aligned}
$$

for all $c \in C_0$ and $(e, d) \in E \rtimes D$, which proves that $(E \rtimes D, \partial)$ is a precrossed C_0-module. □

Proposition 6. *For a given precrossed C_0-module $\partial\colon A \to C_0$, let P be the smallest ideal containing the set*

$$\{(\partial(a) * a') - a * a', (\partial(a) \cdot a') - a' \mid a, a' \in A\}\,.$$

Then,

$$
\begin{array}{rcl}
\overline{\partial} : A/P & \longrightarrow & C_0 \\
a + P & \longmapsto & \partial(a)
\end{array}
$$

is a crossed C_0-module with the action

$$
\begin{aligned}
c \cdot (a + P) &= c \cdot a + P \\
c * (a + P) &= c * a + P
\end{aligned}
$$

for all $c \in C_0$, $(a + P) \in A/P$.

Proof. Since the crossed module conditions are already satisfied, we only need to prove that the action of C_0 on A/P is well-defined.

Let $p = \partial(a) \cdot a' - a' \in P$. Then we have

$$
\begin{aligned}
c \cdot p &= c \cdot \big(\partial(a) \cdot a' - a'\big) \\
&= c \cdot (\partial(a) \cdot a') - c \cdot a' \\
&= (c + \partial(a)) \cdot a' - c \cdot a' \\
&= (c + \partial(a) - c) \cdot (c \cdot a') - c \cdot a' \\
&= \partial(c \cdot a) \cdot (c \cdot a') - c \cdot a' \in P\,,
\end{aligned}
$$

and similarly, by Proposition (3.10) in [4], we have

$$c \cdot \big(\partial(a) * a' - a * a'\big) = \partial(a) * a' - a * a' \,,$$

for all $c \in C_0$, $* \in \Omega_2'$. ☐

Remark 2. We have the functor

$$(\ \ )^{cr} : \mathrm{PXMod} \to \mathrm{XMod}$$

which assigns the crossed module $\bar{\partial}\colon A/P \to C_0$ for a given precrossed module $\partial\colon A \to C_0$.

Theorem 3.3. $\Big(\big((E \rtimes D)/P, \bar{\partial}\big), i_E, i_D\Big)$ *is the coproduct of crossed* C_0*-modules* (E, ∂_E) *and* (D, ∂_D) *where*

$$i_E(e) = (e, 0) + P, \quad i_D(d) = (0, d) + P,$$

for all $e \in E$, $d \in D$.

Proof. Consider the diagram

$$\begin{tikzcd}
(E, \partial_E) \arrow[dr, "i_E"] \arrow[ddr, bend right, "J_E"'] & & (D, \partial_D) \arrow[dl, "i_D"'] \arrow[ddl, bend left, "J_D"] \\
& (E \rtimes D/P, \bar{\partial}) & \\
& (X, \partial_X) &
\end{tikzcd}$$

in XMod/C$_0$. Define

$$\begin{array}{rcl} h : E \rtimes D/P & \longrightarrow & X \\ \overline{(e,d)} & \longmapsto & J_E(e) + J_D(d). \end{array}$$

First of all, h is a morphism in $\mathbb{C}$ since

$$\begin{aligned}
h\big(\overline{(e,d) + (e',d')}\big) &= h(\overline{e + d \cdot e', d + d'}) \\
&= J_E(e + \partial_D(d) \cdot e') + J_D(d + d') \\
&= J_E(e) + J_E(\partial_D(d) \cdot e') + J_D(d + d') \\
&= J_E(e) + \partial_D(d) \cdot J_E(e') + J_D(d + d') \\
&= J_E(e) + (\partial_X J_D(d)) \cdot J_E(e') + J_D(d + d') \\
&= J_E(e) + J_D(d) + J_E(e') - J_D(d) + J_D(d) + J_D(d') \\
&= J_E(e) + J_D(d) + J_E(e') + J_D(d') \\
&= h\overline{(e,d)} + h\overline{(e',d')},
\end{aligned}$$

and

$$
\begin{aligned}
& h\overline{(e,d)} * h\overline{(e',d')} \\
& \quad = h(\overline{(e,d) * (e',d')}) \\
& \quad = h(\overline{e * e' + d * e' + e * d', d * d'}) \\
& \quad = J_E(e * e' + d * e' + e * d') + J_D(d * d') \\
& \quad = J_E(e * e') + J_E(d * e') + J_E(e * d') + J_D(d * d') \\
& \quad = J_E(e * e') + J_E(\partial_D(d) * e') + J_E(e * \partial_D(d')) + J_D(d * d') \\
& \quad = J_E(e * e') + \partial_D(d) * J_E(e') + J_E(e) * \partial_D(d') + J_D(d * d') \\
& \quad = J_E(e * e') + \partial_X(J_D(d)) * J_E(e') + J_E(e) * \partial_X(J_D(d)) + J_D(d * d') \\
& \quad = J_E(e) * J_E(e') + J_D(d) * J_E(e') + J_E(e) * J_D(d') + J_D(d) * J_D(d') \\
& \quad = (J_E(e) + J_D(d)) * J_E(e') + (J_E(e) + J_D(d)) * J_D(d') \\
& \quad = (J_E(e) + J_D(d)) * (J_E(e') + J_D(d')) \\
& \quad = h\overline{(e,d)} + h\overline{(e',d')},
\end{aligned}
$$

for all $\overline{(e,d)}, \overline{(e',d')} \in E \rtimes D/P$.

Since J_E, J_D are morphisms in $\mathrm{XMod/C_0}$, we have

$$
\begin{aligned}
\partial_X h\,\overline{(e,d)} &= \partial_X(J_E(e) + J_D(d)) \\
&= \partial_X J_E(e) + \partial_X J_D(d) \\
&= \partial_E(e) + \partial_D(d) \\
&= \bar{\partial}\ \overline{(e,d)}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
h(c \cdot \overline{(e,d)}) &= h\overline{(c \cdot e, c \cdot d)} \\
&= J_E(c \cdot e) + J_D(c \cdot d) \\
&= c \cdot J_E(e) + c \cdot J_D(d) \\
&= c \cdot (J_E(e) + J_D(d)) \\
&= c \cdot h\overline{(e,d)}\,,
\end{aligned}
$$

for all $c \in C_0$, $\overline{(e,d)} \in E \rtimes D/P$, that yields h is a crossed C_0-module morphism. On the other hand, it is easy to prove that h is unique. □

Remark 3. A category $\mathcal{D}$ is said to be (finitely) "cocomplete" if it has all (finite) colimits. On the other hand, the cocompleteness can be characterized in several ways as follows. For a category $\mathcal{D}$, the following are equivalent:

- $\mathcal{D}$ is finitely cocomplete.
- $\mathcal{D}$ has coequalizers and coproducts.
- $\mathcal{D}$ has pushouts and the initial object.

Consequently, recalling Theorems 3.2 and 3.3, we already proved the following:

Corollary 1. *The category* $\mathrm{XMod/C_0}$ *is (finitely) cocomplete.*

Acknowledgments. The second author was supported by the projects Group Techniques and Quantum Information (MUNI/G/1211/2017), and Mathematical Structures 9 (MUNI/A/0885/2019) by Masaryk University Grant Agency (GAMU). The authors are thankful to James F. Peters for his kind interest on the paper, and also to reviewers for their helpful comments.

REFERENCES

[1] M. Alp, Pullbacks of crossed modules and cat^1-groups, *Turkish J. Math.*, **22** (1998), 273–281.
[2] M. Alp, Pullbacks of crossed modules and Cat^1-commutative algebras, *Turkish J. Math.*, **30** (2006), 237–246.
[3] M. Alp and B. Davvaz, Pullback and pushout crossed polymodules, *Proc. Indian Acad. Sci., Math. Sci.*, **125** (2015), 11–20.
[4] Y. Boyaci, J. M. Casas, T. Datuashvili and E. Ö. Uslu, Actions in modified categories of interest with application to crossed modules, *Theory Appl. Categ.*, **30** (2015), 882–908.
[5] R. Brown, Coproducts of crossed P-modules: Applications to second homotopy groups and to the homology of groups, *Topology*, **23** (1984), 337–345.
[6] R. Brown, From groups to groupoids: A brief survey, *Bull. Lond. Math. Soc.*, **19** (1987), 113–134.
[7] R. Brown, Modelling and computing homotopy types: I, *Indag. Math. (N.S.)*, **29** (2018), 459–482.
[8] R. Brown and C. D. Wensley, On finite induced crossed modules and the homotopy 2-type of mapping cones, *Theory Appl. Categ.*, **1** (1995), 54–70.
[9] J. M. Casas, R. F. Casado, E. Khmaladze and M. Ladra, More on crossed modules in Lie, Leibniz, associative and diassociative algebras, *J. Algebra Appl.*, **16** (2017), 1750107, 17 pp.
[10] J. Casas, T. Datuashvili and M. Ladra, Actors in categories of interest, arXiv:math/0702574.
[11] J. M. Casas, T. Datuashvili and M. Ladra, Universal strict general actors and actors in categories of interest, *Appl. Categ. Struct.*, **18** (2010), 85–114.
[12] J. M. Casas and M. Ladra, Colimits in the crossed modules category in Lie algebras, *Georgian Math. J.*, **7** (2000), 461–474.
[13] K. Emir and S. Çetin, Limits in modified categories of interest, *Bull. Iran. Math. Soc.*, **43** (2017), 2617–2634.
[14] K. Emir and H. Gülsün Akay, Pullback crossed modules in the category of racks, *Hacet. J. Math. Stat.*, **48** (2019), 140–149.
[15] P. J. Higgins, Groups with multiple operators, *Proc. Lond. Math. Soc.*, **6** (1956), 366–416.
[16] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, *J. Pure Appl. Algebra*, **24** (1982), 179–202.
[17] S. MacLane and J. H. C. Whitehead, On the 3-type of a complex, *Proc. Natl. Acad. Sci. U.S.A.*, **36** (1950), 41–48.
[18] G. Orzech, Obstruction theory in algebraic categories. I, *J. Pure Appl. Algebra*, **2** (1972), 287–314.
[19] T. Porter, Extensions, crossed modules and internal categories in categories of groups with operations, *Proc. Edinb. Math. Soc.*, **30** (1987), 373–381.
[20] N. Shammu, *Algebraic and Categorical Structure of Categories of Crossed Modules of Algebras*, University College of North Wales, 1992.
[21] J. H. C. Whitehead, Combinatorial homotopy. II, *Bull. Amer. Math. Soc.*, **55** (1949), 453–496.

Received April 2020; revised May 2020.

E-mail address: aaytekin@pau.edu.tr
E-mail address: emir@math.muni.cz