New Analytical Solutions of Conformable Time Fractional Bad and Good Modified Boussinesq Equations

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Abstract

The main purpose of this article is to obtain the new solutions of fractional bad and good modified Boussinesq equations with the aid of auxiliary equation method, which can be considered as a model of shallow water waves. By using the conformable wave transform and chain rule, nonlinear fractional partial differential equations are converted into nonlinear ordinary differential equations. This is an important impact because both Caputo definition and Riemann–Liouville definition do not satisfy the chain rule. By using conformable fractional derivatives, reliable solutions can be achieved for conformable fractional partial differential equations.

1 Introduction

In 1695, since L' Hospital asked the question, what might be a derivative order 1/2. Many researchers tried to find a definition of fractional derivative after this question. Most of the works focused on an integral form of the fractional derivative. Although most famous approaches are the Caputo definition and Riemann–Liouville definition, these two definitions have some drawbacks. For example,

- Riemann–Liouville definition does not satisfy $D^\alpha 1 = 0$ when α is not a natural number.

- Caputo definition assumes that the function is differentiable.

- Both definitions do not satisfy the derivative of the product of two functions.

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- Both definitions do not satisfy the derivative of the quotient of two functions.

- Both of them do not satisfy the chain rule.

- Both of them do not satisfy the index rule.

Recently, Khalil et al. [5] introduced a new definition of the integral and conformable fractional derivatives. So using the conformable fractional derivative, we can overcome the aforementioned drawbacks of the existing definitions. Now let us give the definition and some properties of conformable fractional derivative and integral.

Definition 1.1. Let $f : [0, \infty) \to$ is a function of α-th order "conformable functional derivate", which is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$, $\alpha \in (0, 1)$.

Definition 1.2. Starting from $a \geq 0$, the conformable integral of a function f is defined in [5] as

$$I_\alpha^a(f)(s) = \int_a^s \frac{f(t)}{t^{1-\alpha}} dt. \tag{2}$$

Similarly, the definition of conformable fractional partial derivative is given by Atangana et al. [2].

Definition 1.3. Let f be a function with n variables such as $x_1, ..., x_n$, the conformable partial derivative of f order $\alpha \in (0, 1]$ in x_i is defined as follows

$$\frac{d^\alpha}{dx_i^\alpha} f(x_1, ..., x_n) = \lim_{\varepsilon \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, ..., x_n) - f(x_1, ..., x_n)}{\varepsilon}. \tag{3}$$

Recently, many studies have been performed in various fields such as applied mathematics, physics and engineering related to fractional calculations [12–16]. Phenomena related to nonlinear partial differential equations (NLPDEs) have emerged in many areas such as physics, mechanics and chemistry to investigate the exact solutions for NLPDEs. In recent years, there are a lot of workings with NLPDEs. For example, Whitham [9] studied variational methods and applications on water waves. Sirendaoreji et al. [7] used the auxiliary equation method for solving NLPDEs. Zhang and Xia [11] studied a generalised new auxiliary equation method and its applications to NLPDEs. Tasbozan et al. [8] studied the Sine-Gordon expansion method to obtain the analytical results for Drinfeld-Sokolov-Wilson system. Yomba [10] discussed the exact results for the nonlinear Klein-Gordon equation and generalised nonlinear Camassa–Holm equation using a generalised auxiliary equation method. Eslami and Mirzazadeh [4] used the first integral method to obtain the exact solutions of the nonlinear Schrödinger equation.

The properties of this new definition [5] are given below.

Theorem 1.4. *Let f, g functions and $\alpha \in (0, 1]$ are α-differentiable at a point $t > 0$, then*

1. $T_\alpha(mf + ng) = mT_\alpha(f) + nT_\alpha(g)$ *for all $m, n \in$.*

2. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all p.*

3. $T_\alpha(f \cdot g) = fT_\alpha(g) + gT_\alpha(f)$.

4. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

5. $T_\alpha(c) = 0$ *for all constant functions $f(t) = c$.*

6. *If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df(t)}{dt}$.*

2 Fractional Bad–Good Modified Boussinesq Equations

In the 1870s, the Boussinesq equation [3] was first introduced by Joseph Boussinesq. This equation corresponds to the shallow water wave model that arises narrow channels and coastlines. This equation has been used to describe the motions of long waves in shallow water under gravity forces. Subsequently, it was applied to many other areas of mathematical physics dealing with wave phenomena. In this paper, we consider bad and good modified Boussinesq equations where the fractional terms are in a conformable sense.

Bad Modified Boussinesq Equation:

$$D_t^{(2\alpha)}u - D_x^2 u - D_x^4 u - 3D_x^2\left(u^2\right) + 3D_x\left(u^2 D_x u\right) = 0,$$

Good Modified Boussinesq Equation:

$$D_t^{(2\alpha)}u - D_x^2 u + D_x^4 u - 3D_x^2\left(u^2\right) + 3D_x\left(u^2 D_x u\right) = 0.$$

In this study, we use the auxiliary equation method to obtain a solution set for the above-mentioned equations using a conformable fractional derivative.

3 Description of Auxiliary Equation Method

Auxiliary equation method [7] has been used to obtain exact solutions for NLPDEs. Auxiliary equation method can be applied in NLPDEs if the considered equation consists only even-order or only odd-order partial derivative terms. This method depends on the following differential equation

$$\left(\frac{dz}{d\xi}\right)^2 = az^2(\xi) + bz^3(\xi) + cz^4(\xi), \tag{4}$$

First mentioned by Sirendaoreji [7]. Clearly, we can express the solution procedure of the method step by step as follows.

Step 1. The general form of a nonlinear conformable fractional differential equation can be regarded as

$$P\left(u, D_t^\alpha u, D_x u, D_t^{(2\alpha)}u, D_x^2 u, ...\right) = 0, \tag{5}$$

where the arguments and subscripts of polynomial P show partial derivatives, and $D_t^{(2\alpha)}$ means two times conformable derivative of the function $u(x,t)$.

Step 2. Using the wave transformation with chain rule [1]

$$u(x,t) = U(\xi), \xi = x - w\frac{t^\alpha}{\alpha}, \tag{6}$$

where w denotes the velocity of the wave. With the aid of this transformation, fractional derivatives of Eq. (5) can be rewritten as:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha} = \frac{d(.)}{d\xi}, \frac{\partial(.)}{\partial x} = -w\frac{d(.)}{d\xi}, ... \tag{7}$$

Using the transformation given in Eq. (6) inside Eq. (5), we obtain the following ordinary differential equation

$$G(U, U', U'', U''', ...) = 0, \tag{8}$$

where the derivatives are respected to ξ.

Table 1 Solutions of Eq. (4) with $\Delta = b^2 - 4ac$ and $\varepsilon = \pm 1$

No	$z(\xi)$	
1	$\dfrac{-ab\text{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac\left(1 + \varepsilon \tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}$	$a > 0$
2	$\dfrac{ab\text{csch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac\left(1 + \varepsilon \coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}$	$a > 0$
3	$\dfrac{2a\text{sech}(\sqrt{a}\xi)}{\varepsilon\sqrt{\Delta} - b\text{sech}(\sqrt{a}\xi)}$	$a > 0,\ \Delta > 0$
4	$\dfrac{2a\sec\left(\sqrt{-a}\xi\right)}{\varepsilon\sqrt{\Delta} - b\sec\left(\sqrt{-a}\xi\right)}$	$a < 0,\ \Delta > 0$
5	$\dfrac{2a\text{csch}(\sqrt{a}\xi)}{\varepsilon\sqrt{-\Delta} - b\text{csch}(\sqrt{a}\xi)}$	$a > 0,\ \Delta < 0$
6	$\dfrac{2a\csc\left(\sqrt{-a}\xi\right)}{\varepsilon\sqrt{\Delta} - b\csc\left(\sqrt{-a}\xi\right)}$	$a < 0,\ \Delta > 0$
7	$\dfrac{-a\text{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{ac}\tanh\left(\frac{\sqrt{a}}{2}\xi\right)}$	$a > 0,\ c > 0$
8	$\dfrac{-a\sec^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{-ac}\tan\left(\frac{\sqrt{-a}}{2}\xi\right)}$	$a < 0,\ c > 0$
9	$\dfrac{a\text{csch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{ac}\coth\left(\frac{\sqrt{a}}{2}\xi\right)}$	$a > 0,\ c > 0$
10	$\dfrac{-a\csc^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{-ac}\cot\left(\frac{\sqrt{-a}}{2}\xi\right)}$	$a < 0,\ c > 0$
11	$-\dfrac{a}{b}\left[1 + \varepsilon\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right]$	$a > 0,\ \Delta = 0$
12	$-\dfrac{a}{b}\left[1 + \varepsilon\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right]$	$a > 0,\ \Delta = 0$
13	$\dfrac{4ae^{\varepsilon\sqrt{a}\xi}}{\left(e^{\varepsilon\sqrt{a}\xi} - b\right)^2 - 4ac}$	$a > 0$
14	$\dfrac{\pm 4ae^{\varepsilon\sqrt{a}\xi}}{1 - 4ace^{2\varepsilon\sqrt{a}\xi}}$	

Step 3. Now, consider $U(\xi)$ is a sum of serial such as

$$U(\xi) = \sum_{i=0}^{n} a_i z^i(\xi), \tag{9}$$

where $z(\xi)$ is the solution of the nonlinear differential equation (4), a, b, c, w, a_i are the real constants and n is a positive integer to be determined by the balancing procedure [6].

Step 4. Balancing the linear and nonlinear terms of the highest order in the ordinary differential equation, Eq. (8) gives the result for the parameter n. Then, we place Eq. (9) into the ordinary differential equation Eq. (8). After this procedure, we get an equation consisting of the powers of $z(\xi)$. All coefficients of $z(\xi)$ are equated to 0 in the final equation. This procedure arouses the system of algebraic equations including a, b, c, w, a_i. Solving this system, with respect to these parameters and using the exact solutions of Eq. (4) in Table 1, gives the analytical solutions. Some solutions of Eq. (4) are given in Table 1.

4 Implementation of the Method

We consider the time-fractional Bad–Good modified Boussinesq equations

$$D_t^{(2\alpha)}u - D_x^2 u - D_x^4 u - 3D_x(u^2 D_x u) = 0, \tag{10}$$

$$D_t^{(2\alpha)}u - D_x^2 u + D_x^4 u - 3D_x(u^2 D_x u) = 0,\tag{11}$$

Using the wave transform (6) and integrating both equations twice, the equations turn into the ordinary differential equation as follows.

$$-U'' + \left(w^2 - 1\right)U - 3U^2 + U^3 = 0,\tag{12}$$

$$U'' + \left(w^2 - 1\right)U - 3U^2 + U^3 = 0,\tag{13}$$

where prime denotes the derivative of the functions with respect to ξ.

Now using the balancing procedure in Eqs. (12)and (13) yields $n = 1$. Thus, the unknown function $U(\xi)$ can be considered as

$$U(\xi) = a_0 + a_1 z(\xi).\tag{14}$$

Placing Eq. (14) into Eqs (12) and (13) and using Eq. (4) led to an algebraic equation with respect to $z(\xi)$. Equating all the coefficients of same powers of to 0 arouses an algebraic equation system. Solving this system gives the following solution sets.

Solutions for Bad modified Boussinesq equation:

SET:

Using obtained constants from Set, with the aid of Table 1, the new wave solutions of Bad modified Boussinesq equation (10) can be given as follows

$$u_{1,2}(x,t) = -\frac{4(1-w^2)\sec h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]^2}{8 + (1-w^2)\left(1 \pm \tan h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]\right)^2},$$

$$u_{3,4}(x,t) = -\frac{4(-1+w^2)\csc h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]^2}{8 - (-1+w^2)\left(1 \pm \cot h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]\right)^2}$$

$$u_{5,6}(x,t) = \frac{2(-1+w^2)\sec h\left[\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]}{\pm\sqrt{4 - 2(-1+w^2)} + 2\sec h\left[\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]},$$

$$u_{7,8}(x,t) = \frac{2(-1+w^2)\sec\left[\sqrt{1-w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]}{\pm\sqrt{4 - 2(-1+w^2)} + 2\sec\left[\sqrt{1-w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]},$$

$$u_{9,10}(x,t) = \frac{2(-1+w^2)\csc h\left[\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]}{\pm\sqrt{-4 + 2(-1+w^2)} + 2\csc h\left[\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]},$$

$$u_{11,12}(x,t) = \frac{2(-1+w^2)\csc\left[\sqrt{1-w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]}{\pm\sqrt{4 - 2(-1+w^2)} + 2\csc\left[\sqrt{1-w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]},$$

$$u_{13,14}(x,t) = \frac{(1-w^2)\sec h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]^2}{-2 \pm \sqrt{2}\sqrt{(-1+w^2)}\tan h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]},$$

$$u_{15,16}(x,t) = \frac{(1-w^2)\sec\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]^2}{-2 \pm \sqrt{2}\sqrt{(1-w^2)}\tan\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^\alpha w}{\alpha}\right)\right]},$$

$$u_{17,18}(x,t) = \frac{(-1+w^2)\csc h\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]^2}{-2 \pm \sqrt{2}\sqrt{(-1+w^2)}\coth\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{19,20}(x,t) = \frac{(1-w^2)\csc\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]^2}{-2 \pm \sqrt{2}\sqrt{(1-w^2)}\cot\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{21,22}(x,t) = \frac{4a_1 e^{\pm\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)}(-1+w^2)}{\left(2a_1 + e^{\pm\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)}\right)^2 - 2a_1^2(-1+w^2)}.$$

Solutions for Good modified Boussinesq equation:
SET:

$$a_0 = 0,\ a_1 = 1 - w^2,\ b = 2a_1,\ c = -\frac{a_1^2}{2}.$$

The new wave solutions of Good modified Boussinesq equation (11) with the aid of Table 1 can be given as follows

$$u_{1,2}(x,t) = \frac{4(-1+w^2)\sec h\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]^2}{8 + (1-w^2)\left(1 \pm \tanh\left[\frac{1}{2}\sqrt{1-w^2}(x - \frac{t^{\alpha_w}}{\alpha})\right]\right)^2},$$

$$u_{3,4}(x,t) = \frac{4(1-w^2)\csc h\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]^2}{8 + (1-w^2)\left(1 \pm \coth\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]\right)^2},$$

$$u_{5,6}(x,t) = \frac{2(1-w^2)\sec h\left[\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]}{\pm\sqrt{4 + 2(1-w^2)} - 2\sec h\left[\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{7,8}(x,t) = \frac{2(1-w^2)\sec\left[\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]}{\pm\sqrt{4 + 2(1-w^2)} - 2\sec\left[\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{9,10}(x,t) = \frac{2(1-w^2)\csc h\left[\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]}{\pm\sqrt{-4 - 2(1-w^2)} - 2\csc h\left[\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{11,12}(x,t) = \frac{2(1-w^2)\csc\left[\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]}{\pm\sqrt{4 + 2(1-w^2)} - 2\csc\left[\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{13,14}(x,t) = \frac{(-1+w^2)\sec h\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]^2}{2 \pm \sqrt{2}\sqrt{w^2-1}\tanh\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{15,16}(x,t) = \frac{(-1+w^2)\sec\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]^2}{2 \pm \sqrt{2}\sqrt{1-w^2}\tan\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^{\alpha_w}}{\alpha}\right)\right]},$$

$$u_{17,18}(x,t) = \frac{(1-w^2)\csc h\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha}w}{\alpha}\right)\right]^2}{2 \pm \sqrt{2}\sqrt{w^2-1}\cot h\left[\frac{1}{2}\sqrt{1-w^2}\left(x - \frac{t^{\alpha}w}{\alpha}\right)\right]},$$

$$u_{19,20}(x,t) = \frac{(-1+w^2)\csc\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^{\alpha}w}{\alpha}\right)\right]^2}{2 \pm \sqrt{2}\sqrt{1-w^2}\cot\left[\frac{1}{2}\sqrt{-1+w^2}\left(x - \frac{t^{\alpha}w}{\alpha}\right)\right]},$$

$$u_{21,22}(x,t) = \frac{4a_1 e^{\pm\sqrt{1-w^2}\left(x - \frac{t^{\alpha}w}{\alpha}\right)}(1-w^2)}{\left(-2a_1 + e^{\pm\sqrt{1-w^2}\left(x - \frac{t^{\alpha}w}{\alpha}\right)}\right)^2 + 2a_1^2(1-w^2)}.$$

5 Conclusion

In this study, an auxiliary equation method is used to obtain the new exact solutions of Bad and Good Boussinesq equations. The study indicates that the auxiliary equation method is direct, effective and understandable and can be used for solving other NLPDEs in mathematical physics. Moreover, using the conformable fractional derivative, one can obtain analytical solutions of the NLPDEs which cannot be solved by Caputo and Riemann–Liouville definitions. In addition, many transactions were made with the program code of the Mathematica program.

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