



A STUDY ON ABSOLUTE EULER TOTIENT SERIES SPACE AND CERTAIN MATRIX TRANSFORMATIONS

Merve İLKHAN, Department of Mathematics, Duzce University, Turkey, merveilkhan@duzce.edu.tr

( <https://orcid.org/0000-0002-0831-1474>)

*G. Canan HAZAR GÜLEÇ, Department of Mathematics, Pamukkale University, Turkey, gchazar@pau.edu.tr

( <https://orcid.org/0000-0002-8825-5555>)

Received: 27.04.2020, Accepted: 04.06.2020

*Corresponding author

Research Article

DOI: 10.22531/muglajsci.727517

Abstract

Recently, many authors have focused on the studies related to sequence and series spaces. In the literature the simple and fundamental method is to construct new sequence and series spaces by means of the matrix domain of triangular matrices on the classical sequence spaces. Based on this approach, in this study, we introduce a new series space $|\phi_z|_p$ as the set of all series summable by absolute summability method $|\Phi, z_n|_p$, where $\Phi = (\phi_{nk})$ denotes Euler totient matrix, $z = (z_n)$ is a sequence of non-negative terms and $p \geq 1$. Also, we show that the series space $|\phi_z|_p$ is linearly isomorphic to the space of all p -absolutely summable sequences ℓ_p for $p \geq 1$. Moreover, we determine some topological properties and α , β and γ -duals of this space and give Schauder basis for the space $|\phi_z|_p$. Finally, we characterize the classes of the matrix operators from the space $|\phi_z|_p$ to the classical spaces $\ell_\infty, c, c_0, \ell_1$ for $1 \leq p < \infty$ and vice versa.

Keywords: Absolute Series Spaces, Matrix Operators, BK Spaces.

MUTLAK EULER TOTİENT SERİ UZAYI VE BAZI MATRİS DÖNÜŞÜMLERİ ÜZERİNE BİR ÇALIŞMA

Özet

Son zamanlarda birçok yazar dizi ve seri uzayları ile ilgili çalışmalara yoğunlaşmışlardır. Literatürde basit ve temel yaklaşım klasik dizi uzayları üzerinde üçgensel matrislerin matris etki alanı yardımıyla yeni dizi ve seri uzayları inşa etmektir. Bu çalışmada, bu yaklaşımdan yola çıkarak $|\Phi, z_n|_p$ mutlak toplanabilme metodu ile toplanabilen tüm serilerin uzayı olan yeni bir $|\phi_z|_p$ seri uzayı tanımlanmıştır, burada $\Phi = (\phi_{nk})$ Euler totient matrisini gösterir, $z = (z_n)$ terimleri negatif olmayan bir dizidir ve $p \geq 1$ dir. $|\phi_z|_p$ seri uzayının tüm mutlak p - toplanabilen dizilerin ℓ_p , $p \geq 1$, uzayına izomorf olduğu gösterilmiştir. Ayrıca, bu uzayın bazı topolojik özellikleri ile α , β and γ - dualleri belirlenmiştir ve $|\phi_z|_p$ uzayı için Schauder bazı verilmiştir. Son olarak, $|\phi_z|_p$ uzayından $\ell_\infty, c, c_0, \ell_1$ klasik dizi uzaylarına ve $\ell_\infty, c, c_0, \ell_1$ klasik dizi uzaylarından $|\phi_z|_p$ uzayına bazı matris operatörleri karakterize edilmiştir.

Anahtar Kelimeler: Mutlak Seri Uzayları, Matris Operatörleri, BK Uzayları.

Cite

İlhan, M., Hazar Güleç, G. C., (2020). "A study on absolute euler totient series space and certain matrix transformations", *Mugla Journal of Science and Technology*, 6(1), 112-119.

1. Introduction

Let ω be the space of all real valued sequences. Each linear subspace of ω is called a sequence space. We write ψ, ℓ_∞, c and c_0 for the sequence spaces of all finite, bounded, convergent, null sequences and also by bs, cs and ℓ_p ($1 \leq p < \infty$), we denote the spaces of all bounded, convergent and p -absolutely convergent series, respectively. Throughout the paper q denotes the conjugate of $p > 1$, i.e., $1/p + 1/q = 1$, and $1/q = 0$ for $p = 1$.

For the sequence spaces X and Y , define the set $S(X:Y)$ by

$$S(X:Y) = \{u = (u_k) \in \omega : xu = (x_k u_k) \in Y \quad (1) \\ \text{for all } x \in X\}.$$

With the notation in (1), α, β and γ -duals of a sequence space X , which are denoted by X^α, X^β and X^γ respectively, are defined by $X^\alpha = S(X : \ell_1)$, $X^\beta = S(X : cs)$ and $X^\gamma = S(X : bs)$.

Let $A = (a_{nk})$ be an infinite matrix of real numbers and X, Y be non-empty subsets of ω . We say that A defines a matrix mapping from X to Y , and we denote it by $A : X \rightarrow Y$, if for every sequence $x = (x_k) \in X$, $Ax = (A_n(x))$, the A -transform of x , is in Y , where the

series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ convergences for all n . By (X, Y) , we denote the class of all such matrices.

For any subset X of ω , the matrix domain X_A is introduced by

$$X_A = \{x \in \omega : Ax \in X\}. \quad (2)$$

A BK space $X \subset \omega$ is a Banach space with continuous coordinates $P_n : X \rightarrow \mathbb{C}$, where $P_n(x) = x_n$ for all $x \in X$, $n \geq 1$ and \mathbb{C} denotes the complex field. Also, a BK -space $X \supset \psi$ is said to have AK if every sequence $x \in X$ has a unique representation $x = \sum_{k=1}^{\infty} x_k e^{(k)}$, where $e^{(k)}$ denotes the sequence with $e_k^{(k)} = 1$ and $e_j^{(k)} = 0$ for $j \neq k$ [1]. For example, the sequence spaces ℓ_{∞} , c and c_0 are BK -spaces with the norm given by $\|x\|_{\ell_{\infty}} = \sup_k |x_k|$ and ℓ_p is a BK -space with the norm $\|x\|_{\ell_p} = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$, $1 \leq p < \infty$. Moreover, the spaces c_0 and ℓ_p ($1 \leq p < \infty$) have the property AK .

If $A = (a_{nk})$ is an infinite triangle matrix, i.e., $a_{nn} \neq 0$, and $a_{nk} = 0$ for $k > n$, there exists its unique inverse [2].

For a given $m \in \mathbb{N}$ with $m > 1$, Euler totient function φ is defined as the number of positive integers less than m that are coprime with m and $\varphi(1) = 1$.

If two numbers m and n are coprime, then $\varphi(mn) = \varphi(m)\varphi(n)$ and also $m = \sum_{d|m} \varphi(d)$ holds.

Consider the infinite matrix $\Phi = (\phi_{nk})$ such that

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & , \text{ if } k|n \\ 0 & , \text{ if } k \nmid n. \end{cases}$$

Schoenberg [3] has proved that this matrix is regular and defined that a sequence (x_n) of real numbers is φ -convergent to $\xi \in \mathbb{R}$ if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{d|n} \varphi(d) x_d = \xi.$$

This regular matrix is called as Euler totient matrix operator in [4] and some new sequence spaces have been introduced by using this matrix.

For any given $m \in \mathbb{N}$ with $m > 1$, Möbius function μ is defined as

$$\mu(m) = \begin{cases} (-1)^r & \text{if } m = p_1 p_2 \dots p_r, \text{ where } p_1, p_2, \dots, p_r \text{ are} \\ & \text{non equivalent prime numbers} \\ 0 & \text{if } p^2 | m \text{ for some prime number } p \end{cases}$$

and $\mu(1) = 1$. Also, if two numbers m and n are coprime, then $\mu(mn) = \mu(m)\mu(n)$ and $\sum_{d|m} \mu(d) = 0$ holds except for $m = 1$.

Let $\sum x_n$ be infinite series with n th partial sum s_n and (z_n) be a sequence of non-negative terms. The series $\sum x_n$ is said to be summable $|A, z_n|_p$, $p \geq 1$, if

$$\sum_{n=1}^{\infty} z_n^{p-1} |\Delta A_n(s)|^p < \infty,$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, for $n \geq 1$ [5]. This method includes the most of well known absolute summability methods. For example, if we take A as matrix of weighted mean (\overline{N}, t_n) (resp. $z_n = T_n/t_n$), then summability $|A, z_n|_p$ reduces to summability methods $|\overline{N}, t_n, z_n|_p$ (resp. $|\overline{N}, t_n|_p$ [6]) [7], where $t_n > 0$ for all n and $T_n = \sum_{k=0}^n t_k \rightarrow \infty$ as $n \rightarrow \infty$. Further, if A is matrix of Nörlund mean (N, t_n) , then summability $|A, z_n|_p$ is same as the summability $|N, t_n|_p$ given by Borwein and Cass [8] with $z_n = n$ for $n \geq 1$, which also includes absolute Cesàro summability $|C, \alpha|_p$ of Flett [9], where (t_n) is a sequence of complex numbers with $T_n = \sum_{k=0}^n t_k \neq 0$, $t_0 \neq 0$, $T_{-n} = 0$ for $n \geq 1$. In addition to all these classical methods, if we take A as Euler totient matrix $\Phi = (\phi_{nk})$, we obtain a new absolute summability method $|\Phi, z_n|_p$.

Many authors have constructed sequence spaces by means of the matrix domain of triangles on the classical sequence spaces. For some of the papers and applications, we refer to [10-31] and references therein.

In this paper, we introduce a new series space by using the Euler totient matrix and determine α , β and γ -duals of this space. Finally, we characterize the classes of matrix operators between the classical spaces ℓ_{∞} , c , c_0 , ℓ_1 and this new space.

2. The Series Space $|\phi_z|_p$

Now, we introduce the series space $|\phi_z|_p$ as the set of all series summable by absolute summability method $|\Phi, z_n|_p$ as follows.

$$|\phi_z|_p = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} z_n^{p-1} |\Delta \Phi_n(s)|^p < \infty \right\},$$

where $(\Phi_n(s))$ Euler totient transform of the sequence (s_n) , that is, $\Phi_n(s) = \sum_{k=1}^{\infty} \phi_{nk} s_k$.

Note that since (s_n) is the sequence of partial sum of infinite series $\sum x_n$, we can write Euler totient transform $(\Phi_n(s))$ of the sequence (s_n) by

$$\begin{aligned} \Phi_n(s) &= \sum_{k=1}^{\infty} \phi_{nk} s_k = \sum_{k=1}^n \phi_{nk} \sum_{j=1}^k x_j \\ &= \sum_{j=1}^n x_j \sum_{\substack{k=j \\ k|n}}^n \frac{\varphi(k)}{n}. \end{aligned}$$

Thus, we obtain that

$$|\phi_z|_p = \{ x = (x_n) \in \omega :$$

$$\sum_{n=1}^{\infty} z_n^{p-1} \left| \sum_{j=1}^{n-1} x_j \left(\sum_{\substack{k=j \\ k|n}}^n \frac{\varphi(k)}{n} - \sum_{\substack{k=j \\ k|n-1}}^{n-1} \frac{\varphi(k)}{n-1} \right) + x_n \frac{\varphi(n)}{n} \right|^p < \infty \}.$$

If we define the matrices $E^{(p)} = (e_{nk}^{(p)})$, $1 \leq p < \infty$ and $F = (f_{nk})$ by

$$e_{nk}^{(p)} = \begin{cases} -z_n^{1/q}, k = n-1 \\ z_n^{1/q}, k = n \\ 0, \text{ otherwise} \end{cases} \quad (3)$$

and

$$f_{nk} = \begin{cases} \frac{1}{n} \sum_{j=k, j|n}^n \varphi(j), 1 \leq k \leq n \\ 0, k > n, \end{cases} \quad (4)$$

then, we can write that $(E^{(p)} \circ F)_n(x) = z_n^{1/q} (F_n(x) - F_{n-1}(x))$ for $n \geq 1$, where $F_n(x) = \sum_{j=1}^n x_j \sum_{\substack{k=j \\ k|n}}^n \frac{\varphi(k)}{n}$ and $F_0(x) = 0$.

So we may restate $|\phi_z|_p = (\ell_p)_{E^{(p)} \circ F}$ according to the notation of matrix domain (2).

Also, since the matrices $E^{(p)}$ and F are triangles, they have the unique inverses and we denote these inverses by $(E^{(p)})^{-1} = \hat{E}^{(p)}$ and $F^{-1} = \hat{F}$ for brevity. Further, we can calculate these matrices $\hat{E}^{(p)} = (\hat{e}_{nk}^{(p)})$ and $\hat{F} = (\hat{f}_{nk})$ by

$$\hat{e}_{nk}^{(p)} = \begin{cases} z_k^{-1/q}, 1 \leq k \leq n \\ 0, k > n \end{cases} \quad (5)$$

and

$$\hat{f}_{nk} = \begin{cases} \frac{\mu\left(\frac{n}{k}\right)k}{\varphi(n)}, k|n \\ -\frac{\mu\left(\frac{n-1}{k}\right)k}{\varphi(n-1)}, k|n-1 \\ \frac{\mu(n)}{\varphi(n)} - \frac{\mu(n-1)}{\varphi(n-1)}, k = 1 \\ 0, \text{ otherwise.} \end{cases} \quad (6)$$

Theorem 2.1 Let $1 \leq p < \infty$ and the matrices $E^{(p)}$ and F be defined by (3) and (4), respectively. Then, the space $|\phi_z|_p$ is a BK space with respect to the norm

$$\|x\|_{|\phi_z|_p} = \|E^{(p)} \circ F(x)\|_{\ell_p}$$

and norm isomorphic to the space ℓ_p , that is, $|\phi_z|_p \cong \ell_p$.

Proof. Since ℓ_p is a BK space and $E^{(p)} \circ F$ is a triangle matrix and $|\phi_z|_p = (\ell_p)_{E^{(p)} \circ F}$, the space $|\phi_z|_p$ is a BK space by Theorem 4.3.2 in [2].

Further, consider the transformations $F : |\phi_z|_p \rightarrow (\ell_p)_{E^{(p)}}$ and $E^{(p)} : (\ell_p)_{E^{(p)}} \rightarrow \ell_p$. Since F and $E^{(p)}$ are linear bijections, then, it is clear that composite function $E^{(p)} \circ F$ is a linear bijective operator. In fact, if $(E^{(p)} \circ F)(x) = \theta$, then $x = \theta$, so $E^{(p)} \circ F$ is injective. Also, let $u = (u_k) \in \ell_p$ be given. Then, since $y = (y_n) = (\sum_{k=1}^n z_k^{-1/q} u_k) \in (\ell_p)_{E^{(p)}}$, we get

$$x = (x_n) = \left(\sum_{k|n} \frac{\mu\left(\frac{n}{k}\right)k}{\varphi(n)} y_k - \sum_{k|n-1} \frac{\mu\left(\frac{n-1}{k}\right)k}{\varphi(n-1)} y_k \right) \in |\phi_z|_p.$$

This gives that $u = (E^{(p)} \circ F)(x) \in \ell_p$, so $E^{(p)} \circ F$ is surjective. Furthermore, $E^{(p)} \circ F$ preserves the norm since

$$\|E^{(p)} \circ F(x)\|_{\ell_p} = \|x\|_{|\phi_z|_p}.$$

Note that the collection of all finite subsets of \mathbb{N} is denoted by \mathcal{N} .

Lemma 2.2 [32]

a) $T = (t_{nk}) \in (\ell_1, c)$ if and only if

$$\lim_n t_{nk} \text{ exists for each } k \geq 1 \quad (7)$$

and

$$\sup_{n,k} |t_{nk}| < \infty. \quad (8)$$

b) $T = (t_{nk}) \in (\ell_1, \ell_\infty)$ if and only if (8) holds.

c) Let $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, c)$ if and only if (7) holds and

$$\sup_n \sum_{k=1}^{\infty} |t_{nk}|^q < \infty. \quad (9)$$

d) Let $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, \ell_\infty)$ if and only if (9) holds.

e) Let $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, \ell_1)$ if and only if

$$\sup_{N \in \mathcal{N}} \sum_k \left| \sum_{n \in N} t_{nk} \right|^q < \infty.$$

Lemma 2.3 [33] Let $1 \leq p < \infty$. $T = (t_{nk}) \in (\ell_1, \ell_p)$ if and only if

$$\sup_k \sum_{n=1}^{\infty} |t_{nk}|^p < \infty.$$

Using following notations and Lemmas 2.2-2.3, we state following theorem related to α, β and γ -duals of the series space $|\phi_z|_p$.

$$\Omega_1 = \{ a = (a_j) \in \omega : \lim_m \left(\sum_{j=r}^m \sum_{k=r}^j a_j \hat{f}_{jk} \right) \text{ exists, } r \geq 1 \},$$

$$\Omega_2 = \left\{ a = (a_j) \in \omega : \sup_{m,r} \left| \sum_{j=r}^m \sum_{k=r}^j a_j \hat{f}_{jk} \right| < \infty \right\},$$

$$\Omega_3 = \{ a = (a_j) \in \omega : \sup_m \sum_{r=1}^m \left| z_r^{-1/q} \sum_{j=r}^m \sum_{k=r}^j a_j \hat{f}_{jk} \right|^q < \infty \},$$

$$\Omega_4 = \left\{ a = (a_j) \in \omega : \sup_r \sum_{n=r}^{\infty} \left| \sum_{k=r}^n a_n \hat{f}_{nk} \right| < \infty \right\},$$

$$\Omega_5 = \{ a = (a_j) \in \omega : \sup_{N \in \mathbb{N}'} \sum_r \left| \sum_{n \in N} \sum_{k=r}^n a_n \hat{f}_{nk} z_r^{-1/q} \right|^q < \infty \}$$

Theorem 2.4 Let $\hat{F} = (\hat{f}_{nk})$ be defined by (6). Then, we have:

a) $(|\phi_z|_p)^\beta = \Omega_1 \cap \Omega_3$ for $1 < p < \infty$ and $(|\phi_z|_1)^\beta = \Omega_1 \cap \Omega_2$.

b) $(|\phi_z|_p)^\gamma = \Omega_3$ for $1 < p < \infty$ and $(|\phi_z|_1)^\gamma = \Omega_2$.

c) $(|\phi_z|_p)^\alpha = \Omega_5$ for $1 < p < \infty$ and $(|\phi_z|_1)^\alpha = \Omega_4$.

Proof. a) Let $1 < p < \infty$. $a \in (|\phi_z|_p)^\beta$ if and only if $ax \in cs$ for every $x \in |\phi_z|_p$. Let $y = F(x)$. Then, $u \in \ell_p$, where $u_n = z_n^{1/q} (y_n - y_{n-1})$ for $n \geq 1$, $y_0 = 0$, and also we have $y_n = \sum_{k=1}^n z_k^{-1/q} u_k$. Since we have

$$\begin{aligned} x_n &= \sum_{k|n} \frac{\mu\left(\frac{n}{k}\right)k}{\varphi(n)} y_k - \sum_{k|n-1} \frac{\mu\left(\frac{n-1}{k}\right)k}{\varphi(n-1)} y_k \\ &= \sum_{k=1}^n \hat{f}_{nk} y_k, \end{aligned}$$

we obtain that

$$\begin{aligned} \sum_{j=1}^m a_j x_j &= \sum_{j=1}^m a_j \sum_{k=1}^j \hat{f}_{jk} y_k \\ &= \sum_{k=1}^m \left\{ \sum_{j=k}^m a_j \hat{f}_{jk} \right\} y_k \\ &= \sum_{r=1}^m z_r^{-1/q} \left(\sum_{j=r}^m \sum_{k=r}^j a_j \hat{f}_{jk} \right) u_r \\ &= \sum_{r=1}^m d_{mr} u_r, \end{aligned}$$

where the matrix $D = (d_{mr})$ is given by

$$d_{mr} = \begin{cases} z_r^{-1/q} \left(\sum_{j=r}^m \sum_{k=r}^j a_j \hat{f}_{jk} \right), & 1 \leq r \leq m \\ 0, & r > m. \end{cases} \quad (10)$$

So it is written by part c) of Lemma 2.2 that $a \in (|\phi_z|_p)^\beta$ iff $D \in (\ell_p, c)$, or equivalently, $a \in \Omega_1 \cap \Omega_3$, which completes the proof.

Since the proof for $p = 1$ is similar by using part a) of Lemma 2.2, we omit the detail.

b) Let $1 < p < \infty$. Then, $a \in (|\phi_z|_p)^\gamma$ if and only if $ax \in bs$ for every $x \in |\phi_z|_p$. Also, $x \in |\phi_z|_p$ iff $u \in \ell_p$, where $u_n = z_n^{1/q} (y_n - y_{n-1})$, $y_0 = 0$ and $y_n = \sum_{j=1}^n x_j \sum_{k=j}^n \frac{\varphi(k)}{n}$ for $n \geq 1$. Thus, since we have

$$\sum_{r=1}^m a_r x_r = \sum_{r=1}^m d_{mr} u_r,$$

where $D = (d_{mr})$ is defined by (10), this implies that $a \in (|\phi_z|_p)^\gamma$ iff $D \in (\ell_p, \ell_\infty)$. Hence, it follows from part d) of Lemma 2.2 that $a \in \Omega_3$ as asserted.

Since the proof for $p = 1$ is similar by using part b) of Lemma 2.2, we omit the detail.

c) Let $1 < p < \infty$. Then, $a \in (|\phi_z|_p)^\alpha$ if and only if $ax \in \ell_1$ for every $x \in |\phi_z|_p$. Then, we get

$$\begin{aligned} a_n x_n &= a_n \sum_{k=1}^n \hat{f}_{nk} y_k = a_n \sum_{k=1}^n \hat{f}_{nk} \sum_{r=1}^k z_r^{-1/q} u_r \\ &= \sum_{r=1}^n \sum_{k=r}^n a_n \hat{f}_{nk} z_r^{-1/q} u_r = \delta_n(u), \end{aligned}$$

where $\delta_n = (\delta_{nr})$ is defined by

$$\delta_{nr} = \sum_{k=r}^n a_n \hat{f}_{nk} z_r^{-1/q}.$$

So, $ax \in \ell_1$ for every $x \in |\phi_z|_p$ if and only if $\delta(u) \in \ell_1$ for every $u \in \ell_p$, or equivalently, $a \in (|\phi_z|_p)^\alpha$ iff $\delta \in (\ell_p, \ell_1)$, which gives $a \in \Omega_5$ from Lemma 2.2, as desired.

Since the proof for $p = 1$ is similar by using Lemma 2.3, we omit the detail.

Theorem 2.5 Let $1 \leq p < \infty$, $\hat{F} = (\hat{f}_{nk})$ and $\tau^{(r)} = (\tau_j^{(r)})$ be defined by (6) and

$$\tau_j^{(r)} = \begin{cases} z_r^{-1/q} \sum_{k=r}^j \hat{f}_{jk}, & r \leq j \\ 0, & r > j. \end{cases}$$

respectively. Then, the sequence $(\tau_j^{(r)})$ is a Schauder base of the space $|\phi_z|_p$.

Proof. It is known that the sequence $(e^{(n)})$ is a Schauder base for the space ℓ_p , where $e^{(n)}$ is a sequence with 1 in n -th place and zeros elsewhere. Because of the transformation $E^{(p)} \circ F$ defined in the proof of Theorem 2.1 is an isomorphism, the inverse image $(E^{(p)} \circ F)^{-1}$ of $(e^{(n)})$ is a Schauder basis for $|\phi_z|_p$. In fact, if $x \in |\phi_z|_p$,

then there exists $u \in \ell_p$ such that $u = (E^{(p)} \circ F)(x)$, so we can deduce from Theorem 2.1 that

$$\left\| x - \sum_{r=1}^m x_r \tau^{(r)} \right\|_{|\phi_z|_p} = \left\| u - \sum_{r=1}^m u_r e^{(r)} \right\|_{\ell_p} \rightarrow 0$$

as $m \rightarrow \infty$, where $(E^{(p)} \circ F)^{-1}(e^{(r)}) = \tau^{(r)}$, $r \geq 1$. Furthermore, every $x \in |\phi_z|_p$ has a unique representation of the form $x = \sum_{r=1}^{\infty} x_r \tau^{(r)}$.

3. Some Matrix Operators

In this section, we firstly characterize the matrix classes from the space $|\phi_z|_p$ to the classical spaces $\ell_{\infty}, c, c_0, \ell_1$ for $1 \leq p < \infty$.

Lemma 3.1 [32]

a) $T = (t_{nk}) \in (\ell_1, c_0)$ if and only if (8) holds and

$$\lim_n t_{nk} = 0 \text{ for each } k \geq 1. \quad (11)$$

b) Let $1 < p < \infty$. $T = (t_{nk}) \in (\ell_p, c_0)$ if and only if (9) and (11) hold.

Theorem 3.2 Let define the matrix $R^1 = (r_{nk}^1)$ with a matrix $T = (t_{nk})$ as

$$r_{nk}^1 = \lim_{m \rightarrow \infty} \sum_{j=k}^m t_{nj} \sum_{l=k}^j \hat{f}_{jl}$$

for all $n, k \in \mathbb{N}$.

• $T \in (|\phi_z|_1, \ell_{\infty})$ if and only if

$$R^1 = (r_{nk}^1) \text{ is well defined for all } n, k \in \mathbb{N} \quad (12)$$

$$\sup_{m,k} \left| \sum_{j=k}^m t_{nj} \sum_{l=k}^j \hat{f}_{jl} \right| < \infty \text{ for each } n \in \mathbb{N} \quad (13)$$

$$\sup_{n,k} |r_{nk}^1| < \infty. \quad (14)$$

• $T \in (|\phi_z|_1, c)$ if and only if (12), (13) and (14) hold, and

$$\lim_{n \rightarrow \infty} r_{nk}^1 \text{ exists for each } k \in \mathbb{N}.$$

• $T \in (|\phi_z|_1, c_0)$ if and only if (12), (13) and (14) hold, and

$$\lim_{n \rightarrow \infty} r_{nk}^1 = 0 \text{ for each } k \in \mathbb{N}.$$

• $T \in (|\phi_z|_1, \ell_1)$ if and only if (12) and (13) hold, and

$$\sup_k \sum_n |r_{nk}^1| < \infty.$$

Proof. The proof is given only for the first case since others can be proved similarly.

• $T \in (|\phi_z|_1, \ell_{\infty})$ if and only if $Tx \in \ell_{\infty}$ for all $x \in |\phi_z|_1$. Since the series $\sum_{k=0}^{\infty} t_{nk} x_k$ is convergent, we have that $(t_{nk}) \in (|\phi_z|_1)^{\beta}$ for each fixed $n \in \mathbb{N}$. By Theorem 2.4, we obtain that

$$\lim_{m \rightarrow \infty} \sum_{j=k}^m \sum_{l=k}^j t_{nj} \hat{f}_{jl}$$

exists for each $n, k \in \mathbb{N}$ and

$$\sup_{m,k} \left| \sum_{j=k}^m \sum_{l=k}^j t_{nj} \hat{f}_{jl} \right| < \infty \text{ for each } n \in \mathbb{N}.$$

That is, (12) and (13) hold. Now, we prove the necessity and sufficiency of (14). Consider the linear operator $E^{(1)} \circ F : |\phi_z|_1 \rightarrow \ell_1$. Let $y = Fx$ and $v = \Delta y = (E^{(1)} \circ F)x$ for any $x \in |\phi_z|_1$. Then we have $y_n = \sum_{k=1}^n v_k$. Hence it follows that

$$\begin{aligned} \sum_{k=1}^m t_{nk} x_k &= \sum_{k=1}^m t_{nk} \sum_{j=1}^k \hat{f}_{kj} y_j \\ &= \sum_{k=1}^m t_{nk} \sum_{j=1}^k \hat{f}_{kj} \sum_{l=1}^j v_l \\ &= \sum_{k=1}^m t_{nk} \sum_{j=1}^k \sum_{l=j}^k \hat{f}_{kl} v_j \\ &= \sum_{j=1}^m \left(\sum_{k=j}^m t_{nk} \sum_{l=j}^k \hat{f}_{kl} \right) v_j \\ &= \sum_{j=1}^m \tilde{r}_{mj}^1 v_j, \end{aligned}$$

where

$$\tilde{r}_{mj}^1 = \begin{cases} \sum_{k=j}^m t_{nk} \sum_{l=j}^k \hat{f}_{kl} & , 1 \leq j \leq m \\ 0 & , j > m \end{cases}$$

for each $n \in \mathbb{N}$. Also, it can be deduced by (12) and (13) that $\tilde{R}^1 = (\tilde{r}_{mj}^1) \in (\ell_1, c)$. Then the series $\tilde{R}_m^1(v) = \sum_{j=1}^{\infty} \tilde{r}_{mj}^1 v_j$ converges uniformly in m for all $v \in \ell_1$ and so we have $\lim_{m \rightarrow \infty} \tilde{R}_m^1(v) = \sum_{j=1}^{\infty} \lim_{m \rightarrow \infty} \tilde{r}_{mj}^1 v_j$. Thus, we obtain that

$$\begin{aligned} T_n(x) &= \lim_{m \rightarrow \infty} \tilde{R}_m^1(v) = \sum_{j=1}^{\infty} \left(\lim_{m \rightarrow \infty} \tilde{r}_{mj}^1 \right) v_j \\ &= \sum_{j=1}^{\infty} r_{nj}^1 v_j = R_n^1(v). \end{aligned}$$

This yields that $Tx \in \ell_{\infty}$ for $x \in |\phi_z|_1$ if and only if $R^1 v \in \ell_{\infty}$ for $v \in \ell_1$. We conclude that $T \in (|\phi_z|_1, \ell_{\infty})$ if and only (12) and (13) hold and also $R^1 \in (\ell_1, \ell_{\infty})$ which means (14).

Theorem 3.3 Let $1 < p < \infty$ and define the matrix $R^p = (r_{nk}^p)$ with a matrix $T = (t_{nk})$ as

$$r_{nk}^p = z_k^{-1/q} \lim_{m \rightarrow \infty} \sum_{j=k}^m t_{nj} \sum_{l=k}^j \hat{f}_{jl}$$

for all $n, k \in \mathbb{N}$.

• $T \in (|\phi_z|_p, \ell_\infty)$ if and only if (12) holds and

$$\sup_m \sum_{k=1}^m \left| z_k^{-1/q} \sum_{j=k}^m t_{nj} \sum_{l=k}^j \hat{f}_{jl} \right|^q < \infty \text{ for each } n \in \mathbb{N}, \quad (15)$$

$$\sup_n \sum_k |r_{nk}^p|^q < \infty. \quad (16)$$

• $T \in (|\phi_z|_p, c)$ if and only if (12), (15) and (16) hold, and

$$\lim_{n \rightarrow \infty} r_{nk}^p \text{ exists for each } k \in \mathbb{N}.$$

• $T \in (|\phi_z|_p, c_0)$ if and only if (12), (15) and (16) hold, and

$$\lim_{n \rightarrow \infty} r_{nk}^p = 0 \text{ for each } k \in \mathbb{N}.$$

• $T \in (|\phi_z|_p, \ell_1)$ if and only if (12) and (15) hold, and

$$\sup_{N \in \mathbb{N}} \sum_k \left| \sum_{n \in \mathbb{N}} r_{nk}^p \right|^q < \infty.$$

Proof. The proof is given only for the first case since others can be proved similarly.

• $T \in (|\phi_z|_p, \ell_\infty)$ if and only if $Tx \in \ell_\infty$ for all $x \in |\phi_z|_p$. Since the series $\sum_{k=1}^\infty t_{nk} x_k$ is convergent, we have that $(t_{nk}) \in (|\phi_z|_p)^\beta$ for each fixed $n \in \mathbb{N}$. From Theorem 2.4, we obtain that (12) holds and

$$\sup_m \sum_{k=1}^m \left| z_k^{-1/q} \sum_{j=k}^m \sum_{l=k}^j t_{nj} \hat{f}_{jl} \right|^q < \infty,$$

for each $n \in \mathbb{N}$.

Now, we prove the necessity and sufficiency of (16). Consider the linear operator $E^{(p)} \circ F : |\phi_z|_p \rightarrow \ell_p$ defined by $(E^{(p)} \circ F)_n(x) = z_n^{1/q} (F_n(x) - F_{n-1}(x))$, $n \geq 1$ and $F_0(x) = 0$. Let $y = Fx$ and $v = (E^{(p)} \circ F)x$ for any $x \in |\phi_z|_p$. Then we have $y_n = \sum_{k=1}^n z_k^{-1/q} v_k$. Hence it follows that

$$\begin{aligned} \sum_{k=1}^m t_{nk} x_k &= \sum_{k=1}^m t_{nk} \sum_{j=1}^k \hat{f}_{kj} y_j \\ &= \sum_{k=1}^m t_{nk} \sum_{j=1}^k \hat{f}_{kj} \sum_{l=1}^j z_l^{-1/q} v_l \\ &= \sum_{k=1}^m t_{nk} \sum_{j=1}^k \sum_{l=j}^k \hat{f}_{kl} z_j^{-1/q} v_j \\ &= \sum_{j=1}^m \left(z_j^{-1/q} \sum_{k=j}^m t_{nk} \sum_{l=j}^k \hat{f}_{kl} \right) v_j \\ &= \sum_{j=1}^m \tilde{r}_{mj}^p v_j, \end{aligned}$$

$$\text{where } \tilde{r}_{mj}^p = \begin{cases} z_j^{-1/q} \sum_{k=j}^m t_{nk} \sum_{l=j}^k \hat{f}_{kl} & , 1 \leq j \leq m \\ 0 & , j > m \end{cases}$$

for each $n \in \mathbb{N}$. Also, it can be deduced by (12) and (15) that $\tilde{R}^p = (\tilde{r}_{mj}^p) \in (\ell_p, c)$. Then the series $\tilde{R}_m^p(v) = \sum_{j=1}^\infty \tilde{r}_{mj}^p v_j$ converges uniformly in m for all $v \in \ell_p$ and so we have $\lim_{m \rightarrow \infty} \tilde{R}_m^p(v) = \sum_{j=1}^\infty \lim_{m \rightarrow \infty} \tilde{r}_{mj}^p v_j$. Thus, we obtain that

$$\begin{aligned} T_n(x) &= \lim_{m \rightarrow \infty} \tilde{R}_m^p(v) = \sum_{j=1}^\infty \left(\lim_{m \rightarrow \infty} \tilde{r}_{mj}^p \right) v_j = \sum_{j=1}^\infty r_{nj}^p v_j \\ &= R_n^p(v). \end{aligned}$$

This yields that $Tx \in \ell_\infty$ for $x \in |\phi_z|_p$ if and only if $R^p v \in \ell_\infty$ for $v \in \ell_p$. We conclude that $T \in (|\phi_z|_p, \ell_\infty)$ if and only (12) and (15) hold and also $R^p \in (\ell_p, \ell_\infty)$ which means (16).

Now, we give the characterization of the matrix classes from the classical spaces $\ell_\infty, c, c_0, \ell_1$ to the space $|\phi_z|_p$ for $1 \leq p < \infty$. We need the following lemma to prove our results.

Lemma 3.4 [32]

a) $T = (t_{nk}) \in (\ell_\infty, \ell_1) = (c, \ell_1) = (c_0, \ell_1)$ if and only if

$$\sup_{K \in \mathbb{N}} \sum_{n=1}^\infty \left| \sum_{k \in K} t_{nk} \right| < \infty.$$

b) Let $p > 1$. $T = (t_{nk}) \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p)$ if and only if

$$\sup_{K \in \mathbb{N}} \sum_{n=1}^\infty \left| \sum_{k \in K} t_{nk} \right|^p < \infty.$$

Theorem 3.5 Let $T = (t_{nk})$ be an infinite matrix.

• $T \in (\ell_\infty, |\phi_z|_1) = (c, |\phi_z|_1) = (c_0, |\phi_z|_1)$ if and only if

$$\sup_{K \in \mathbb{N}} \sum_{n=1}^\infty \left| \sum_{k \in K} \left\{ \sum_{j=1}^n \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right\} \right| < \infty.$$

• $T \in (\ell_1, |\phi_z|_1)$ if and only if

$$\sup_k \sum_{n=1}^\infty \left| \sum_{j=1}^n \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right| < \infty.$$

Proof. The proof is given only for the matrix class $(\ell_\infty, |\phi_z|_1)$ since the other cases can be proved similarly.

Consider the matrix $S^1 = (s_{nk}^1)$ defined as

$$s_{nk}^1 = \sum_{j=1}^n \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} < \infty.$$

for all $n, k \in \mathbb{N}$. Let $x = (x_n) \in \ell_\infty$. We obtain the following equality:

$$\sum_{k=1}^\infty s_{nk}^1 x_k$$

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^n \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right) x_k \\
 &= \sum_{j=1}^n \sum_{k=1}^{\infty} t_{jk} x_k \sum_{l=j, l|n}^n \frac{\varphi(l)}{n} \\
 &\quad - \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} t_{jk} x_k \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \\
 &= F_n(Tx) - F_{n-1}(Tx).
 \end{aligned}$$

This implies that $S_n^1(x) = (E^{(1)} \circ F)_n(Tx)$ for all $n \in \mathbb{N}$. Hence, it follows that $Tx \in |\phi_z|_1$ for any $x \in \ell_{\infty}$ if and only if $S^1x \in \ell_1$ for any $x \in \ell_{\infty}$. We conclude that

$$\sup_{K \in \mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{k \in K} \left\{ \sum_{j=1}^n \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right\} \right| < \infty$$

since we have $S^1 \in (\ell_{\infty}, \ell_1)$.

Theorem 3.6 Let $T = (t_{nk})$ be an infinite matrix and $1 < p < \infty$.

• $T \in (\ell_{\infty}, |\phi_z|_p) = (c, |\phi_z|_p) = (c_0, |\phi_z|_p)$ if and only if

$$\sup_{K \in \mathcal{N}} \sum_{n=1}^{\infty} \left| \sum_{k \in K} \left\{ \sum_{j=1}^n z_n^{1/q} \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right\} \right|^p < \infty.$$

• $T \in (\ell_1, |\phi_z|_p)$ if and only if

$$\sup_k \sum_{n=1}^{\infty} \left| \sum_{j=1}^n z_n^{1/q} \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right|^p < \infty.$$

Proof. The proof is given only for the matrix class $(\ell_1, |\phi_z|_p)$ since the other cases can be proved similarly.

Let $p > 1$. Consider the matrix $S^p = (s_{nk}^p)$ defined as

$$s_{nk}^p = \sum_{j=1}^n z_n^{1/q} \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk}$$

for all $n, k \in \mathbb{N}$. Let $x = (x_n) \in \ell_1$. We obtain the following equality:

$$\begin{aligned}
 &\sum_{k=1}^{\infty} s_{nk}^p x_k \\
 &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^n z_n^{1/q} \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right) x_k
 \end{aligned}$$

$$\begin{aligned}
 &= z_n^{1/q} \left(\sum_{j=1}^n \sum_{k=1}^{\infty} t_{jk} x_k \sum_{l=j, l|n}^n \frac{\varphi(l)}{n} \right. \\
 &\quad \left. - \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} t_{jk} x_k \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) \\
 &= z_n^{1/q} (F_n(Tx) - F_{n-1}(Tx)).
 \end{aligned}$$

This implies that $S_n^p(x) = (E^{(p)} \circ F)_n(Tx)$ for all $n \in \mathbb{N}$. Hence, it follows that $Tx \in |\phi_z|_p$ for any $x \in \ell_1$ if and only if $S^p x \in \ell_p$ for any $x \in \ell_1$. We conclude that

$$\sup_k \sum_{n=1}^{\infty} \left| \sum_{j=1}^n z_n^{1/q} \left(\sum_{l=j, l|n}^n \frac{\varphi(l)}{n} - \sum_{l=j, l|n-1}^{n-1} \frac{\varphi(l)}{n-1} \right) t_{jk} \right|^p < \infty,$$

since we have $S^p \in (\ell_1, \ell_p)$.

4. Conclusion

In this paper new series spaces are introduced by using a new summability method derived by Euler totient matrix. After determining dual spaces and some topological properties of the resulting spaces, characterization of certain matrix classes on these spaces are obtained.

5. References

- [1] Altay, B. Başar, F. and Malkowsky, E., "Matrix transformations on some sequence spaces related to strong Cesàro summability and boundedness", *Appl. Math. Comput*, 211(2), 255-264, 2009.
- [2] Wilansky, A., *Summability Through Functional Analysis*, North-Holland Mathematical Studies, vol.85, Elsevier Science Publisher, 1984.
- [3] Schoenberg, I., "The integrability of certain functions and related summability methods", *Amer. Math. Monthly*, 66, 361-375, 1959.
- [4] İlhan, M. and Kara, E.E., "A new Banach space defined by Euler totient matrix operator", *Oper. Matrices*, 13(2), 527-544, 2019.
- [5] Sarıgöl, M.A., "On the local properties of factored Fourier series", *Appl. Math. Comp.*, 216, 3386-3390, 2010.
- [6] Bor, H., "On $|N, p_n|_k$ summability factors of infinite series", *Tamkang J. Math.*, 16, 13-20, 1985.
- [7] Sulaiman, W.T., "On summability factors of infinite series", *Proc. Amer. Math. Soc.*, 115, 313-317, 1992.
- [8] Borwein, D. and Cass, F.P., "Strong Nörlund summability", *Math. Zeitschr.*, 103, 94-111, 1968.
- [9] Flett, T.M., "On an extension of absolute summability and some theorems of Littlewood and Paley", *Proc. London Math. Soc.*, 7, 113-141, 1957.
- [10] Başarır, M. and Kara, E.E., "On some difference sequence spaces of weighted means and compact operators", *Ann. Funct. Anal.*, 2, 114-129, 2011.
- [11] Başarır, M. and Kara, E.E., "On compact operators on the Riesz B^m -difference sequence spaces", *Iran. J. Sci. Technol. Trans. A Sci.*, 35, 279-285, 2011.

- [12] Bektaş, Ç.A., Et, M. and Çolak, R., "Generalized difference sequence spaces and their dual spaces", *J. Math. Anal. Appl.*, 292(2), 423-432, 2004.
- [13] Et, M., "On some difference sequence spaces" *Turk. J. Math.*, 17, 18-24, 1993.
- [14] Hazar, G.C. and Sarigöl, M.A., "Absolute Cesàro series spaces and matrix operators", *Acta App. Math.*, 154, 153-165, 2018.
- [15] Kara, E.E. and Başarır, M., "On compact operators and some Euler $B(m)$ difference sequence spaces", *J. Math. Anal. Appl.*, 379(2), 499-511, 2011.
- [16] Kara, E.E. and İlhan, M., "Some properties of generalized Fibonacci sequence spaces", *Linear Multilinear Algebra.*, 64(11), 2208-2223, 2016.
- [17] Kirişçi, M. and Başar, F., "Some new sequence spaces derived by the domain of generalized difference matrix", *Comput. Math. Appl.*, 60, 1299-1309, 2010.
- [18] Kirişçi, M., "Riesz type integrated and differentiated sequence spaces", *Bull. Math. Anal. Appl.*, 7(2), 14-27, 2015.
- [19] Mursaleen, M. and Noman, A.K., "Compactness by the Hausdorff measure of noncompactness", *Nonlinear Anal.*, 73(8), 2541-2557, 2010.
- [20] Mursaleen, M. and Noman A.K., "Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means", *Comput. Math. Appl.*, 60(5), 1245-1258, 2010.
- [21] Mursaleen, M. and Noman A.K., "On the spaces of λ -convergent and bounded sequences", *Thai J. Math.*, 8(2), 311-329, 2012.
- [22] Mohiuddine, S.A., "An application of almost convergence in approximation theorems", *Appl. Math. Lett.*, 24, 1856-1860, 2011.
- [23] Mohiuddine, S.A. and Alotaibi, A., "Weighted almost convergence and related infinite matrices", *J. Inequal. Appl.*, Article Number.15, 2018.
- [24] Hazar, G.C. and Sarigöl, M.A., "On absolute Nörlund spaces and matrix operators", *Acta Math. Sin. (Engl. Ser.)*, 34(5), 812-826, 2018.
- [25] Sarigöl, M.A., "Spaces of Series Summable by Absolute Cesàro and Matrix Operators", *Comm. Math Appl.*, 7(1), 11-22, 2016.
- [26] İlhan, M. Demiriz, S. and Kara, E.E., "A new paranormed sequence space defined by Euler totient function", *Karaelmas Sci. Eng. J.*, 9(2), 277-282, 2019.
- [27] Ellidokuzoğlu, H.B. and Demiriz, S., "Euler-Riesz difference sequence spaces", *Turk. J. Math. Comput. Sci.*, 7, 63-72, 2017.
- [28] Demiriz, S. and Çakan, C. "Some topological and geometrical properties of a new difference sequence space", *Abstr. Appl. Anal.*, 2011, Article ID 213878, 14 pages.
- [29] Duyar, O. Demiriz, S. and Özdemir, O. "On some new generalized difference sequence spaces of nonabsolute type", *J. Math.*, 2014, Article ID 876813, 13 pages.
- [30] Ercan, S. and Bektaş Ç.A., "On λ -convergence and λ -boundedness of m th order", *Comm. Statist. Theory Methods*, 2019, doi:10.1080/03610926.2019.1692032.
- [31] Ercan, S., "On λ_r -convergence and λ_r -boundedness", *J. Adv. Phys.*, 7(1), 123-129, 2018.
- [32] Stieglitz, M. and Tietz, H., "Matrixtransformationen von folgenräumen eine ergebnisüberischt", *Math Z.*, 154, 1-16, 1977.
- [33] Maddox, I.J., *Elements of functinal analysis*, Cambridge University Press, London, New York, 1970.