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Existence of positive solutions for nonlinear three-point problems on time scales

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Abstract

In this paper, by using fixed point theorems in cones, we study the existence of at least one, two and three positive solutions of a nonlinear second-order three-point boundary value problem for dynamic equations on time scales. As an application, we also give some examples to demonstrate our results.

MSC: 34B18; 39A10

Keywords: Time scales; Boundary value problems; Positive solutions; Cone; Fixed point theorems

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [11] and is a rapidly expanding area of research. The study of time scales has led to many important applications, e.g., in the study of insect population models, neural networks, heat transfer and epidemic models [1]. Some basic definitions and theorems on time scales can be found in the book [5] and another excellent source on time scales is the book [6]. The existence problems of positive solutions for the TPBVP, especially on time scales, have attracted many authors' attention and concern (see [2,3,7,9,10,12,14,15]).

We are interested in the existence of multiple positive solutions of the following three-point boundary value problem (TPBVP):

$$\begin{cases} u^{\Delta\nabla}(t) + h(t) f(t, u(t)) = 0, \quad t \in [t_1, t_3] \subset \mathbb{T}, \\ u^{\Delta}(t_1) = 0, \quad \alpha u(t_3) + \beta u^{\Delta}(t_3) = u^{\Delta}(t_2), \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0 \leqslant t_1 < t_2 < t_3$, $\alpha > 0$ and $\beta > 1$.

We have organized the paper as follows. In Section 2, we give some lemmas which are needed later. In Section 3, we apply the Krassnoselskii's fixed point theorem to prove the existence of at least one positive solution to the TPBVP (1.1). In Section 4, conditions for the existence of at least two positive solutions to the TPBVP (1.1) are discussed by using Avery–Henderson fixed point theorem. In Section 5, to prove the existence of at least three positive solutions

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to the TPBVP (1.1) we will use the Legget–Williams fixed point theorem. The results are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

2. Preliminaries

In this section, we will employ several lemmas to prove the main results in this paper. These lemmas are based on the linear TPBVP

$$\begin{cases} u^{\Delta\nabla}(t) + y(t) = 0, & t \in [t_1, t_3] \subset \mathbb{T}, \\ u^{\Delta}(t_1) = 0, & \alpha u(t_3) + \beta u^{\Delta}(t_3) = u^{\Delta}(t_2). \end{cases} \tag{2.1}$$

Lemma 1. *Let $\alpha \neq 0$. Then, for $y \in C_{ld}[t_1, t_3]$ the TPBVP* (2.1) *has the unique solution*

$$u(t) = \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) y(s)\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} y(s)\nabla s - \int_{t_1}^{t} (t-s)y(s)\nabla s.$$

Proof. From $u^{\Delta\nabla}(t) + y(t) = 0$, we have

$$u(t) = u(t_1) + u^{\Delta}(t_1)(t - t_1) - \int_{t_1}^{t} (t-s)y(s)\nabla s.$$

By using the first boundary condition, we get

$$u(t) = u(t_1) - \int_{t_1}^{t} (t-s)y(s)\nabla s := A - \int_{t_1}^{t} (t-s)y(s)\nabla s.$$

From the other boundary condition, we obtain

$$\alpha A - \alpha \int_{t_1}^{t_3} (t_3 - s)y(s)\nabla s - \beta \int_{t_1}^{t_3} y(s)\nabla s = -\int_{t_1}^{t_2} y(s)\nabla s.$$

Since

$$A = \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) y(s)\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} y(s)\nabla s,$$

the equality

$$u(t) = \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) y(s)\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} y(s)\nabla s - \int_{t_1}^{t} (t-s)y(s)\nabla s$$

yields. □

Lemma 2. *Let $\alpha > 0$ and $\beta \geqslant 1$. If $y \in C_{ld}([t_1, t_3], [0, \infty))$, then the unique solution u of the TPBVP* (2.1) *satisfies*

$$u(t) \geqslant 0, \quad t \in [t_1, t_3] \subset \mathbb{T}.$$

Proof. It is clear that $u(t)$ is decreasing on $[t_1, t_3]$. Therefore, if $u(t_3) \geqslant 0$, then $u(t) \geqslant 0$ for $t \in [t_1, t_3]$.

$$\begin{aligned} u(t_3) &= \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) y(s)\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} y(s)\nabla s - \int_{t_1}^{t_3} (t_3-s)y(s)\nabla s \\ &= \frac{\beta - 1}{\alpha}\int_{t_1}^{t_2} y(s)\nabla s + \frac{\beta}{\alpha}\int_{t_2}^{t_3} y(s)\nabla s \\ &\geqslant 0. \end{aligned}$$

Hence the result holds. □

Let $\mathscr{B}$ denote the Banach space $C_{ld}[t_1, t_3]$ with the norm $\|u\| = \sup_{t\in[t_1,t_3]}|u(t)|$. Define the cone $P \subset \mathscr{B}$ by

$$P = \{u \in \mathscr{B}: u(t) \geqslant 0,\ u \text{ is concave and } u^{\Delta}(t_1) = 0\}. \tag{2.2}$$

Lemma 3. *If $u \in P$, then*

$$u(t) \geqslant \frac{t_3 - t}{t_3}\|u\|, \quad t \in [t_1, t_3] \subset \mathbb{T}, \tag{2.3}$$

where $\|u\| = \sup_{t\in[t_1,t_3]}|u(t)|$.

Proof. Since u is concave on $[t_1, t_3]$, $u^{\Delta}(t)$ is decreasing on $[t_1, t_3] \subset \mathbb{T}^k$. Then $u^{\Delta}(t) \leqslant u^{\Delta}(t_1) = 0$ for $t \in [t_1, t_3] \subset \mathbb{T}^k$ and $u(t)$ is decreasing on $[t_1, t_3] \subset \mathbb{T}$. Hence $\|u\| = \sup_{t\in[t_1,t_3]}|u(t)| = u(t_1)$.

Let

$$g(t) = u(t) - \frac{t_3 - t}{t_3}\|u\|, \quad t \in [t_1, t_3] \subset \mathbb{T}. \tag{2.4}$$

Since $g^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) \leqslant 0$, we know that the graph of g is concave on $[t_1, t_3] \subset \mathbb{T}$. We get

$$g(t_1) = \frac{t_1}{t_3}u(t_1) \geqslant 0$$

and

$$g(t_3) = u(t_3) \geqslant 0.$$

From the concavity of g, we have

$$g(t) \geqslant 0, \quad t \in [t_1, t_3] \subset \mathbb{T}. \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$u(t) \geqslant \frac{t_3 - t}{t_3}\|u\|, \quad t \in [t_1, t_3] \subset \mathbb{T}. \qquad \square$$

The solutions of the TPBVP (1.1) are the fixed points of the operator A defined by

$$Au(t) = \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s\right) h(s) f(s, u(s))\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} h(s) f(s, u(s))\nabla s - \int_{t_1}^{t} (t - s)h(s) f(s, u(s))\nabla s.$$

3. Existence of at least one positive solution

We will assume the following hypotheses:

(H1) $h \in C_{ld}([t_1, t_3], [0, \infty))$ and there exists $t_0 \in [t_1, t_3]$ such that $h(t_0) > 0$.

(H2) $f : [t_1, t_3] \times [0, \infty) \to [0, \infty)$ is continuous such that $f(t, .) > 0$ on any subset of $\mathbb{T}$ containing t_0.

We will need also the following (Krasnoselskii's) fixed point theorem [8] to prove the existence at least one positive solution to TPBVP (1.1).

Theorem 1 (*Guo and Lakshmikantham [8]*). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A: K \cap (\overline{\Omega_2}\backslash\Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leqslant \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geqslant \|u\|$ *for* $u \in K \cap \partial\Omega_2$;
or
(ii) $\|Au\| \geqslant \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \leqslant \|u\|$ *for* $u \in K \cap \partial\Omega_2$ *hold. Then A has a fixed point in* $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Theorem 2. *Let* $\alpha > 0$ *and* $\beta \geqslant 1$. *Assume conditions* $(H1)$, $(H2)$ *are satisfied. In addition, suppose there exist numbers* $0 < r < R < \infty$ *such that*

$$f(s,u) \leqslant \frac{1}{k_1} u \quad \text{if } 0 \leqslant u \leqslant r$$

and

$$f(s,u) \geqslant \frac{t_3}{k_2(t_3 - t_2)} u \quad \text{if } R \leqslant u < \infty,$$

where

$$k_1 = \int_{t_2}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) h(s) \nabla s + \int_{t_1}^{t_2} \left(t_3 + \frac{\beta - 1}{\alpha} - s \right) h(s) \nabla s$$

and

$$k_2 = \int_{t_1}^{t_2} \left(t_3 + \frac{\beta - 1}{\alpha} - s \right) h(s) \nabla s.$$

Then the TPBVP (1.1) *has at least one positive solution.*

Proof. Define the cone P as in (2.2). From (H1), (H2), Lemmas 2 and 3, $AP \subset P$. It is also easy to check that $A: P \to P$ is completely continuous. If $u \in P$ with $\|u\| = r$, then we get

$$\begin{aligned}
\|Au\| &= \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) h(s) f(s,u(s)) \nabla s - \frac{1}{\alpha} \int_{t_1}^{t_2} h(s) f(s,u(s)) \nabla s \\
&= \int_{t_2}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) h(s) f(s,u(s)) \nabla s + \int_{t_1}^{t_2} \left(t_3 + \frac{\beta - 1}{\alpha} - s \right) h(s) f(s,u(s)) \nabla s \\
&\leqslant \int_{t_2}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s \right) h(s) \frac{1}{k_1} u(s) \nabla s + \int_{t_1}^{t_2} \left(t_3 + \frac{\beta - 1}{\alpha} - s \right) h(s) \frac{1}{k_1} u(s) \nabla s \\
&\leqslant \|u\|.
\end{aligned}$$

So, if we set

$$\Omega_1 := \{u \in C_{ld}([t_1, t_3], \mathbb{R}) \colon \|u\| < r\},$$

then $\|Au\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_1$.

Let us now set

$$\Omega_2 := \left\{ u \in C_{ld}([t_1, t_3], \mathbb{R}) : \|u\| < \frac{t_3}{t_3 - t_2} R \right\}.$$

Then for $u \in P$ with $\|u\| = (t_3/(t_3 - t_2))R$, we have

$$u(t) \geqslant u(t_2) \geqslant \frac{t_3 - t_2}{t_3} \|u\| = R, \quad t \in [t_1, t_2].$$

Therefore from (2.3), we have

$$\begin{aligned}\|Au\| &= \int_{t_1}^{t_3}\left(t_3+\frac{\beta}{\alpha}-s\right)h(s)f(s,u(s))\nabla s-\frac{1}{\alpha}\int_{t_1}^{t_2}h(s)f(s,u(s))\nabla s\\ &= \int_{t_2}^{t_3}\left(t_3+\frac{\beta}{\alpha}-s\right)h(s)f(s,u(s))\nabla s+\int_{t_1}^{t_2}\left(t_3+\frac{\beta-1}{\alpha}-s\right)h(s)f(s,u(s))\nabla s\\ &\geqslant \int_{t_1}^{t_2}\left(t_3+\frac{\beta-1}{\alpha}-s\right)h(s)\frac{t_3}{k_2(t_3-t_2)}u(s)\nabla s\\ &\geqslant \|u\|.\end{aligned}$$

Hence, $\|Au\|\geqslant\|u\|$ for $u\in P\cap\partial\Omega_2$. Thus by the first part of Theorem 1, A has a fixed point u in $P\cap(\overline{\Omega_2}\backslash\Omega_1)$. Therefore, the TPBVP (1.1) has at least one positive solution. □

4. Existence of at least two positive solutions

In this section, we apply the Avery–Henderson fixed point theorem [4] to prove the existence of at least two positive solutions to the nonlinear TPBVP (1.1).

Theorem 3 (*Avery and Henderson [4]*). *Let P be a cone in a real Banach space E. Set*

$$P(\phi,r)=\{u\in P:\phi(u)<r\}.$$

If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0)=0$ such that, for some positive constants r and M,

$$\phi(u)\leqslant\theta(u)\leqslant\eta(u)\quad\text{and}\quad\|u\|\leqslant M\phi(u)$$

for all $u\in\overline{P(\phi,r)}$. Suppose that there exist positive numbers $p<q<r$ such that

$$\theta(\lambda u)\leqslant\lambda\theta(u)\quad\text{for all } 0\leqslant\lambda\leqslant 1\quad\text{and}\quad u\in\partial P(\theta,q).$$

If $A:\overline{P(\phi,r)}\to P$ is a completely continuous operator satisfying

(i) *$\phi(Au)>r$ for all $u\in\partial P(\phi,r)$,*
(ii) *$\theta(Au)<q$ for all $u\in\partial P(\theta,q)$,*
(iii) *$P(\eta,p)\neq\emptyset$ and $\eta(Au)>p$ for all $u\in\partial P(\eta,p)$, then A has at least two fixed points u_1 and u_2 such that*

$$p<\eta(u_1)\quad\text{with } \theta(u_1)<q\quad\text{and}\quad q<\theta(u_2)\quad\text{with } \phi(u_2)<r.$$

Theorem 4. *Assume $(H1)$, $(H2)$ hold and $\alpha>0$, $\beta>1$. Suppose there exist positive numbers $p<q<r$ such that the function f satisfies the following conditions*:

(i) *$f(s,u)>rM$ for $s\in[t_1,t_2]$ and $u\in[r,rt_3/(t_3-t_2)]$,*
(ii) *$f(s,u)<qm$ for $s\in[t_1,t_3]$ and $u\in[0,qt_3/(t_3-t_2)]$,*
(iii) *$f(s,u)>pM$ for $s\in[t_1,t_2]$ and $u\in[p(t_3-t_2)/t_3,p]$*

for some positive constants m and M. Then the TPBVP (1.1) has at least two positive solutions u_1 and u_2 such that

$$u_1(t_1)>p\quad\text{with } u_1(t_2)<q\quad\text{and}\quad u_2(t_2)>q\quad\text{with } u_2(t_2)<r.$$

Proof. Define the cone P as in (2.2). From (H1), (H2), Lemmas 2 and 3, $AP\subset P$. Moreover, A is a completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(u):=u(t_2),\quad\theta(u):=u(t_2),\quad\eta(u):=u(t_1).$$

For each $u \in P$,

$$\phi(u) = \theta(u) \leqslant \eta(u).$$

In addition, for each $u \in P$, $\phi(u) = u(t_2) \geqslant ((t_3 - t_2)/t_3)\|u\|$. Thus

$$\|u\| \leqslant \frac{t_3}{t_3 - t_2}\phi(u) \quad \text{for all } u \in P. \tag{4.1}$$

Also, $\theta(0) = 0$ and for all $u \in P$, $\lambda \in [0, 1]$ we have $\theta(\lambda u) = \lambda\theta(u)$.

We now verify that all of the conditions of Theorem 3 are satisfied.

If $u \in \partial P(\phi, r)$, from (4.1) we have $r = u(t_2) \leqslant u(s) \leqslant \|u\| \leqslant rt_3/(t_3 - t_2)$ for $s \in [t_1, t_2]$. Define

$$M := \frac{\alpha}{\beta - 1}\left(\int_{t_1}^{t_2} h(s)\nabla s\right)^{-1}. \tag{4.2}$$

Then from assumption (i), we have

$$\begin{aligned}
\phi(Au) &= \int_{t_1}^{t_3}\left(t_3 + \frac{\beta}{\alpha} - s\right)h(s)f(s, u(s))\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} h(s)f(s, u(s))\nabla s \\
&\quad - \int_{t_1}^{t_2}(t_2 - s)h(s)f(s, u(s))\nabla s \\
&= \int_{t_1}^{t_2}\left(t_3 - t_2 + \frac{\beta - 1}{\alpha}\right)h(s)f(s, u(s))\nabla s + \int_{t_2}^{t_3}\left(t_3 + \frac{\beta}{\alpha} - s\right)h(s)f(s, u(s))\nabla s \\
&\geqslant \frac{\beta - 1}{\alpha}\int_{t_1}^{t_2} h(s)f(s, u(s))\nabla s \\
&> \frac{\beta - 1}{\alpha} rM \int_{t_1}^{t_2} h(s)\nabla s \\
&= r.
\end{aligned}$$

Thus, condition (i) of Theorem 3 holds.

If $u \in \partial P(\theta, q)$, by (4.1) we have $0 \leqslant u(s) \leqslant \|u\| \leqslant qt_3/(t_3 - t_2)$ for $s \in [t_1, t_3]$. Define

$$m := \left(\int_{t_1}^{t_3}\left(t_3 + \frac{\beta}{\alpha} - s\right)h(s)\nabla s\right)^{-1}. \tag{4.3}$$

Then from assumption (ii), we have

$$\begin{aligned}
\theta(Au) &= \int_{t_1}^{t_3}\left(t_3 + \frac{\beta}{\alpha} - s\right)h(s)f(s, u(s))\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2} h(s)f(s, u(s))\nabla s \\
&\quad - \int_{t_1}^{t_2}(t_2 - s)h(s)f(s, u(s))\nabla s \\
&\leqslant \int_{t_1}^{t_3}\left(t_3 + \frac{\beta}{\alpha} - s\right)h(s)f(s, u(s))\nabla s \\
&< qm\int_{t_1}^{t_3}\left(t_3 + \frac{\beta}{\alpha} - s\right)h(s)\nabla s \\
&= q.
\end{aligned}$$

Hence condition (ii) of Theorem 3 holds.

Since $0 \in P$ and $p>0$, $P(\eta, p) \neq \emptyset$. If $u \in \partial P(\eta, p)$, from (2.3) we have $p(t_3-t_2)/t_3 \leqslant u(t_2) \leqslant u(s) \leqslant \|u\| = p$ for $s \in [t_1, t_2]$. Then from assumption (iii), we obtain

$$\begin{aligned}\eta(Au) &= \int_{t_1}^{t_3}\left(t_3+\frac{\beta}{\alpha}-s\right)h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\int_{t_1}^{t_2}h(s)f(s,u(s))\nabla s\\ &= \int_{t_1}^{t_3}(t_3-s)h(s)f(s,u(s))\nabla s + \frac{\beta-1}{\alpha}\int_{t_1}^{t_2}h(s)f(s,u(s))\nabla s\\ &\quad + \frac{\beta}{\alpha}\int_{t_2}^{t_3}h(s)f(s,u(s))\nabla s\\ &> \frac{\beta-1}{\alpha}pM\int_{t_1}^{t_2}h(s)\nabla s\\ &= p.\end{aligned}$$

Since all conditions of Theorem 3 are satisfied, the TPBVP (1.1) has at least two positive solutions u_1 and u_2 such that

$$u_1(t_1) > p \quad \text{with } u_1(t_2) < q \quad \text{and} \quad u_2(t_2) > q \quad \text{with } u_2(t_2) < r. \qquad \square$$

Example 1. Let $\mathbb{T} = [0, 1] \cup \{1 + 1/3^n : n \in \mathbb{N}_0\}$. We consider the following TPBVP

$$\begin{cases} u^{\Delta\nabla}(t) + t\mathrm{e}^{u/(u+1)} = 0, & t \in [0, 2] \subset \mathbb{T},\\ u^{\Delta}(0) = 0, & \frac{1}{2}u(2) + 3u^{\Delta}(2) = u^{\Delta}(\frac{3}{2}). \end{cases} \tag{4.4}$$

Taking $t_1 = 0$, $t_2 = \frac{3}{2}$, $t_3 = 2$, $\alpha = \frac{1}{2}$, $\beta = 3$ and $h(t) = t$, we have $m = 78/1129$ and $M = \frac{1}{41}$.

$$f(t, u) = f(u) := e^{u/(u+1)}$$

is continuous and increasing. Now we check that the conditions of Theorem 4 are satisfied. Let $r = 110$. Since $f(110) \approx 2.69$, we have

$$f(u) \geqslant 2.69 > rM \approx 2.68, \quad u \in [r, 4r].$$

It means that condition (i) of Theorem 4 is satisfied. Suppose $q = 40$. As $f(4q) \approx 2.70$, we get

$$f(u) \leqslant 2.70 < qm \approx 2.76, \quad u \in [0, 4q]$$

so that condition (ii) of Theorem 4 is met. Set $p = 32$. Since $f(\frac{p}{4}) \approx 2.43$, we obtain

$$f(u) \geqslant 2.43 > pM \approx 0.78, \quad u \in \left[\frac{p}{4}, p\right]$$

and condition (iii) of Theorem 4 is holds. So, the TPBVP (4.4) has at least two positive solutions u_1 and u_2 satisfying

$$u_1(0) > 32 \quad \text{with } u_1(\tfrac{3}{2}) < 40 \quad \text{and} \quad u_2(\tfrac{3}{2}) > 40 \quad \text{with } u_2(\tfrac{3}{2}) < 110.$$

5. Existence of at least three positive solutions

We will use the Legget–Williams fixed point theorem [13] to prove the existence of at least three positive solutions to the nonlinear TPBVP (1.1).

Theorem 5 (*Legget and Williams [13]*). *Let P be a cone in the real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\},$$

$$P(\psi, a, b) := \{x \in P : a \leqslant \psi(x), \|x\| \leqslant b\}.$$

Suppose $A:\overline{P_r}\to\overline{P_r}$ *be a completely continuous operator and* ψ *be a nonnegative continuous concave functional on* P *with* $\psi(u)\leqslant\|u\|$ *for all* $u\in\overline{P_r}$. *If there exists* $0<p<q<l\leqslant r$ *such that the following condition hold*:

(i) $\{u\in P(\psi,q,l):\psi(u)>q\}\neq\emptyset$ *and* $\psi(Au)>q$ *for all* $u\in P(\psi,q,l)$;
(ii) $\|Au\|<p$ *for* $\|u\|\leqslant p$;
(iii) $\psi(Au)>q$ *for* $u\in P(\psi,q,r)$ *with* $\|Au\|>l$, *then* A *has at least three fixed points* u_1,u_2 *and* u_3 *in* $\overline{P_r}$ *satisfying*

$$\|u_1\|<p,\quad \psi(u_2)>q,\quad p<\|u_3\|\quad \text{with } \psi(u_3)<q.$$

Theorem 6. *Assume* $(H1)$, $(H2)$ *hold and* $\alpha>0$, $\beta>1$. *Suppose that there exist constants* $0<p<q<qt_3/(t_3-t_2)\leqslant r$ *such that*

(i) $f(s.u)\leqslant rm$ *for* $s\in[t_1,t_3]$ *and* $u\in[0,r]$;
(ii) $f(s.u)>qM$ *for* $s\in[t_1,t_2]$ *and* $u\in[q,qt_3/(t_3-t_2)]$;
(iii) $f(s.u)<pm$ *for* $s\in[t_1,t_3]$ *and* $u\in[0,p]$.

Then the TPBVP (1.1) *has at least three positive solutions* u_1,u_2 *and* u_3 *satisfying*

$$u_1(t_1)<p,\quad u_2(t_2)>q,\quad u_3(t_1)>p\quad \text{with } u_3(t_2)<q.$$

Proof. The conditions of Theorem 5 will be shown to be satisfied. Define the nonnegative continuous concave functional $\psi:P\to[0,\infty)$ to be $\psi(u):=u(t_2)$, the cone P as in (2.2), M as in (4.2) and m as in (4.3). We have $\psi(u)\leqslant\|u\|$ for all $u\in P$. If $u\in\overline{P_r}$, then $\|u\|\leqslant r$ and from assumption (i) $f(s,u)\leqslant rm$. Then we have

$$\begin{aligned}\|Au\| &= \int_{t_1}^{t_3}\left(t_3+\frac{\beta}{\alpha}-s\right)h(s)f(s,u(s))\nabla s-\frac{1}{\alpha}\int_{t_1}^{t_2}h(s)f(s,u(s))\nabla s\\ &\leqslant rm\int_{t_1}^{t_3}\left(t_3+\frac{\beta}{\alpha}-s\right)h(s)\nabla s\\ &= r.\end{aligned}$$

Thus, we have $A:\overline{P_r}\to\overline{P_r}$. Since $qt_3/(t_3-t_2)\in P(\psi,q,qt_3/(t_3-t_2))$ and $\psi(qt_3/(t_3-t_2))=qt_3/(t_3-t_2)>q$, $\{u\in P(\psi,q,qt_3/(t_3-t_2)):\psi(u)>q\}\neq\emptyset$. For $u\in P(\psi,q,qt_3/(t_3-t_2))$, we have $q\leqslant u(t_2)\leqslant u(s)\leqslant\|u\|\leqslant qt_3/(t_3-t_2)$ for $s\in[t_1,t_2]$. Using assumption (ii), we obtain

$$\begin{aligned}\psi(Au) &= \int_{t_1}^{t_3}\left(t_3+\frac{\beta}{\alpha}-s\right)h(s)f(s,u(s))\nabla s-\frac{1}{\alpha}\int_{t_1}^{t_2}h(s)f(s,u(s))\nabla s\\ &\quad -\int_{t_1}^{t_2}(t_2-s)h(s)f(s,u(s))\nabla s\\ &\geqslant \frac{\beta-1}{\alpha}\int_{t_1}^{t_2}h(s)f(s,u(s))\nabla s\\ &> \frac{\beta-1}{\alpha}qM\int_{t_1}^{t_2}h(s)\nabla s\\ &= q.\end{aligned}$$

Hence, condition (i) of Theorem 5 holds.

If $\|u\| \leqslant p$, then $f(s,u) < pm$, $s \in [t_1, t_3]$ from assumption (iii). We obtain

$$\begin{aligned}\|Au\| &= \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s\right) h(s) f(s,u(s)) \nabla s - \frac{1}{\alpha} \int_{t_1}^{t_2} h(s) f(s,u(s)) \nabla s \\ &< pm \int_{t_1}^{t_3} \left(t_3 + \frac{\beta}{\alpha} - s\right) h(s) \nabla s \\ &= p.\end{aligned}$$

Consequently, condition (ii) of Theorem 5 holds.

For condition (iii) of Theorem 5, we suppose that $u \in P(\psi, q, r)$ with $\|Au\| > qt_3/(t_3 - t_2)$. Then, from (2.3) we get

$$\psi(Au) = Au(t_2) \geqslant \frac{t_3 - t_2}{t_3} \|Au\| > q.$$

Because all of the hypotheses of the Legget–Williams fixed point theorem are satisfied, the nonlinear TPBVP (1.1) has at least three positive solutions. $\square$

Example 2. Let $\mathbb{T} = [0,1] \cup [2,3]$. We consider the following TPBVP:

$$\begin{cases} u^{\Delta\nabla}(t) + \dfrac{2006u^5}{u^5 + 2007} = 0, \quad t \in [0,3] \subset \mathbb{T}, \\ u^{\Delta}(0) = 0, \quad u(3) + 2u^{\Delta}(3) = u^{\Delta}\left(\frac{5}{2}\right). \end{cases} \tag{5.1}$$

Taking $t_1 = 0$, $t_2 = \frac{5}{2}$, $t_3 = 3$, $\alpha = 1$, $\beta = 2$ and $h(t) = 1$, we have $m = \frac{1}{7}$ and $M = \frac{2}{3}$.

$$f(t,u) = f(u) := \frac{2006u^5}{u^5 + 2007}$$

is continuous and increasing on $[0, \infty)$. If we take $p = \frac{1}{2}$, $q = 2000$ and $r = 15\,000$, then

$$0 < p < q < \frac{qt_3}{t_3 - t_2} \leqslant r.$$

Now we check that the conditions of Theorem 6 are satisfied. From $\lim_{u\to\infty} f(u) = 2006$,

$$f(u) \leqslant 2006 < rm \approx 2142.85, \quad u \in [0, r]$$

so that condition (i) of Theorem 6 is met.

Since $f(2000) \approx 2006$, we have

$$f(u) > qM \approx 1333.33, \quad u \in [q, 6q].$$

It means that condition (ii) of Theorem 6 is satisfied.

Lastly, as $f(\frac{1}{2}) \approx 0.0312$,

$$f(u) < pm \approx 0.0714, \quad u \in [0, p]$$

and condition (iii) of Theorem 6 is holds. Finally, the TPBVP (5.1) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$u_1(0) < \tfrac{1}{2}, \quad u_2(\tfrac{5}{2}) > 2000, \quad u_3(0) > \tfrac{1}{2} \quad \text{with } u_3(\tfrac{5}{2}) < 2000.$$

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