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On Quaternion-Gaussian Fibonacci Numbers and Their Properties

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Abstract

We study properties of Gaussian Fibonacci numbers. We start with some basic identities. Thereafter, we focus on properties of the quaternions that accept gaussian Fibonacci numbers as coefficients. Using the Binet form we prove fundamental relations between these numbers. Moreover, we investigate whether the quaternions newly defined provide existing some important identities such as Cassini's identity for quaternions.

1 Introduction

The investigation of Gaussian numbers is a research topic of great interest. A Gaussian number is a complex number whose real and imaginary parts are both integers, the set of these numbers is denoted by $\mathbb{Z}[i]$ ($i^2 = -1$). Gaussian numbers form the lattice of all points with integral rational coordinates on the plane. As known such numbers were first considered in 1832 by C.F. Gauss [6] and also discovered the properties of the set of complex integers. In 1963, A.F. Horadam [10] introduced the complex Fibonacci numbers and Fibonacci quaternions. In 1965, J. H. Jordan [11] investigated gaussian numbers and extend some relationships are well-known about Fibonacci sequences to the gaussian Fibonacci numbers. Also, Jordan derived some identities involving

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the terms of gaussian Fibonacci and Lucas sequences. Then, we have given some of these below (see, for example [11]):

$$\sum_{l=0}^{n} GF_l = GF_{n+2} - 1, \quad \sum_{l=0}^{n} GL_l = GL_{n+2} - (1+2i), \tag{1}$$

$$GF_{n+1}GF_{n-1} - GF_n^2 = (-1)^n(2-i), \tag{2}$$

$$GL_{n+1}GL_{n-1} - GL_n^2 = 5(-1)^{n+1}(2-i), \tag{3}$$

$$GF_n^2 + GF_{n+1}^2 = F_{2n}(1+2i), \tag{4}$$

$$GF_nGL_n = F_{2n-1}(1+2i) \tag{5}$$

and

$$GL_n = GF_{n+1} + GF_{n-1}, \tag{6}$$

where GF_n and GL_n are the n-th gaussian Fibonacci and gaussian Lucas numbers, respectively, and F_n is the n-th Fibonacci number.

Gaussian Fibonacci numbers GF_n are defined by $GF_n = GF_{n-1} + GF_{n-2}$, where $GF_0 = i$, $GF_1 = 1$ and $n \geq 2$. It is clear that these numbers are closely related to Fibonacci numbers, it means that $GF_n = F_n + iF_{n-1}$ and $GF_{-n} = (-1)^{n-1}(F_n - iF_{n+1})$. Some elements of this sequence can be given according to the following table:

0	1	2	3	4	5	...	n	...
i	1	$1+i$	$2+i$	$3+2i$	$5+3i$	...	$F_n + iF_{n-1}$	...

Table 1. Gaussian Fibonacci numbers.

In 1977, G. Berzsenyi [2] presented a natural manner of extension of the Fibonacci numbers into the complex plane and gave the applicability of the generalization. If α and β are the roots of the characteristic equation for gaussian Fibonacci sequence, one can give Binet formula of this sequence as follows.

$$GF_n = \frac{1}{\alpha - \beta}\{(1-i\beta)\alpha^n - (1-i\alpha)\beta^n\}. \tag{7}$$

The purpose of this study is to introduce quaternion-gaussian Fibonacci numbers and investigate their properties. Classically, quaternions are hypercomplex numbers written with the help of three imaginary units. Quaternions are represented as

$$q = (a_0, a_1, a_2, a_3) = a_0 + a_1 i + a_2 j + a_3 k, \tag{8}$$

where a_t $(t = 0, 1, 2, 3)$ are real numbers and the vectors i, j, k obey the famous multiplication rules defined by Hamilton [9] in 1866 for quaternions. Furthermore, the multiplication of their basis elements can be done according to the following table:

$\cdot$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Table 2. Multiplication of quaternionic units

The conjugate of any quaternion q is given by negating the three imaginary components. Also, the norm and absolute value for any quaternion q are defined as $||q|| = q \cdot \overline{q} = \overline{q} \cdot q$ and $|q| = \sqrt{||q||}$, respectively. Therefore, there is a multiplicative inverse of each non-zero quaternion.

Quaternions have been a topic of intensive investigation ever since they were introduced. Different types of quaternions have been studied by many mathematicians. The reader can refer to [3, 4, 5, 7, 8, 10, 13, 14] for properties and applications of different types of quaternions. One of the earliest studies on this subject belong to A.F. Horadam, in [10] the author defined Fibonacci quaternions and gave some quaternions recurrence relations. In [7] and [8], Halici gave the Binet formulas, generating functions, some identities for Fibonacci and complex Fibonacci quaternions, respectively. In [13], Polatli and Kesim investigated the quaternions with generalized Fibonacci and Lucas number components. In [5], Bolat and Ipek studied Pell and Pell-Lucas quaternions. Szynal-Liana and Wloch introduced Jacobsthal quaternions and gave some identities related with these quaternions [14]. In [12], the authors studied the generalized gaussian Fibonacci numbers.

Inspired by all of these studies, we have described a new type of quaternion sequence and called the elements of this sequence as quaternion-gaussian Fibonacci numbers.

2 Quaternion-Gaussian Fibonacci Numbers

We begin with a formal definition of general term in quaternion-gaussian Fibonacci number sequence.

Definition 2.1. Let us define quaternion-gaussian Fibonacci number sequence as follows

$$\{GFQ_n\}_{n\geq 0} = \{GFQ_0, GFQ_1, \ldots, GFQ_n, \ldots\}.$$

The general term GFQ_n is

$$GFQ_n = (GF_n, GF_{n+1}, GF_{n+2}, GF_{n+3}) \in \mathbb{C}^4, \tag{9}$$

where GF_n is the n-th gaussian Fibonacci number and the initial values are

$$GFQ_0 = (i, 1, 1+i, 2+i) \;\text{ and }\; GFQ_1 = (1, 1+i, 2+i, 3+2i). \tag{10}$$

Notice that using the gaussian Fibonacci numbers we can write the following recursive relation

$$GFQ_n + GFQ_{n+1} = GFQ_{n+2}, \quad n \geq 0. \tag{11}$$

In the following theorem we give a formula which gives any element of the sequence. This formula is known as the Binet formula.

Theorem 2.1. *For the elements GFQ_n, we have*

$$GFQ_n = \frac{1}{\alpha - \beta}\{GFQ_1(\alpha^n - \beta^n) + GFQ_0(\alpha^{n-1} - \beta^{n-1})\} \tag{12}$$

where $GFQ_1 = (1, 1+i, 2+i, 3+2i)$ *and* $GFQ_0 = (i, 1, 1+i, 2+i)$.

Proof. The characteristic equation of gaussian Fibonacci sequence has the roots α and β. Then,

$$GFQ_n = A\alpha^n + B\beta^n \tag{13}$$

is the solution for the recurrence relation

$$GFQ_n + GFQ_{n+1} = GFQ_{n+2}. \tag{14}$$

So, writing $n = 0$ and $n = 1$ in the identity (2.5) and solving this system, we get the values A and B as follows.

$$A = \frac{1}{\alpha - \beta}(GFQ_1 - \beta GFQ_0), \ B = \frac{1}{\alpha - \beta}(\alpha GFQ_0 - GFQ_1). \tag{15}$$

If these values are written in the general solution. Hence, it gives that

$$\begin{aligned} GFQ_n &= \frac{1}{\alpha - \beta}\{(GFQ_1 - \beta GFQ_0)\alpha^n + (\alpha GFQ_0 - GFQ_1)\beta^n\} \\ &= \frac{1}{\alpha - \beta}\{GFQ_1(\alpha^n - \beta^n) + GFQ_0(\alpha^{n-1} - \beta^{n-1})\} \end{aligned} \tag{16}$$

Then, the result is obtained. □

Note that we can also write the formula (12) as follows:

$$GFQ_n = F_n GFQ_1 + F_{n-1} GFQ_0, \ n \geq 1. \tag{17}$$

The formula (12) is more useful in obtaining some identities. Furthermore, with the help of this formula, we can also derive the numbers with the negative indices. We give the following corollary without proof.

Corollary. *For the negative integers, we have*

$$GFQ_{-n} = (-1)^n (F_{n+1} GFQ_0 - F_n GFQ_1) \tag{18}$$

where $GFQ_1 = (1, 1+i, 2+i, 3+2i)$ *and* $GFQ_0 = (i, 1, 1+i, 2+i)$.

Theorem 2.2. *The sum of first n of numbers GFQ_n is as follows:*

$$\sum_{l=0}^{n} GFQ_l = GFQ_{n+2} - (i + 4k). \tag{19}$$

Proof. Since $GFQ_n = GFQ_{n+1} - GFQ_{n-1}$ for all $n \geq 1$, we can write the following equations:

$$\begin{aligned} GFQ_0 &= GFQ_1 - GFQ_{-1}, \\ GFQ_1 &= GFQ_2 - GFQ_0, \\ GFQ_2 &= GFQ_3 - GFQ_1. \end{aligned}$$

And with this way of processing, we can write

$$\begin{aligned} GFQ_{n-1} &= GFQ_n - GFQ_{n-2}, \\ GFQ_n &= GFQ_{n+1} - GFQ_{n-1}. \end{aligned}$$

Then, as a result of necessary calculations, we get

$$\sum_{l=0}^{n} GFQ_l = GFQ_{n+2} - GFQ_1 = GFQ_{n+2} - (i + 4k). \tag{20}$$

□

Corollary. *The sum of the first n of the double-indexed numbers GFQ_{2n} is*

$$\sum_{l=0}^{n} GFQ_{2l} = GFQ_{2n+1} - (k - i). \tag{21}$$

Proof. The proof can be easily made similar to Theorem 2.4. □

As known generating functions provide a powerful tool for solving linear homogeneous recurrence relations with constant coefficients. The generating function is a function that corresponds to the Binet formula, which finds the desired elements of the sequence providing the recursive relation.

In the following theorem we give generating function for the numbers GFQ_n with coefficients from gaussian Fibonacci sequence.

Theorem 2.3. *The generating function of the sequence* $\{GFQ_n\}_{n\geq 0}$ *is as follows.*

$$g(t) = \sum_{n=0}^{\infty} GFQ_n t^n = \frac{2i + 3k + (k - i)t}{1 - t - t^2}. \tag{22}$$

Proof. Let us show the generating function of the sequence $\{GFQ_n\}_{n\geq 0}$ by $g(t)$:

$$g(t) = GFQ_0 + GFQ_1 t + GFQ_2 t^2 + \ldots + GFQ_n t^n + \ldots \tag{23}$$

Let us multiply the generating function by t and t^2, respectively, in order to benefit from the recursive relation. Then by calculating the equations $tg(t)$ and $t^2 g(t)$, and also using the recurrence relation, the desired formula is obtained:

$$(1 - t - t^2)g(t) = GFQ_0 + (GFQ_1 - GFQ_0)t. \tag{24}$$

Then, the result is followed. □

It should be note that using the generating function of the sequence

$$\{GFQ_n\}_{n\geq 0} = \{GFQ_0, GFQ_1, \ldots, GFQ_n, \ldots\} \tag{25}$$

generating function of the following sequence

$$A_n = GFQ_0 + GFQ_1 + \ldots + GFQ_n \tag{26}$$

can also be given. Now we give this function with the following corollary.

Corollary. *The generating function of the numbers* A_n *is as follows:*

$$\sum_{n=0}^{\infty} A_n t^n = \frac{GFQ_0 + tGFQ_{-1}}{1 - 2t + t^3}. \tag{27}$$

Proof. Using Eq. (22), the generating function of sequence $\{GFQ_n\}_{n\geq 0}$ is

$$\frac{GFQ_0 + (GFQ_1 - GFQ_0)t}{1 - t - t^2} = \frac{GFQ_0 + tGFQ_{-1}}{1 - t - t^2},$$

then the generating function of the sequence $\sum_{l=0}^{n} GFQ_l$ has the form

$$\begin{aligned}\sum_{n=0}^{\infty} A_n t^n &= \frac{\sum_{n=0}^{\infty} GFQ_n t^n}{1 - t} \\ &= \frac{GFQ_0 + tGFQ_{-1}}{(1 - t)(1 - t - t^2)} \\ &= \frac{GFQ_0 + tGFQ_{-1}}{1 - 2t + t^3},\end{aligned}$$

where $GFQ_{-1} = k - i$. □

Now, using the elements of the gaussian Fibonacci sequence, let's define the following tridiagonal matrix with complex coefficients and order $(n \times n)$:

$$K_n = \begin{bmatrix} 1 & i & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n} . \tag{28}$$

Next, we state determinant formulas for the sequence $\{GFQ_n\}_{n \in \mathbb{N}}$.

Corollary. *For the elements GFQ_n of the quaternion-gaussian Fibonacci sequence, we have*

$$\det(K_n) = Sc(GFQ_n) = GF_n, \tag{29}$$

where $Sc(GFQ_n)$ is the scalar part of the n-th quaternion-gaussian Fibonacci number.

Proof. Let's calculate with the induction. For $n = 1$ and $n = 2$, we have respectively,

$$\det(K_1) = 1 = GF_1 = Sc(GFQ_1) \tag{30}$$

and

$$\det(K_2) = 1 + i = GF_2 = Sc(GFQ_2). \tag{31}$$

Let's assume that this claim is correct for $n-1$ and $n-2$. So,

$$\det(K_{n-1}) = GF_{n-1}, \quad \det(K_{n-2}) = GF_{n-2}. \tag{32}$$

Then, we get

$$\begin{aligned} \det(K_n) &= \det(K_{n-1}) + \det(K_{n-2}) \\ &= GF_{n-1} + GF_{n-2} \\ &= GF_n = Sc(GFQ_n). \end{aligned}$$

Therefore, the truth of the claim is proved.

Furthermore, with the help of this matrix the other desired components of the GFQ_n can be found. □

Theorem 2.4. *Let n be integer and $n \geq 1$. Then, it holds that*

$$\begin{pmatrix} GFQ_n & GFQ_{n-1} \\ GFQ_{n+1} & GFQ_n \end{pmatrix} = \begin{pmatrix} GFQ_1 & GFQ_0 \\ GFQ_2 & GFQ_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} . \tag{33}$$

Proof. For $n = 1$, it is clear that the equation is correct. According to the assume, we know that this claim is true for n:

$$\begin{pmatrix} GFQ_n & GFQ_{n-1} \\ GFQ_{n+1} & GFQ_n \end{pmatrix} = \begin{pmatrix} GFQ_1 & GFQ_0 \\ GFQ_2 & GFQ_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}. \tag{34}$$

The, for $n+1$ we write the following equation:

$$\begin{aligned} \begin{pmatrix} GFQ_1 & GFQ_0 \\ GFQ_2 & GFQ_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} &= \begin{pmatrix} GFQ_1 & GFQ_0 \\ GFQ_2 & GFQ_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} GFQ_n & GFQ_{n-1} \\ GFQ_{n+1} & GFQ_n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} GFQ_{n+1} & GFQ_n \\ GFQ_{n+2} & GFQ_{n+1} \end{pmatrix}. \end{aligned}$$

Thus, the claim is proved. □

In the next theorem, we give one of the very important relation on the Fibonacci-like numbers which is known as Cassini's identity. Cassini's identity helps to quantify how much deviation from geometry in each term of the sequence. In this sense, in [1] the author investigated the quaternionic Fibonacci Cassini's identity with the maple's help.

Now, with the help of Theorem 2.4, we give the Cassini's identity in the following theorem.

Theorem 2.5. *The Cassini's identity for the numbers GFQ_n is as follows:*

$$GFQ_n^2 - GFQ_{n+1}GFQ_{n-1} = 5(-1)^{n-1}(2-j). \tag{35}$$

Proof. We will use the following equation

$$\begin{pmatrix} GFQ_n & GFQ_{n-1} \\ GFQ_{n+1} & GFQ_n \end{pmatrix} = \begin{pmatrix} GFQ_1 & GFQ_0 \\ GFQ_2 & GFQ_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}. \tag{36}$$

Let's take determinant for both sides of the equality (36). And let's calculate the values GFQ_1^2 and GFQ_2GFQ_0. Thus, using definition 2.1 we have $GFQ_0 = 2i + 3k$, $GFQ_1 = i + 4k$ and $GFQ_2 = 3i + 7k$. Following the quaternion multiplication rules, the desired values are found

$$GFQ_1^2 = -17, \; GFQ_2GFQ_0 = -27 + 5j. \tag{37}$$

The, we have

$$\begin{aligned} GFQ_n^2 - GFQ_{n+1}GFQ_{n-1} &= (GFQ_1^2 - GFQ_2GFQ_0)(-1)^{n-1} \\ &= 5(-1)^{n-1}(2-j). \end{aligned}$$

Thus, the proof is completed. □

Corollary. *For the numbers GFQ_n and $n \geq 0$, we have*

$$\begin{pmatrix} GFQ_{n+1} & GFQ_n \\ GFQ_{n+2} & GFQ_{n+1} \end{pmatrix} = \begin{pmatrix} i+4k & 2i+3k \\ 3i+7k & i+4k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n. \tag{38}$$

Proof. The accuracy of the claim can be easily seen by the induction method. □

Theorem 2.6. *For the numbers GFQ_n, we have*

$$GFQ_n^2 + GFQ_{n+1}^2 = -15F_{2n+3}, \tag{39}$$

where F_n is the n-th Fibonacci number.

Proof. Let us calculate the numbers GFQ_n^2 and GFQ_{n+1}^2 one by one. Then using the Eq. (17), we get

$$\begin{aligned} GFQ_n^2 &= (F_n GFQ_1 + F_{n-1} GFQ_0)^2 \\ &= F_n^2 GFQ_1^2 + F_n F_{n-1}(GFQ_1 GFQ_0 + GFQ_0 GFQ_1) \\ &+ F_{n-1}^2 GFQ_0^2. \end{aligned} \tag{40}$$

Calculating

$$GFQ_1^2 = -17, \ GFQ_0^2 = -13,$$

and

$$GFQ_1 GFQ_0 = -14 + 5j, \ GFQ_0 GFQ_1 = -14 - 5j$$

and by placing these values in the Eq. (40), we obtain that

$$GFQ_n^2 = -(17F_n^2 + 28F_n F_{n-1} + 13F_{n-1}^2). \tag{41}$$

Similarly, we get

$$FQ_{n+1}^2 = -(17F_{n+1}^2 + 28F_n F_{n+1} + 13F_n^2). \tag{42}$$

Thus, we obtain

$$\begin{aligned} GFQ_n^2 + GFQ_{n+1}^2 &= -(17F_{2n+1} + 28F_{2n} + 13F_{2n-1}) \\ &= -(45F_{2n} + 30F_{2n-1}) \\ &= -(30F_{2n+1} + 15F_{2n}) \\ &= -(15F_{2n+2} + 15F_{2n+1}) \\ &= -15F_{2n+3}. \end{aligned}$$

So, we get the desired result. □

The next theorem gives the norm value of elements GFQ_n.

Theorem 2.7. *The norm value of GFQ_n is*

$$||GFQ_n|| = GFQ_n \cdot \overline{GFQ_n} = 3F_{2n+2}(1+2i), \tag{43}$$

where F_n is the n-th Fibonacci number.

Proof. Using the formula that gives the norm values of these numbers, let's first write the following equation.

$$||GFQ_n|| = GF_n^2 + GF_{n+1}^2 + GF_{n+2}^2 + GF_{n+3}^2. \tag{44}$$

Now, let us calculate each term on the right side of the Eq. (44). Then, we can write the following equations by the aid of formula in Eq. (7):

$$GF_n^2 = \frac{1}{(\alpha-\beta)^2}\{(1-i\beta)^2\alpha^{2n} + (1-i\alpha)^2\beta^{2n} - 2(1-i\alpha)(1-i\beta)(\alpha\beta)^n\} \tag{45}$$

If we calculate the other terms in the same way and write in the Eq. (44), then some simplifications can be seen. Thus, we obtain that

$$\begin{aligned}(\alpha-\beta)^2\cdot||GFQ_n|| = {} & (1-i\beta)^2\alpha^{2n} + (1-i\alpha)^2\beta^{2n} - 2(2-i)(-1)^n \\ & + (1-i\beta)^2\alpha^{2n+2} + (1-i\alpha)^2\beta^{2n+2} - 2(2-i)(-1)^{n+1} \\ & + (1-i\beta)^2\alpha^{2n+4} + (1-i\alpha)^2\beta^{2n+4} - 2(2-i)(-1)^{n+2} \\ & + (1-i\beta)^2\alpha^{2n+6} + (1-i\alpha)^2\beta^{2n+6} - 2(2-i)(-1)^{n+3}.\end{aligned}$$

Expanding the terms in parenthesis proves the formula

$$(\alpha-\beta)^2\cdot||GFQ_n|| = \{(\sigma\alpha^n)^2(1+\alpha^2+\alpha^4+\alpha^6) + (\mu\beta^n)^2(1+\beta^2+\beta^4+\beta^6)\},$$

where $\sigma = 1-i\beta$ and $\mu = 1-i\alpha$. Also, by taking advantage of the repeated relations of α and β, we can give this norm value as follows. Furthermore, using the equations,

$$\alpha^2 = F_n\alpha + F_{n-1},\ \beta^2 = F_n\beta + F_{n-1},\ \alpha\beta = -1,\ \alpha+\beta = 1, \tag{46}$$

we get

$$\begin{aligned}(\alpha-\beta)^2\cdot||GFQ_n|| &= 3\{\sigma^2(4\alpha^{2n+1}+3\alpha^{2n}) + \mu^2(4\beta^{2n+1}+3\beta^{2n})\} \\ &= 3\{(1-\beta^2)(4\alpha^{2n+1}+3\alpha^{2n}) + (1-\alpha^2)(4\beta^{2n+1}+3\beta^{2n})\} \\ &\quad + 6i\{4\alpha^{2n} - 3\alpha^{2n-1} + 4\beta^{2n} - 3\beta^{2n-1}\} \\ &= 3\{\alpha^{2n-2}(7\alpha+4) + \beta^{2n-2}(7\beta+4)\} \\ &\quad + 6i\{\alpha^{2n-1}(4\alpha-3) + \beta^{2n-1}(4\beta-3)\}.\end{aligned}$$

Then, the result is obtained. $\square$

Especially, we can also calculation this norm using the equality $GF_n = F_n + iF_{n-1}$. In this case, the norm value in relation to Fibonacci and Lucas numbers is as follows

$$GF_n^2 = (F_n + iF_{n-1})^2 = F_n^2 - F_{n-1}^2 + 2i(F_nF_{n-1}). \tag{47}$$

Then,

$$\begin{aligned} GF_n^2 + GF_{n+1}^2 &= F_n^2 - F_{n-1}^2 + 2i(F_nF_{n-1}) + F_{n+1}^2 - F_n^2 + 2i(F_{n+1}F_n) \\ &= F_{2n+1} - F_{2n-1} + 2iF_nL_n \\ &= F_{2n+1} - F_{2n-1} + 2iF_{2n}. \end{aligned}$$

and

$$||GFQ_n|| = (F_{2n+4} + F_{2n})(1 + 2i) = 3F_{2n+2}(1 + 2i).$$

3 Conclusion

In this study, we first focused on gaussian Fibonacci numbers and showed that these numbers provide some important identities. Then, we have defined the quaternions that accept these numbers as coefficients. We would like to point out that these quaternions will be important in the study of other quaternions and some other identities can be also given involving these numbers. In the future, we will study this type of quaternion in a generalized Gaussian number, for example, the Horadam gaussian numbers.

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