Mathematical & Computational Applications, Vol. 1, No. 1, pp. 21-28, 1996

THE LIFTS OF LAGRANGE AND HAMILTON EQUATIONS TO THE EXTENDED VECTOR BUNDLES

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ABSTRACT

In this study, the vertical and complete lifts of Lagrange and Hamilton equations on the extended vector bundle have been given. Furthermore, some results on higher order mechanical systems on the extended vector bundles have been obtained.

Key words: Extended Vector Bundle, Lagrangenian and Hamiltonian, Vertical and Complete Lifts.

§0 INTRODUCTION AND NOTATIONS

In the previous studies [1]-[13][1], the complete and vertical lifts of the differential elements defined on a manifold and a vector bundle were examined carefully. Lagrange-Euler and Hamilton equations on the almost tangent and cotangent manifold were studied in the other previous studies [14]-[17].

Let E (also M) be a configuration manifold and $\pi: E \to M$ be surjective submersion. Then $\pi=(E,\pi,M)$ has a vector bundle structure[2].

It is well-known that Lagrangian and Hamiltonian formalisms can be intrinsically characterized by geometric structures canonically associated to the extended vector bundle and vector bundle. The dynnamical equations for both theories may be expressed in the following symbolical equation;

(1) $\iota(X)\omega = V$

If we are studying the Hamiltonian theory then equation (1) is the intrinsical version of Hamilton equations when $\omega = -d\lambda$, $V=dH$, where λ is the Liouville form canonically constructed on the cotangent vector bundle $T\pi^*$ of the vector bundle π and $H: TE^* \to R$ is Hamilton function.

Concerning the Lagrangian description we may derive the dynamical equations from variational considerations or by pulling back the form to the tangent bundle TE. But

[1] The numbers in brackets [ ] refer to References at the end of the paper.

this is only possible when the Lagrangian $L: TE \rightarrow R$ involved in the theory is a regular function. For the regular Lagrangians, equation (1) takes the form

$$\iota(X)\omega = dE_L$$

where $\omega = -(leg^*)^{-1}(d\lambda)$, $dE_L = V(L) - L$ with $leg: TE \rightarrow TE^*$ being the Legendre transformation, V the Liouville vertical vector field on TE and E_L the energy associated to the function L.

In this paper, all manifolds and mappings are assumed to be differentiable of class C^{∞}, unless otherwise stated and the sum is taken over repeated indices.

§1. THE LIFTS OF TENSOR FIELDS

Let $f: E \rightarrow \mathbf{R}$ be a function defined on the vector bundle $\pi=(E,\pi,M)$. Then the vertical, the complete and the complete-vertical lift of order (r,s) $(0\leq r\leq k,\ 0\leq s\leq k-r)$ of a function f defined on the vector bundle π to its the extended vector bundle $\pi^k=({}^kE,\pi^k,{}^kM)$ with respect to its adapted coordinate system $\{ x^{ri}, u^{r\alpha} : 0\leq r\leq k \}$ are defined

$$f_k^{V^k} = f^{V^k}|_{{}^kE} = f|_{\tau_{0_E}(\tau_{1_E}\ldots\tau_{k\text{-}1_E}({}^kE))} \circ \tau_{0_E}|_{\tau_{1_E}(\tau_{2_E}\ldots\tau_{k\text{-}1_E}({}^kE))} \circ \ldots \circ \tau_{k\text{-}1_E}|_{{}^kE}$$

$$f_k^{C^k} = (f_{k-1}^{C^{k-1}})^C|_{{}^kE} = (\partial_{ri} f_{k-1}^{C^{k-1}})^V|_{{}^kE}\, x^{r+1i} + (\partial_{r\alpha} f_{k-1}^{C^{k-1}})^V|_{{}^kE}\, u^{r+1\alpha}.$$

$$f_{r+s}^{C^rV^s} = f_k^{C^rV^s} = f_r^{C^r}|_{\tau_{r_E}(\tau_{r+1_E}\ldots\tau_{k\text{-}1_E}({}^kE))} \circ \tau_{r_E}|_{\tau_{r+1_E}(\tau_{r+2_E}\ldots\tau_{k\text{-}1_E}({}^kE))} \circ \ldots$$
$$\ldots \circ \tau_{k\text{-}1_E}|_{{}^kE}$$

respectively[2].

PROPOSITION 1. For all the functions $f,g \in \Im_0^0(\pi)$ and integer numbers $0\leq r\leq k$;

i) $(f+g)_r^{V^r} = f_r^{V^r} + g_r^{V^r}$, $(f+g)_r^{C^r} = f_r^{C^r} + g_r^{C^r}$, $f_k^{C^rV^s} = f_k^{V^sC^r}$

ii) $(f.g)_r^{V^r} = f_r^{V^r}.g_r^{V^r}$, $(f \cdot g)_r^{C^r} = \sum_{i=0}^{r} \binom{r}{i} f_r^{C^{r-i}V^i}\, g_r^{C^iV^{r-i}}$

iii) $(\frac{\partial f}{\partial x^{0i}})_r^{C^r} = \frac{\partial f_r^{C^r}}{\partial x^{0i}}$, $(\frac{\partial f}{\partial u^{0\alpha}})_r^{C^r} = \frac{\partial f_r^{C^r}}{\partial u^{0\alpha}}$, $(\frac{\partial f}{\partial x^{0i}})_r^{V^r} = \frac{\partial f_r^{C^r}}{\partial x^{ri}}$, $(\frac{\partial f}{\partial u^{0\alpha}})_r^{V^r} = \frac{\partial f_r^{C^r}}{\partial u^{r\alpha}}$

iv) $(\frac{\partial f}{\partial x^{0i}})_k^{C^{k-r}V^r} = \frac{\partial f_k^{C^k}}{\partial x^{ri}}$, $(\frac{\partial f}{\partial u^{0\alpha}})_k^{C^{k-r}V^r} = \frac{\partial f_k^{C^k}}{\partial u^{r\alpha}}$. [2]

Let X be a vector field defined on the vector bundle π and local expression of X $X=X^{0i}\frac{\partial}{\partial x^{0i}} + X^{0\alpha}\frac{\partial}{\partial u^{0\alpha}}$. In that case, the vertical, the complete and the complete-vertical lift of order $(r,s)(0\leq r\leq k\ 0\leq s\leq k-r)$ of a vector field X defined on the vector bundle π to its the extended vector bundle π^k are defined by the induction method for integer number k, then; the local expressions of them are

$$X_k^{V^k} = (X^{oi})_k^{V^k} \frac{\partial}{\partial x^{ki}} + (X^{o\alpha})_k^{V^k} \frac{\partial}{\partial u^{k\alpha}}$$

$$X_k^{C^k} = \sum_{r=0}^{k} \{\binom{k}{r}(X^{oi})_k^{C^r V^{k-r}} \frac{\partial}{\partial x^{ri}} + \binom{k}{r}(X^{o\alpha})_k^{C^r V^{k-r}} \frac{\partial}{\partial u^{r\alpha}}\}$$

$$X_k^{C^r V^s} = \sum_{t=0}^{k} \{\binom{r}{k-t} (X^{0i})_k^{C^{t-s} V^{s+k-t}} \frac{\partial}{\partial x^{ti}} + \binom{r}{k-t}(X^{0\alpha})_k^{C^{t-s} V^{s+k-t}} \frac{\partial}{\partial u^{t\alpha}}\}$$

respectively[2].

PROPOSITION 2. For all vector fields $X,Y\in \Im_0^1(\pi)$ and all function $f\in \Im_0^0(\pi)$

i) $(X+Y)_r^{V^r} = X_r^{V^r} + Y_r^{V^r}$, $(X+Y)_k^{C^k} = X_k^{C^k} + Y_k^{C^k}$

ii) $(fX)_r^{V^r} = f_r^{V^r} X_r^{V^r}$, $(fX)_k^{C^k} = \sum_{r=0}^{k} \binom{k}{r} f_k^{C^r V^{k-r}} X_k^{C^{k-r} V^r}$

iii) $(fX)_k^{C^r V^s} = \sum_{h=0}^{r} \binom{r}{h} f_k^{C^{r-h} V^{s+h}} X_k^{C^h V^{k-h}}$ $\quad 0\leq r,s\leq k\ (r+s=k)$ [2].

Let ω be a 1-form defined on the vector bundle π and $\omega=\omega_{0i}\, dx^{0i} + \omega_{0\alpha}\, dx^{0\alpha}$ be local expression of ω. In that case, the vertical, the complete and the complete-vertical lift of order $(r,s)(0\leq r\leq k\ 0\leq s\leq k-r)$ of a 1-form ω defined on the vector bundle π to its the extended vector bundle π^k are defined by the induction method for integer number k, then; the local expressions of them are

$$\omega_k^{V^k} = (\omega_{0i})_k^{V^k} dx^{0i} + (\omega_{0\alpha})_k^{V^k} du^{0\alpha}$$

$$\omega_k^{C^k} = \sum_{r=0}^{k} \{ (\omega_{0i})_k^{C^{k-r} V^r} dx^{ri} + (\omega_{0\alpha})_k^{C^{k-r} V^r} du^{r\alpha} \}$$

$$\omega_k^{C^r V^s} = \sum_{t=0}^{k} \{ \frac{\binom{r}{t}}{\binom{k}{t}} (\omega_{0i})_k^{C^{r-t} V^{s+t}} dx^{ti} + \frac{\binom{r}{t}}{\binom{k}{t}} (\omega_{r\alpha})_k^{C^{r-t} V^{s+t}} du^{t\alpha} \}$$

respectively[2].

PROPOSITION 3. For all 1-forms $\omega,\theta \in \Im_1^0(\pi)$ and all function $f \in \Im_0^0(\pi)$

i) $(\omega+\theta)_r^{V^r} = \omega_r^{V^r} + \theta_r^{V^r}$, $(\omega+\theta)_k^{C^k} = \omega_k^{C^k} + \theta_k^{C^k}$

ii) $(f\omega)_r^{V^r} = f_r^{V^r}\ \omega_r^{V^r}$, $(f\omega)_k^{C^k} = \sum_{r=0}^{k} \binom{k}{r} f_k^{C^r V^{k-r}}\ \omega_k^{C^{k-r}V^r}$

iii) $(f\omega)_k^{C^r V^s} = \sum_{h=0}^{r} \binom{r}{h} f_k^{C^{r-h}V^{s+h}}\ \omega_k^{C^h V^{k-h}}$ $0 \le r,s \le k$ $(r+s=k)$ [2].

§2. THE LIFTS OF LAGRANGE AND HAMILTON FORMALISMS

If $\beta(t) = (\beta_i, \beta_\alpha): R \to E$ is a curve in E, then we obtain the following Hamilton coordinates $\{ x^{ri}, u^{r\alpha} : 0 \le r \le k,\ 1 \le i \le m,\ 1 \le \alpha \le n\}$ on π^k

$$x^{ri} = \frac{d^r \beta_i(t)}{dt^r}, \quad \frac{d^r x^{0i}}{dt^r} = x^{ri}, \quad u^{r\alpha} = \frac{d^r \beta_\alpha(t)}{dt^r}, \quad \frac{d^r u^{0\alpha}}{dt^r} = u^{r\alpha}.$$

Now, let $J: TTE \to TTE$ be a almost tangent structure on TE then the action of J on vector fields $\xi = (\xi^{0i}, \xi^{1i}, \xi^{0\alpha}, \xi^{1\alpha}) \in \Im_0^1(TE)$ is locally characterized by

$$J(\xi) = \xi^{0i} \frac{\partial}{\partial x^{1i}} + \xi^{0\alpha} \frac{\partial}{\partial u^{1\alpha}}.$$

Moreover, the interior product induced by J is operator ι_J defined by

$$\iota_J(\omega)(X_1, \dots, X_r) = \sum_{i=1}^{r} \omega(X_1, \dots, J(X_i), \dots, X_r);\ \omega \in \Im_r^0(TE), X_i \in \Im_0^1(TE)$$

and exterior vertical derivation d_J is defined by

$$d_J = [\iota_J, d] = \iota_J d - d\iota_J .$$

Let $L: TE \to R$ be a function and consider the following the 2-form on TE and V Liouville vertical vector field

$$\omega_L = -dd_J L \ , \quad V = x^{1i} \frac{\partial}{\partial x^{1i}} + u^{1\alpha} \frac{\partial}{\partial u^{1\alpha}}$$

Then ω_L is symplectic if and only if L is regular. It can be shown that equation (1) takes the form

(2) $\qquad \iota_J(X)\omega_L = dE_L \ , \ dE_L = V(L) - L$

If ω_L is symplectic then (2) admits a unique solution X which is a semispray on TE. If we write (2) in local coordinates then the integral curves of X are solutions of the Euler-Lagrange equations on the vector bundle π:

(3) $$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x^{1i}}-\frac{\partial L}{\partial x^{0i}}+\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u^{1\alpha}}\right)-\frac{\partial L}{\partial u^{0\alpha}}=0\right.$$

Now, if we calculate the vertical lift of the Liouville vector field V then we get

$$V_k^{V^k}=x^{1i}\frac{\partial}{\partial x^{k+1i}}+u^{1\alpha}\frac{\partial}{\partial u^{k+1\alpha}}$$

and $V_k^{V^k}$ is called as <u>the vertical lift of Liouville vector field</u> to the extended vector bundle. Thus if <u>the vertical lift $L_k^{V^k}$ of L regular lagrange</u> is considered then we write

$$\iota_{J_k}(\tilde{X})\tilde{\omega}_{L_k^{V^k}}=dE_{L_k^{V^k}},\ \tilde{\omega}\in\Im_2^0({}^{k+1}E),\ \tilde{X}\in\Im_0^1({}^{k+1}E)$$

$$E_{L_k^{V^k}}=V_k^{V^k}(L_k^{V^k})-L_k^{V^k}=-L_k^{V^k},\ \tilde{\omega}_{L_k^{V^k}}=-dd_{J_k}L_k^{V^k}$$

(4) $$\iota_{J_k}(\tilde{X})dd_{J_k}L_k^{V^k}=dL_k^{V^k}$$

$L_k^{V^k}$ is regular lagrange therefore $\tilde{\omega}_{L_k^{V^k}}$ is symplectic, hence (4) admits a unique solution $\tilde{X}$ which is a semispray on ${}^{k+1}E$. If we write (4) in local coordinates then the integral curves of $\tilde{X}$ are solutions of <u>the vertical lift of Euler-Lagrange equations</u> to the extended vector bundle π^{k+1}:

(5) $$\frac{\partial}{\partial t}\left(\frac{\partial L_k^{V^k}}{\partial x^{1i}}\right)-\frac{\partial L_k^{V^k}}{\partial x^{0i}}+\frac{\partial}{\partial t}\left(\frac{\partial L_k^{V^k}}{\partial u^{1\alpha}}\right)-\frac{\partial L_k^{V^k}}{\partial u^{0\alpha}}=0\ .$$

Moreover, if the complete lift of Liouville vector field V to π^{k+1} is calculate then we get

$$V_k^{C^k}=\sum_{r=0}^{k}\binom{k}{r}x^{r+1i}\frac{\partial}{\partial x^{r+1i}}+\binom{k}{r}u^{r+1\alpha}\frac{\partial}{\partial u^{r+1\alpha}}$$

and $V_k^{C^k}$ is called as <u>the complete lift of the Liouville vector field</u> to π^{k+1}. Now, if the almost tangent structures $J_s:{}^{k+1}E\rightarrow{}^{k+1}E\ 1\leq s\leq k$ is defined by

$$J_s(\tilde{X})=\sum_{r=s}^{k+1}\tilde{X}^{r-si}\frac{\partial}{\partial x^{ri}}+\tilde{X}^{r-s\alpha}\frac{\partial}{\partial u^{r\alpha}};\ \tilde{X}\in\Im_0^1({}^{k+1}E)$$

$$\iota_{J_s}(\tilde{\omega})(\tilde{X}_1,\ldots,\tilde{X}_r)=\sum_{i=1}^{r}\tilde{\omega}(\tilde{X}_1,\ldots,J_s(\tilde{X}_i),\ldots,\tilde{X}_r);\ \tilde{\omega}\in\Im_r^0({}^{k+1}E),\tilde{X}_i\in\Im_0^1({}^{k+1}E)$$

$$d_{J_s}=\left[\iota_{J_s},d\right]=\iota_{J_s}d-d\iota_{J_s}$$

with respect to J_s and the complete lift $L_k^{C^k}$ of lagrange L is considered then we write

$$\iota_{J_k}(\tilde{X})\tilde{\omega}_{L_k^{C^k}} = dE_{L_k^{C^k}}, \tilde{\omega} \in \Im_2^0({}^{k+1}E),\ \tilde{X} \in \Im_0^1({}^{k+1}E),$$

$$\tilde{\omega}_{L_k^{C^k}} = -dd_{J_k} L_k^{C^k}$$

$$E_{L_k^{C^k}} = V_k^{C^k}(L_k^{C^k}) - L_k^{C^k} = (V[L])_k^{C^k} - L_k^{C^k} = (V(L)-L)_k^{C^k}$$

(6) $$\iota_{J_k}(\tilde{X}) dd_{J_k} L_k^{C^k} = d(V(L)-L)_k^{C^k}.$$

Hence $L_k^{C^k}$ is regular lagrange therefore $\tilde{\omega}_{L_k^{C^k}}$ is symplectic, and equation (6) admits a unique solution $\tilde{X}$ which is a semispray on ${}^{k+1}E$. If we write equation (6) in local coordinates then the integral curves of $\tilde{X}$ are solutions of the complete lift of Euler-Lagrange equations to the extended vector bundle π^{k+1}:

(7) $$\sum_{r=0}^{k}(-1)^r \frac{\partial^r}{\partial t^r}\left(\frac{\partial L_k^{C^k}}{\partial x^{ri}}\right) + (-1)^r \frac{\partial^r}{\partial t^r}\left(\frac{\partial L_k^{C^k}}{\partial u^{r\alpha}}\right) = 0$$

Now, if the Hamilton function $H: TE^* \to R$ is considered and the Hamilton vector field on vector bundle then we write

(8) $$X = \frac{\partial H}{\partial x_{1i}}\frac{\partial}{\partial x_{0i}} - \frac{\partial H}{\partial x_{0i}}\frac{\partial}{\partial x_{1i}} + \frac{\partial H}{\partial u_{1\alpha}}\frac{\partial}{\partial u_{0\alpha}} - \frac{\partial H}{\partial u_{0\alpha}}\frac{\partial}{\partial u_{1\alpha}}.$$

If the equation (1) is put in order again then we write

(9) $$\iota_J(X)(-d\lambda) = dH.$$

where λ is Liouville form defined on TE^*. If equation (9) is written with respect to Hamilton coordinates in TE^* then following Hamilton equations on the vector bundle are obtained:

(10) $$\frac{\partial x_{0i}}{\partial t} = \frac{\partial H}{\partial x_{1i}},\ \frac{\partial x_{1i}}{\partial t} = -\frac{\partial H}{\partial x_{0i}},\ \frac{\partial u_{0\alpha}}{\partial t} = \frac{\partial H}{\partial u_{1\alpha}},\ \frac{\partial u_{1\alpha}}{\partial t} = -\frac{\partial H}{\partial u_{0\alpha}}.$$

Now if the vertical lift of Hamilton Vector field X on TE^* is calculated with respect to properties of vertical lifting then we get

$$X_k^{V^k} = \frac{\partial H}{\partial x_{1i}}\frac{\partial}{\partial x_{ki}} - \frac{\partial H}{\partial x_{0i}}\frac{\partial}{\partial x_{k+1i}} + \frac{\partial H}{\partial u_{1\alpha}}\frac{\partial}{\partial u_{k\alpha}} - \frac{\partial H}{\partial x_{0i}}\frac{\partial}{\partial u_{k+1\alpha}}.$$

Furthermore if the equation (1) is put in order again then we write

(11) $$\iota_{J_k}(X_k^{V^k})(-d\lambda_k^{V^k}) = d(H_k^{V^k})$$

where $H_k^{V^k}$ is the vertical lift of H to ${}^{k+1}E^*$ and $\lambda_k^{V^k}$ is the vertical lift of λ to ${}^{k+1}E^*$.

Hence if the equation (11) is written with respect to Hamilton coordinates in ${}^{k+1}E^*$ then we are hold the following the vertical lift of Hamilton equations to the extended vector bundle:

(12) $$\frac{\partial x_{0i}}{\partial t}=\frac{\partial H_k^{V^k}}{\partial x_{1i}},\ \frac{\partial x_{1i}}{\partial t}=-\frac{\partial H_k^{V^k}}{\partial x_{0i}},\ \frac{\partial u_{0\alpha}}{\partial t}=\frac{\partial H_k^{V^k}}{\partial u_{1\alpha}},\ \frac{\partial u_{1\alpha}}{\partial t}=-\frac{\partial H_k^{V^k}}{\partial u_{0\alpha}}.$$

Moreover, if the complete lift of Hamilton vector field X on TE^* is calculated with respect to properties of vertical and complete lifting then we put

$$X_k^{C^k}=\sum_{r=0}^{k}\binom{k}{r}\left(\frac{\partial H_k^{C^rV^{k-r}}}{\partial x_{k-r+1i}}\frac{\partial}{\partial x_{ri}}-\frac{\partial H_k^{C^rV^{k-r}}}{\partial x_{k-ri}}\frac{\partial}{\partial x_{r+1i}}+\frac{\partial H_k^{C^rV^{k-r}}}{\partial u_{k-r+1\alpha}}\frac{\partial}{\partial u_{r\alpha}}-\frac{\partial H_k^{C^rV^{k-r}}}{\partial u_{k-r\alpha}}\frac{\partial}{\partial u_{r+1\alpha}}\right)$$

Furthermore if the equation (1) is put in order again then the following equation is written

(13) $$\iota_{J_k}(X_k^{C^k})(-d\lambda_k^{C^k})=dH_k^{C^k}$$

where $H_k^{C^k}$ is the complete lift of H to ${}^{k+1}E^*$ and $\lambda_k^{C^k}$ is the complete lift of λ to ${}^{k+1}E^*$. Hence if the equation (13) is written with respect to Hamilton coordinates in ${}^{k+1}E^*$ then the following the complete lift of Hamilton equations to the extended vector bundle are hold:

(14) $$\frac{\partial x_{ri}}{\partial t}=\frac{\partial H_k^{C^k}}{\partial x_{r+1i}},\ \frac{\partial x_{r+1i}}{\partial t}=-\frac{\partial H_k^{C^k}}{\partial x_{ri}},\ \frac{\partial u_{r\alpha}}{\partial t}=\frac{\partial H_k^{C^k}}{\partial u_{r+1\alpha}},\ \frac{\partial u_{r+1\alpha}}{\partial t}=-\frac{\partial H_k^{C^k}}{\partial u_{r\alpha}}.$$

§3. CONCLUSION

Lagrangian and Hamiltonian formalisms in generalized classical mechanics and field theory can be intrinsically characterized to the extended vector bundles of a vector bundle. Moreover a geometric approach of Lagrangian and Hamiltonian formalisms involving higher order derivatives is given by the hold results in this study.

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