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Original Article

SITEM for the conformable space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations

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Abstract

In the present paper, new analytical solutions for the space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations are obtained by using the simplified $\tan(\frac{\phi(\xi)}{2})$-expansion method. Here, fractional derivatives are defined in the conformable sense. To show the correctness of the obtained traveling wave solutions, residual error function is defined. It is observed that the new solutions are very close to the exact solutions. The solutions obtained by the presented method have not been reported in former literature.
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Keywords: Space-time fractional Boussinesq equation; (2 + 1)-dimensional breaking soliton equation; Simplified $\tan(\frac{\phi(\xi)}{2})$-expansion method (SITEM); Conformable derivative.

1. Introduction

Nonlinear partial differential equations (NPDEs) have an important role in modeling many events in the fields of science such as physics, engineering, mechanics, chemistry and biology. Obtaining the exact solutions of NPDEs is the best way to better understand these phenomena. The solutions of the NPDEs and stability analysis of the solutions have been investigated by many authors (see, for example, [1–15]). In general, the solutions of NPDEs have been found in the form of traveling wave solutions (see, for example, [1,3–8,13,14,16–21]).

Propagation of long waves on the surface of water with a small amplitude in non-dimensional nonlinear lattices and in nonlinear strings are modeled by the Boussinesq equation [22]. In the Boussinesq water waves approach, the vertical structure of the horizontal and vertical flow velocity of the wave is considered. In coastal and ocean engineering, Boussinesq-type equations are often used in computer models for the simulation of water waves in shallow seas and harbours. Mathematical modeling of tsunami wave and tidal oscillations can be given as an example to the ocean engineering applications of the equation. These models have also applications in engineering such as predicting refraction, diffraction, shoaling and harmonic interaction along around coastal structures [23].

Boussinesq-type equations are nonlinear partial differential equations. Solving nonlinear partial differential equations is a more complex problem than solving linear partial differential equations. To obtain the solutions of the Boussinesq-type equations, researchers have been used some methods such as (G'/G)-expansion method, repeated homogeneous balance method, ansatz method, the hyperbolic tangent method [17,18,20,21,24,25]. Space-time fractional Boussinesq equation with the modified Riemann–Liouville has been solved by using the new simplified bilinear method and the exponential rational function method in [26,27]. Fourier spectral approximation method and modified Kudryashov method have been applied to the time fractional Boussinesq equation in [19,28], respectively.

(2 + 1)-dimensional breaking soliton equation was first given by Calogero and Degasperis [29,30]. (2+1)-dimensional

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interaction of a Riemann wave propagating along the y-axis with a long wave along and the x-axis is modeled by (2 + 1)-dimensional breaking soliton equation. Taking $y = x$, and by integrating the resulting equation in the (2 + 1)-dimensional breaking soliton equation, the equation is transformed to the KdV equation. KdV equation is a mathematically modeling the special waves called solitons on shallow water surfaces. The main characteristic feature of the family of these equations is that the spectral parameter which used in the Lax representations possesses so-called breaking behavior [31]. Modified simple equation method, the Exp-function method, the direct integration and homotopy perturbation method, projective Riccati equation expansion method, the simplest equation method, ansatz method and Lie group method have been applied to construct the exact solutions of the (2 + 1)-dimensional breaking soliton equation [32–37].

Recently, the improved $\tan(\frac{\phi(\xi)}{2})$-expansion method (ITEM) has been studied by several authors [38–40]. In [41], ITEM has been simplified and called simplified ITEM (SITEM). SITEM has been applied to the Kundu–Eckhaus equation and conformable space-time fractional coupled Konopelchenko–Dubrovsky equations for only zero parameter p in [41,42], respectively. In this paper, SITEM for nonzero parameter p is applied to the conformable space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations. Firstly, using the complex transformation, the conformable space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations are transformed into nonlinear ordinary differential equations with integer orders. Solutions of the ordinary differential equations are written in the form of series expansion of the function $\tan(\frac{\phi(\xi)}{2})$ with unknown coefficients. The unknown coefficients of the series expansion are computed. Therefore, new solutions for each model are obtained. The obtained traveling wave solutions are given by the trigonometric, exponential and rational functions. Furthermore, the accuracy of the new solutions is presented by using residual error function (REF).

2. Description of the conformable fractional derivative and its properties

For a function $f : (0, \infty) \to R$, the conformable fractional derivative of f of order $0 < \alpha < 1$ is given as (see, for example, [43])

$$T_t^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

Some important properties of the conformable fractional derivative are as follows:

$$T_t^{\alpha}(af + bg)(t) = aT_t^{\alpha} f(t) + bT_t^{\alpha} g(t), \quad \forall a, b \in R,$$
$$T_t^{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha},$$
$$T_t^{\alpha}(f(g(t))) = t^{1-\alpha} g'(t) f'(g(t)).$$

3. Analytic solutions to the conformable space-time fractional Boussinesq equation

Conformable space-time fractional Boussinesq equation is given by the following formula [18,20]

$$T_t^{\alpha}T_t^{\alpha}u - T_x^{\beta}T_x^{\beta}u - T_x^{\beta}T_x^{\beta}(u^2) + T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u = 0, \quad 0 < \alpha \le 1, \ \ 0 < \beta \le 1. \tag{1}$$

Let us consider the following transformation

$$u(x,t) = U(\xi), \quad \xi = k\frac{t^{\alpha}}{\alpha} + m\frac{x^{\beta}}{\beta}, \tag{2}$$

where k, m are constants. Substituting (2) into Eq. (1) we obtain the following differential equations

$$(k^2 - m^2)U'' - m^2(U^2)'' + m^4U^{(4)} = 0. \tag{3}$$

Integrating of Eq. (3) with zero constant of integration, we have

$$(k^2 - m^2)U - m^2U^2 + m^4U'' = 0. \tag{4}$$

Let us suppose that the solution of Eq. (4) can be written in the form

$$U(\xi) = \sum_{k=0}^{N} A_k\Big[p + \tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^k + \sum_{k=1}^{N} B_k\Big[p + \tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-k}. \tag{5}$$

$\phi(\xi)$ is the solution of the following ordinary differential equation

$$\phi'(\xi) = a\sin(\phi(\xi)) + b\cos(\phi(\xi)) + c, \tag{6}$$

where a, b, c, $A_k(0 \le k \le N)$ and $B_k(1 \le k \le N)$ are constants to be determined. The solution of Eq. (6) is given as follows:

For $b = c, \quad a = 0$,

$$\tan\Big(\frac{\phi}{2}\Big) = b\xi + c_1 - p.$$

For $b = c, \quad a \ne 0$,

$$\tan\Big(\frac{\phi}{2}\Big) = c_1\exp(a\xi) - \frac{b}{a}.$$

For $b \ne c, \quad \Delta = a^2 + b^2 - c^2 > 0$,

$$\tan\Big(\frac{\phi}{2}\Big) = \frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)} - p.$$

For $b \ne c, \quad \Delta = a^2 + b^2 - c^2 = 0$,

$$\tan\Big(\frac{\phi}{2}\Big) = \frac{a}{b-c} + \frac{2}{b-c}\frac{c_2}{c_1 + c_2\xi}.$$

For $b \ne c, \quad \Delta = a^2 + b^2 - c^2 < 0$,

$$\tan\Big(\frac{\phi}{2}\Big) = \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin\Big(\frac{\sqrt{-\Delta}}{2}\xi\Big) + c_2\cos\Big(\frac{\sqrt{-\Delta}}{2}\xi\Big)}{c_1\cos\Big(\frac{\sqrt{-\Delta}}{2}\xi\Big) + c_2\sin\Big(\frac{\sqrt{-\Delta}}{2}\xi\Big)},$$

where c_1 and c_2 are arbitrary constants, $r_1 = (a + p(b-c) + \sqrt{\Delta})/2$ and $r_2 = (a + p(b-c) - \sqrt{\Delta})/2$.

Table 1
REF ($a=3, b=2, c=1, c_1=20, c_2=30, p=4, m=0.2$) for the solution $u_2(x,t)$ in Case 1($\Delta>0$ and $b\neq c$).

x	t	α	β	REF
20	10	0.5	0.75	10^{-19}
−20	10	0.75	1	10^{-19}
2	3	0.5	0.2	10^{-17}
5	5	0.25	0.5	10^{-18}
1	1	0.25	0.5	10^{-16}
7	8	0.5	0.25	10^{-18}
40	40	0.75	0.25	10^{-21}
−20	10	0.5	1	10^{-20}

Substituting Eq. (5) into Eq. (4) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 2. Therefore, Eq. (5) reduces to

$$U(\xi)=A_0+A_1\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]+A_2\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^2+B_1\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-1}+B_2\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-2}. \quad (7)$$

Substituting Eq. (7) into Eq. (4), collecting all the terms with the same power of $\tan(\frac{\phi}{2})$, we can obtain a set of algebraic equations for the unknowns $A_0,\ A_1,\ B_1, A_2,\ B_2,\ k,\ m$:

$$3A_2b^2m^4-2A_2^2m^2-6A_2bcm^4+3A_2c^2m^4=0,$$
$$14pA_2b^2m^4-\dots$$

Solving the algebraic equations in the Mathematica, we can obtain the solutions of the Eq. (1) in the form of Eq. (7).

To show the correctness of the obtained solutions, we define residual error function (REF) as follows

$$REF(x)=T_t^{\alpha}T_t^{\alpha}u-T_x^{\beta}T_x^{\beta}u-T_x^{\beta}T_x^{\beta}(u^2)+T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u,\quad 0<\alpha\le 1,\quad 0<\beta\le 1. \quad (8)$$

Note that the smallness of the REF shows that the approximate solution is close to the exact solution. Therefore, REF can be used to measure correctness of the approximate solution.

Case 1: $A_0=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)$, $A_1=-3(b-c)m^2(a+bp-cp), A_2=\frac{3}{2}(b-c)^2m^2, B_1=0, B_2=0, \xi=\pm\sqrt{m^2-m^4\Delta}\frac{t^{\alpha}}{\alpha}+m\frac{x^{\beta}}{\beta}$:

For $\Delta>0$ and $b\neq c$, $(m\in(-1/\sqrt{\Delta},1/\sqrt{\Delta}))$

$$u_{1,2}(x,t)=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)-6m^2(a+bp-cp)\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]+6m^2\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^2. \quad (9)$$

For the parameters $a=3, b=2, c=1, c_1=20, c_2=30, p=4, m=0.2$ and some values of the x, t, α, β, the values of the REF are given in Table 1.

For $\Delta=0$ and $b\neq c$,

$$u_{3,4}(x,t)=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)-3m^2(a+p(b-c))\Big[p(b-c)+a+2\frac{c_2}{c_1+c_2\xi}\Big]+\frac{3}{2}m^2\Big[p(b-c)+a+2\frac{c_2}{c_1+c_2\xi}\Big]^2. \quad (10)$$

REF for $\Delta=0$ and $b\neq c$ is computed as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

For $\Delta<0$ and $b\neq c$,

$$u_{5,6}(x,t)=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)-3m^2(a+bp-cp)\Big[p(b-c)+a+\sqrt{-\Delta}\frac{-c_1\sin\big(\frac{\sqrt{-\Delta}}{2}\xi\big)+c_2\cos\big(\frac{\sqrt{-\Delta}}{2}\xi\big)}{c_1\cos\big(\frac{\sqrt{-\Delta}}{2}\xi\big)+c_2\sin\big(\frac{\sqrt{-\Delta}}{2}\xi\big)}\Big]+\frac{3}{2}m^2\Big[p(b-c)+a+\sqrt{-\Delta}\frac{-c_1\sin\big(\frac{\sqrt{-\Delta}}{2}\xi\big)+c_2\cos\big(\frac{\sqrt{-\Delta}}{2}\xi\big)}{c_1\cos\big(\frac{\sqrt{-\Delta}}{2}\xi\big)+c_2\sin\big(\frac{\sqrt{-\Delta}}{2}\xi\big)}\Big]^2. \quad (11)$$

REF for $\Delta<0$ and $b\neq c$ is calculated as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

Case 2: $A_0=\frac{1}{2}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2), A_1=-3(b-c)m^2(a+bp-cp), A_2=\frac{3}{2}(b-c)^2m^2, B_1=0, B_2=0, \xi=\pm\sqrt{m^2+m^4\Delta}\frac{t^{\alpha}}{\alpha}+m\frac{x^{\beta}}{\beta}$:

For $\Delta>0$ and $b\neq c$,

$$u_{7,8}(x,t)=\frac{1}{2}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2)-6m^2(a+bp-cp)\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]+6m^2\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^2. \quad (12)$$

For the parameters $a=3, b=6, c=5, c_1=2, c_2=3, p=1/2, m=1/4$ and some values of the x, t, α, β, the values of the REF are given in Table 2.

For $\Delta=0$ and $b\neq c$,

$$u_{9,10}(x,t)=\frac{1}{2}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2)-3m^2(a+bp-cp)\Big[p(b-c)+a+2\frac{c_2}{c_1+c_2\xi}\Big]+\frac{3}{2}m^2\Big[p(b-c)+a+2\frac{c_2}{c_1+c_2\xi}\Big]^2. \quad (13)$$

REF for $\Delta=0$ and $b\neq c$ is computed as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

Table 2
REF ($a=3, b=6, c=5, c_1=2, c_2=3, p=1/2, m=1/4$) for the solution $u_8(x,t)$ in Case 2 ($\Delta>0$ and $b\neq c$).

x	t	α	β	REF
10	10	0.5	0.5	10^{-21}
10	20	0.5	0.25	10^{-24}
20	30	0.75	0.5	10^{-30}
−20	30	0.75	1	10^{-16}
4	5	0.5	0.25	10^{-18}
1	1	0.75	0.5	10^{-16}
7	8	0.75	0.5	10^{-20}
40	40	0.5	0.5	10^{-28}

For $\Delta<0$ and $b\neq c$, $(m\in(-1/\sqrt{-\Delta}, 1/\sqrt{-\Delta}))$

$$u_{11,12}(x,t)=\frac{1}{2}m^2\Bigg(2a^2-b^2+c^2+6abp-6acp+3b^2p^2+\sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\Bigg]-6bcp^2+3c^2p^2\Bigg)-3m^2(a+bp-cp)\left[p(b-c)+a+\frac{3}{2}m^2\times\left[p(b-c)+a+\sqrt{-\Delta}\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right]^2.\right. \tag{14}$$

REF for $\Delta<0$ and $b\neq c$ is computed as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

Case 3: $A_0=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)$, $A_1=0, A_2=0, B_1=-3m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3), B_2=\frac{3}{2}m^2(-b-c+2ap+bp^2-cp^2)^2, \xi=\pm\sqrt{m^2-m^4\Delta}\frac{t^\alpha}{\alpha}+m\frac{x^\beta}{\beta}$:

For $b=c,\quad a=0,$

$$u_{13,14}(x,t)=6b^2m^2(b\xi+c_1)^{-2}. \tag{15}$$

REF for $b=c,\quad a=0$ is computed as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

For $b=c,\quad a\neq 0,$

$$u_{15,16}(x,t)=-3m^2(-2ac+2a^2p)\left(p+c_1\exp(a\xi)-\frac{b}{a}\right)^{-1}+3/2m^2(2c-2ap)^2\left(p+c_1\exp(a\xi)-\frac{b}{a}\right)^{-2}. \tag{16}$$

REF for $b=c,\quad a\neq 0$ is computed as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

For $\Delta>0$ and $b\neq c$, $(m\in(-1/\sqrt{\Delta}, 1/\sqrt{\Delta}))$

$$u_{17,18}(x,t)=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)-3m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3)\times\left[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\right]^{-1}+\frac{3}{2}m^2(-b-c+2ap+bp^2-cp^2)^2\times\left[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\right]^{-2}. \tag{17}$$

For the parameters $a=3, b=6, c=5, c_1=2, c_2=3, p=2, m=1/\sqrt{2\Delta}$ and some values of the x,t,α,β, the values of the REF are given in Table 3.

Table 3
REF ($a=3, b=6, c=5, c_1=2, c_2=3, p=2, m=1/\sqrt{2\Delta}$) for the solution $u_{18}(x,t)$ in Case 3($\Delta>0$ and $b\neq c$).

x	t	α	β	REF
−1	1	0.5	1	10^{-17}
1	1	0.5	0.75	10^{-17}
4	5	0.5	0.5	10^{-19}
10	20	0.5	0.25	10^{-22}
20	30	0.25	0.75	10^{-21}
7	8	0.75	0.5	0
40	40	0.25	0.5	10^{-22}
−5	5	0.25	1	10^{-16}

For $\Delta=0$ and $b\neq c$,

$$u_{19,20}(x,t)=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)-3m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3)\times\left[p+\frac{a}{b-c}+\frac{2}{b-c}\frac{c_2}{c_1+c_2\xi}\right]^{-1}+\frac{3}{2}m^2(-b-c+2ap+bp^2-cp^2)^2\times\left[p+\frac{a}{b-c}+\frac{2}{b-c}\frac{c_2}{c_1+c_2\xi}\right]^{-2}. \tag{18}$$

For the values $x=20, t=30, \alpha=0.25, \beta=0.75$ and some values of the a,b,c,c_1,c_2,p,m, the values of the REF are given in Table 4.

Table 4
REF ($x=20, t=30, \alpha=0.25, \beta=0.75$) for the solution $u_{20}(x,t)$ in Case 3($\Delta=0$ and $b\neq c$).

a	b	c	c_1	c_2	p	m	REF
3	4	5	2	3	2	4	0
6	8	10	2	3	2	4	0
8	15	17	2	3	2	4	0
7	24	25	2	3	2	4	0
5	12	13	2	3	2	4	0
4	8	$\sqrt{80}$	2	3	2	4	10^{-16}
7	8	$\sqrt{113}$	2	3	2	4	10^{-16}
10	80	$\sqrt{6500}$	2	3	2	4	10^{-15}

For $\Delta<0$ and $b\neq c$,

$$u_{21,22}(x,t)=\frac{3}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)-3m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2$$

Table 5
REF for the solution $u_{22}(x,t)$ in Case 3($\Delta < 0$ and $b \neq c$).

a	b	c	c_1	c_2	p	m	REF
1	2	3	2	3	2	4	0
7	6	11	2	3	2	4	0
6	12	14	2	3	2	4	0
5	3	20	2	3	2	4	0
1	9	10	2	3	2	4	0
3	6	9	2	3	2	4	0
7	25	30	2	3	2	4	0
−1	−5	8	2	3	2	4	0

$$- 3acp^2 + b^2p^3 - 2bcp^3 + c^2p^3)$$
$$\times \left[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right]^{-1}$$
$$+\frac{3}{2}m^2(-b-c+2ap+bp^2-cp^2)^2$$
$$\times \left[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right]^{-2}. \quad (19)$$

For the arbitrary values x, t, α, β and for some parameters a, b, c, c_1, c_2, p, m, the computation of values REF are given in Table 5. In the Table, the values of the REF equal to the zero.

4. Analytic solutions to the conformable space-time fractional (2 + 1)-dimensional breaking soliton equation

Conformable space-time fractional (2 + 1)-dimensional breaking soliton equation is presented in the following form [32,33]

$$T_x^\beta T_x^\beta T_x^\beta T_y^\theta u - 2T_y^\theta u T_x^\beta T_x^\beta u - 4T_x^\beta u T_x^\beta T_y^\theta u + T_x^\beta T_t^\alpha u = 0,$$
$$0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 0 < \theta \leq 1. \quad (20)$$

Using the following transformation

$$u(x,y,t) = U(\xi), \quad \xi = k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{y^\theta}{\theta}, \quad (21)$$

where k, m and n are constants, for Eq. (20), we have the following differential equation

$$m^3nU^{(4)} - 6m^2nU'U'' + kmU'' = 0. \quad (22)$$

Integrating of Eq. (22) with zero constant of integration, we write

$$m^3nU''' - 3m^2n(U')^2 + kmU' = 0. \quad (23)$$

Let us suppose that the solution of Eq. (23) can be written as Eq. (5). Substituting Eq. (5) into Eq. (23) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 1. Therefore, Eq. (5) reduces to

$$U(\xi) = A_0 + A_1\left[p + \tan\left(\frac{\phi(\xi)}{2}\right)\right] + B_1\left[p + \tan\left(\frac{\phi(\xi)}{2}\right)\right]^{-1}. \quad (24)$$

Substituting Eq. (24) into Eq. (23), collecting all the terms with the same power of $\tan(\frac{\phi}{2})$, we can obtain a set of algebraic equations for the unknowns A_0, A_1, B_1, k, m, n:

$$6nA_1^2bcm^2 - 3nA_1^2b^2m^2 - 3nA_1^2c^2m^2 - 3nA_1b^3m^3$$
$$+9nA_1b^2cm^3 - 9nA_1b.c^2m^3 + 3nA_1c^3m^3 = 0,$$
$$24npA_1^2bcm^2 - \ldots$$

Solving the algebraic equations in the Mathematica, we can obtain the solutions of the Eq. (20) in the form of Eq. (24).

For the conformable space-time fractional (2 + 1)-dimensional breaking soliton equation, we define residual error function (REF) as follows

$$REF(x) = T_x^\beta T_x^\beta T_x^\beta T_y^\theta u - 2T_y^\theta u T_x^\beta T_x^\beta u - 4T_x^\beta u T_x^\beta T_y^\theta u$$
$$+ T_x^\beta T_t^\alpha u, 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 0 < \theta \leq 1. \quad (25)$$

Case 1: $A_0 = A_0, A_1 = -m(b-c), B_1 = -\frac{m\Delta}{b-c}, p = -\frac{a}{b-c}, \xi = (-4m^2n\Delta\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{y^\theta}{\theta})$:

For $\Delta > 0$ and $b \neq c$,

$$u_1(x,y,t) = A_0 - 2m\left[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\right] - \frac{m\Delta}{2}\left[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\right]^{-1}. \quad (26)$$

For $\Delta < 0$ and $b \neq c$,

$$u_2(x,y,t) = A_0 - m\sqrt{-\Delta}\left[\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right] - \frac{m\Delta}{\sqrt{-\Delta}}\left[\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right]^{-1}. \quad (27)$$

Case 2: $A_0 = A_0, A_1 = -m(b-c), B_1 = 0, p = -\frac{a}{b-c}, \xi = (-m^2n\Delta\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{y^\theta}{\theta})$:

For $\Delta > 0$ and $b \neq c$,

$$u_3(x,y,t) = A_0 - 2m\left[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\right]. \quad (28)$$

For $\Delta < 0$ and $b \neq c$,

$$u_4(x,y,t) = A_0 - m\sqrt{-\Delta}\left[\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right]. \quad (29)$$

REFs in the Case 1 and Case 2 are computed as zero by using the symbolic toolbox in Matlab Programming for arbitrary parameters.

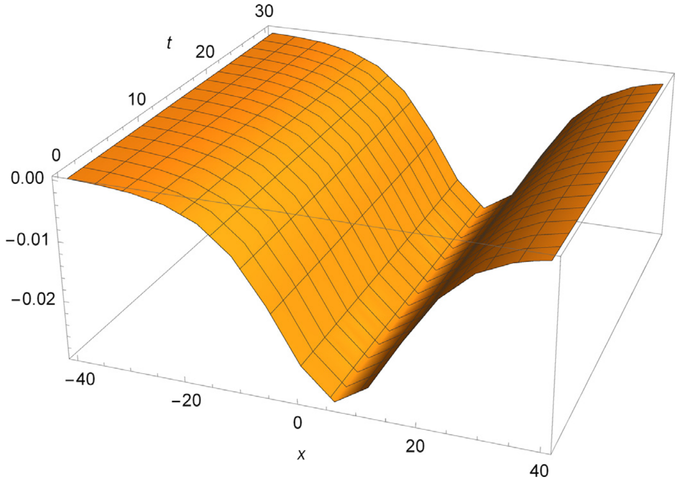

Fig. 1. 3D plot for the obtained traveling wave solution $u_2(x,t)$ in Eq. (9).

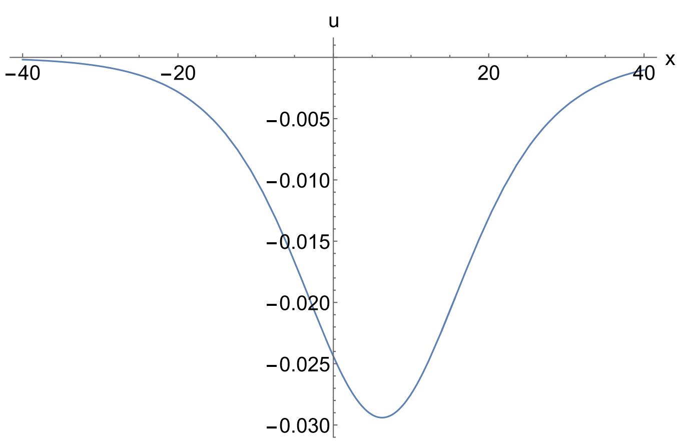

Fig. 2. 2D plot for the obtained traveling wave solution $u_2(x,1)$ in Eq. (9).

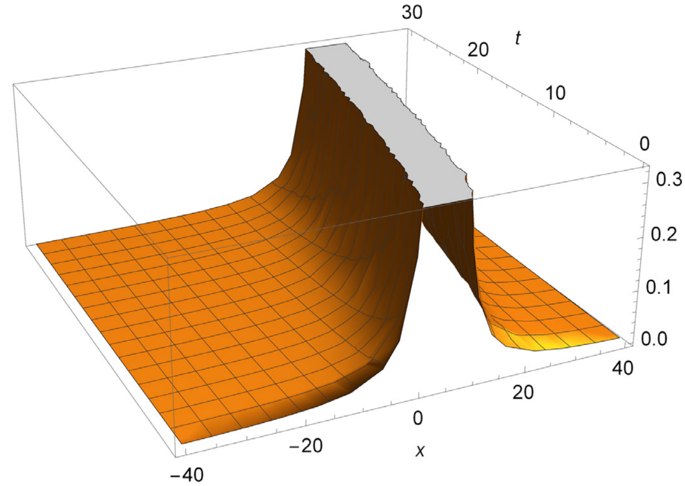

Fig. 3. 3D plot for the obtained traveling wave solution $u_4(x,t)$ in Eq. (10).

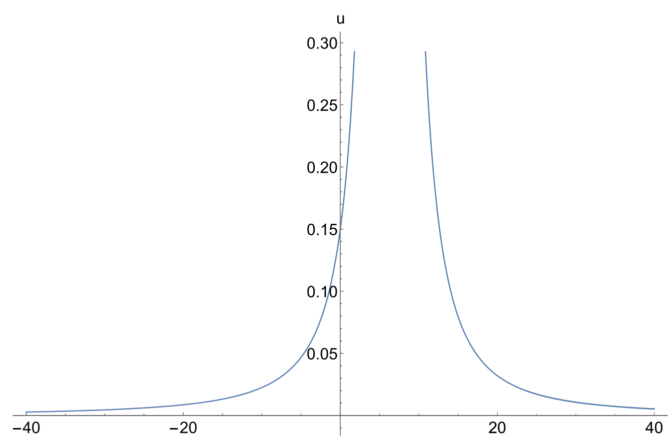

Fig. 4. 2D plot for the obtained traveling wave solution $u_4(x,1)$ in Eq. (10).

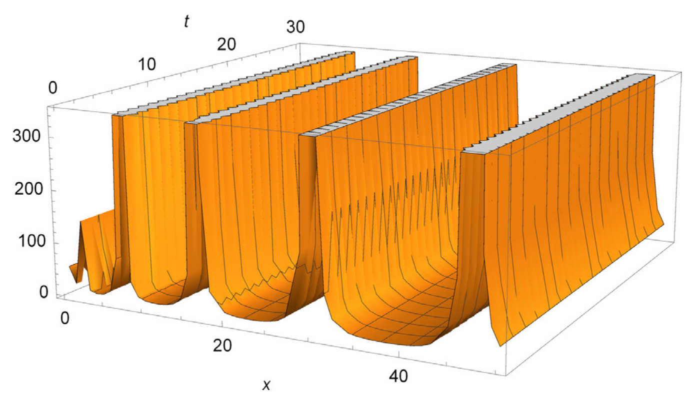

Fig. 5. 3D plot for the obtained traveling wave solution $u_6(x,t)$ in Eq. (11).

5. Results and discussion

Traveling waves have a great importance in solitary wave theory. Three of traveling waves are the solitary waves, the periodic waves and the kink waves [44]. The solitary waves are localized traveling waves and asymptotically zero at large distances. The kink waves rise or descend from one asymptotic state to another. Periodic waves are waves with a repeating continuous pattern. In this paper, we have established the traveling wave solutions of the conformable space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations. These solutions are expressed in terms of the trigonometric, exponential and rational functions involving arbitrary parameters. For the special values of the parameters, physical behaviors of the obtained solutions are given in Figs. 1–16. Figs. 1 and 2 show solitary wave solutions of Eq. (1). Fig. 1 is 3D plot of the traveling wave solution $u_2(x,t)$ in Eq. (9) for $\alpha = 0.75$, $\beta = 1$, $m = -0.2$, $a = 0.5$, $b = 0.5$, $c = 0.1$, $c_1 = 2$, $c_2 = 1$ and $p = 0.02$. Fig. 2 demonstrates the same solution with 2D plot for $-40 \le x \le 40$ at $t = 1$. Figs. 3 and 4 are 3D and 2D plots of the singular soliton of the solution $u_4(x,t)$ in Eq. (10) for $\alpha = 0.75$, $\beta = 1$, $m = -0.2$, $a = 3$, $b = 4$, $c = 5$, $c_1 = 1$, $c_2 = 1$ and $p = 1$, respectively. Figs. 5 and 6 are periodic wave solutions of Eq. (1). In these figures, 3D and 2D plots of the traveling wave solution $u_6(x,t)$ in Eq. (11) are given for $\alpha = 0.5$, $\beta = 0.5$, $m = 0.5$, $a = 1$, $b = 2$, $c = 5$, $c_1 = 5$, $c_2 = 2$ and $p = 1$, respectively. We see that the wave amplitudes go to infinity and the wavelengths increase when x approaches to infinity. Figs. 7 and 8 are 3D and 2D plots of the solitary wave

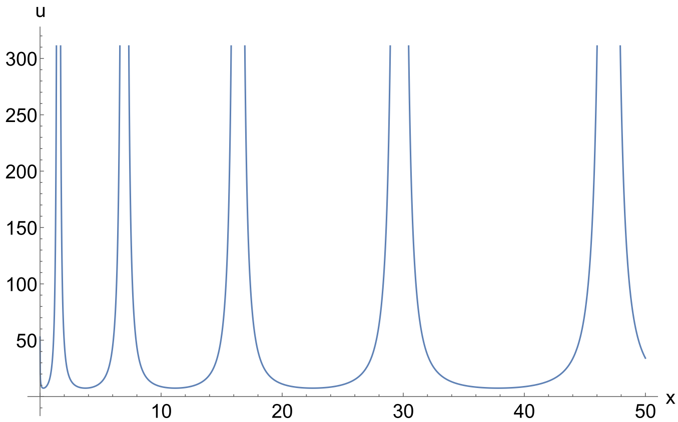

Fig. 6. 2D plot for the obtained traveling wave solution $u_6(x, 1)$ in Eq. (11).

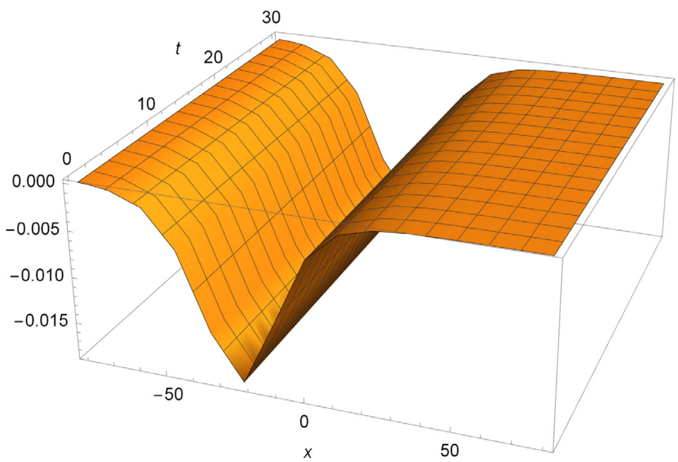

Fig. 7. 3D plot for the obtained traveling wave solution $u_{18}(x, t)$ in Eq. (17).

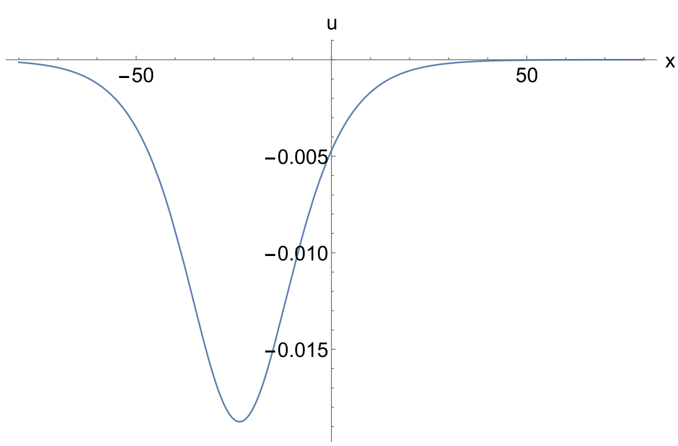

Fig. 8. 2D plot for the obtained traveling wave solution $u_{18}(x, 1)$ in Eq. (17).

solution $u_{18}(x, t)$ in Eq. (17) for $\alpha = 0.5$, $\beta = 1$, $m = \frac{1}{2\sqrt{\Delta}}$, $a = 3$, $b = 6$, $c = 5$, $c_1 = 2$, $c_2 = 3$ and $p = 2$, respectively. Note that 3D graph gives the action of u in the space x at time t and illustrates the change of amplitude and shape for each obtained solitary wave solutions. 2D graph shows the action of u in space x at fixed time $t = 1$.

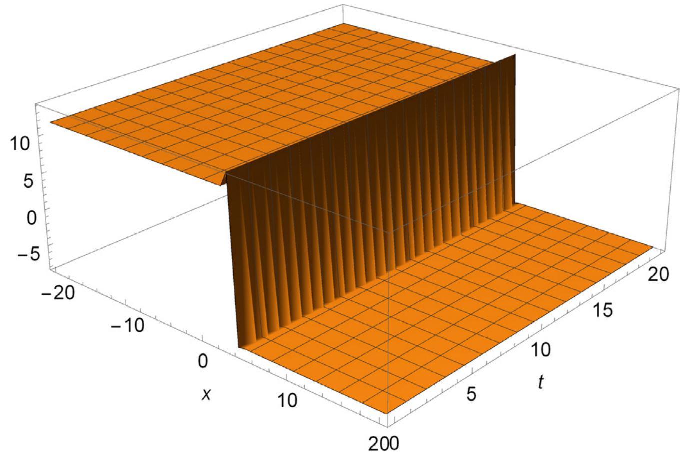

Fig. 9. 3D plot for the obtained traveling wave solution $u_1(x, 1, t)$ in Eq. (26).

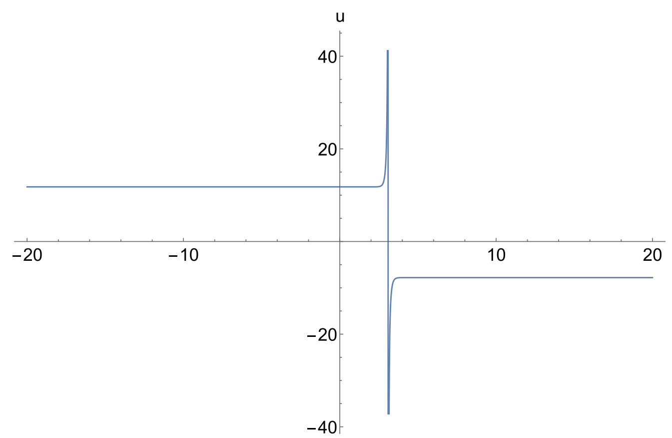

Fig. 10. 2D plot for the obtained traveling wave solution $u_1(x, 1, 1)$ in Eq. (26).

The solutions $u_1(x, y, t)$, $u_2(x, y, t)$, $u_3(x, y, t)$ and $u_4(x, y, t)$ of the Eq. (20) are simulated as traveling wave solutions for various values of the physical parameters in Figs. 9–16. Figs. 9 and 10 are 3D and 2D plots of the solitary wave solutions $u_1(x, 1, t)$ and $u_1(x, 1, 1)$ in Eq. (26) for $\alpha = 0.75$, $\beta = 1$, $\theta = 0.25$, $m = 2$, $n = 0.05$, $a = 3$, $b = 1$, $c = 2$, $c_1 = -1$, $c_2 = 1$ and $A_0 = 2$, respectively. Figs. 11 and 12 are 3D and 2D plots of the periodic wave solutions $u_2(x, 1, t)$ and $u_2(x, 1, 1)$ in Eq. (27) for $\alpha = 0.75$, $\beta = 1$, $\theta = 0.25$, $m = 0.05$, $n = -1$, $a = 1$, $b = 0.2$, $c = 3$, $c_1 = 2$, $c_2 = 2$ and $A_0 = 2$, respectively. Figs. 13 and 14 are 3D and 2D plots of the anti-kink wave solutions $u_3(x, 1, t)$ and $u_3(x, 1, 1)$ in Eq. (28) for $\alpha = 0.75$, $\beta = 1$, $\theta = 0.5$, $m = 0.1$, $n = -0.5$, $a = 5$, $b = 5$, $c = 1$, $c_1 = 0.5$, $c_2 = 2$ and $A_0 = 1$, respectively. Figs. 15 and 16 are 3D and 2D plots of the periodic wave solutions $u_4(x, 1, t)$ and $u_4(x, 1, 1)$ in Eq. (29) for $\alpha = 0.75$, $\beta = 1$, $\theta = 0.25$, $m = 0.1$, $n = -0.5$, $a = 1$, $b = 4$, $c = 5$, $c_1 = 2$, $c_2 = 2$ and $A_0 = 1$, respectively. Note that 3D graph gives the action of u in the spaces x and y at time t and illustrates the change of amplitude and shape for each obtained solitary wave solutions. 2D graph shows

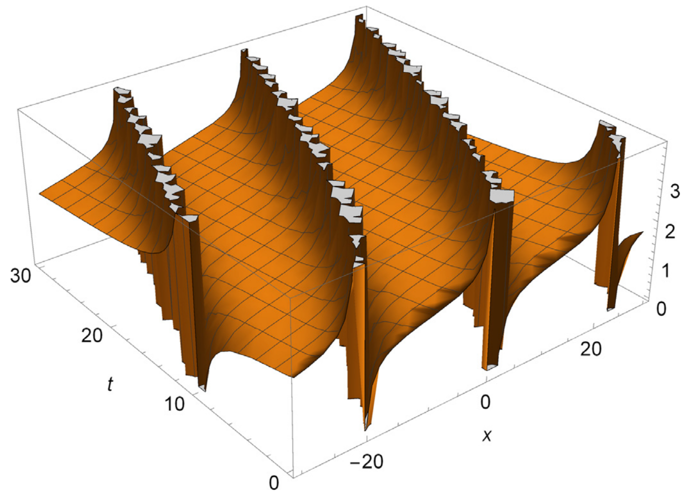

Fig. 11. 3D plot for the obtained traveling wave solution $u_2(x, 1, t)$ in Eq. (27).

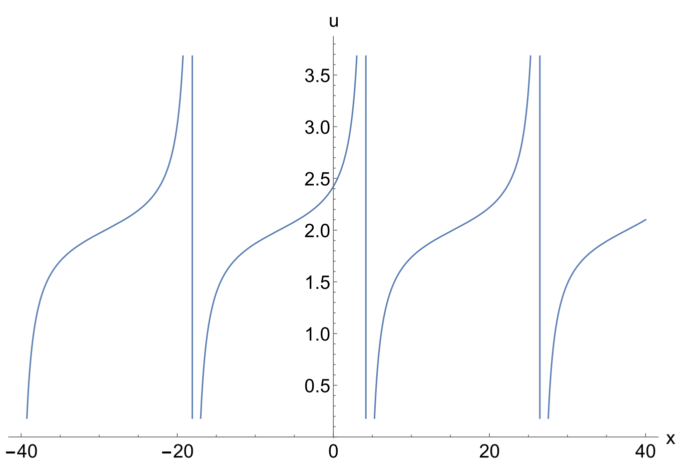

Fig. 12. 2D plot for the obtained traveling wave solution $u_2(x, 1, 1)$ in Eq. (27).

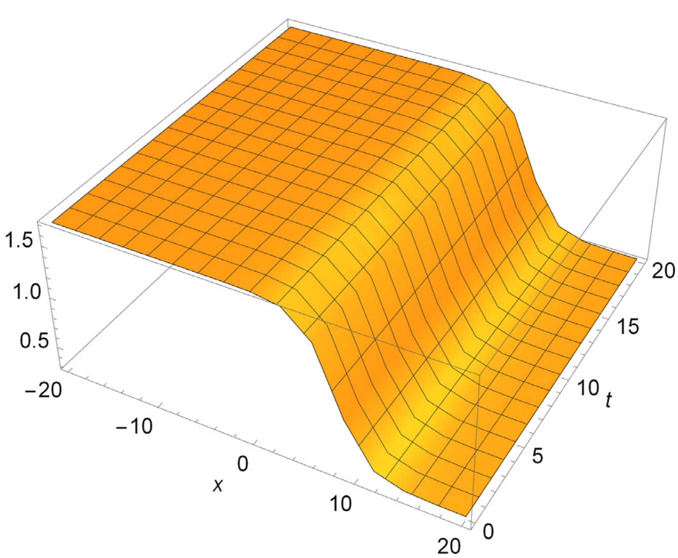

Fig. 13. 3D plot for the obtained traveling wave solution $u_3(x, 1, t)$ in Eq. (28).

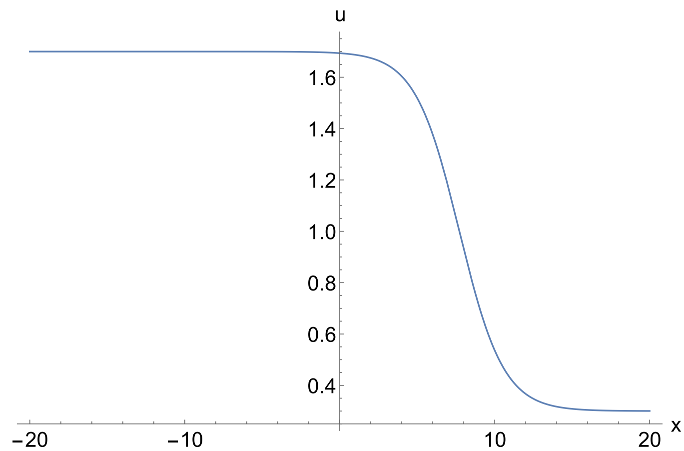

Fig. 14. 2D plot for the obtained traveling wave solution $u_3(x, 1, 1)$ of Eq. (28).

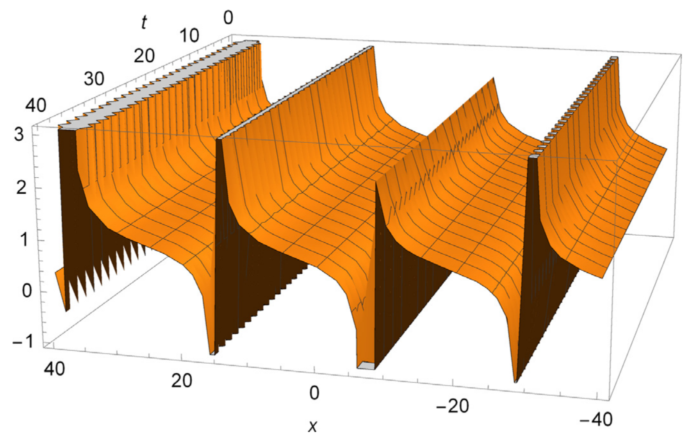

Fig. 15. 3D plot for the obtained traveling wave solution $u_4(x, 1, t)$ of Eq. (29).

the action of u in the space x at fixed $y = 1$ and $t = 1$. All graphics in figures are drawn by the aid of Mathematica 10.

The accuracy of the obtained solutions is shown by using REF. For the solutions $u_2(x, t)$, $u_8(x, t)$ and $u_{18}(x, t)$ in Eq. (9), Eq. (12) and Eq. (17), the some values of the REF are given in Table 1, Table 2 and Table 3, respectively. It is observed that the maximum values of the REFs are 10^{-16}. This means that the obtained solutions are close to the exact solutions. For the solutions in Eqs. (10), (11), (13), (14), (15), (16), (26), (27), (28) and (29), REFs are equal to zero for the arbitrary parameters. These results are obtained by using the symbolic toolbox in Matlab. Therefore, the obtained solutions are the exact solution. For the solution $u_{20}(x, t)$ in Eq. (18), the some values of the REF are given in Table 4. Note that for the integer values of the parameters a, b, c, REF is zero. For the arbitrary values x, t, α, β and for special parameters a, b, c, c_1, c_2, p, m, REF of the solution $u_{22}(x)$ in Eq. (19) is given in Table 5. Note that all values of the REF are equal to the zero.

Generally speaking, the selection of the parameters is important in the calculation of REF. As can be seen from the Tables 1–5, it is possible to obtain the exact solution for some

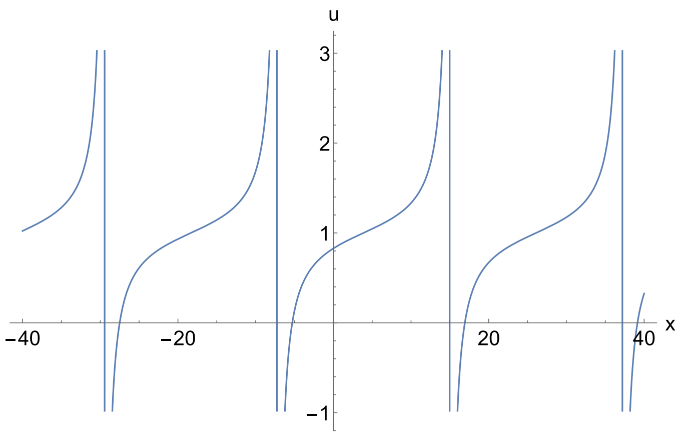

Fig. 16. 2D plot for the obtained traveling wave solution $u_4(x, 1, 1)$ of Eq. (29).

values of the parameter, while it is possible to obtain the solutions close to the exact solutions for others.

6. Conclusion

In this paper, we have derived many traveling wave solutions of the conformable space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations. These solutions are expressed in terms of rational, trigonometric and exponential functions involving arbitrary parameters. When these parameters are special values, physical behaviors of these solutions are presented in Figs. 1–16. Furthermore, REF has been defined to check the accuracy of the solutions. The computation of the REF for the conformable space-time fractional Boussinesq equation shows that some of the results are close to the exact solution, while others are the exact solution depending on the specific selection of the parameters. The solutions of the conformable space-time fractional (2 + 1)-dimensional breaking soliton equation are the exact solutions for the arbitrary parameters. This is because the REF is always calculated as zero regardless of the choice of parameters.

In this article, SITEM method is applied to the conformable space-time fractional Boussinesq and (2 + 1)-dimensional breaking soliton equations. To our knowledge, SITEM has not been studied extensively in the literature. Note that the obtained solutions are new and have not been reported in former literature. SITEM can be also applied to the others fractional or nonfractional, nonlinear PDEs equations with constant coefficients.

Declaration of Competing Interest

Authors have no interests to declare.

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