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Maximal and fractional maximal operators in the Lorentz-Morrey spaces and their applications to the Bochner-Riesz and Schrödinger-type operators

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Abstract

The aim of this paper is to obtain boundedness conditions for the maximal function Mf and to prove the necessary and sufficient conditions for the fractional maximal oparator M_α in the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ which are a new class of functions. We get our main results by using the obtained sharp rearrangement estimates. The obtained results are applied to the boundedness of particular operators such as the Bochner-Riesz operator B_r^δ and the Schrödinger-type operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$ in the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$, where the nonnegative potential V belongs to the reverse Hölder class $B_\infty(\mathbb{R}^n)$.

Subject Classification: *42B20, 42B35, 47G10.*

Keywords: *Maximal operator, Fractional maximal operator, Lorentz-Morrey spaces, Bochner-Riesz operator, Schrödinger-type operators.*

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1. Introduction

Let $B(x,r)$ the open ball centered at x of radius r for $x \in \mathbb{R}^n$ and $|B(x,r)|$ is the Lebesgue measure of $B(x,r)$. The fractional maximal operator is defined at $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$M_\alpha f(x) := \sup_{r>0} |B(x,r)|^{\frac{\alpha}{n}-1} \int_{B(x,r)} |f(y)|\,dy, 0 \leq \alpha < n$$

where the supremum is taken over all the balls centered at x of radius r. Note that in the case $\alpha = 0$ we get the classical Hardy-Littlewood maximal operator $M := M_0$. It is well known that for the maximal operator M the rearrangement inequality

$$cf^{**}(t) \leq (Mf)^*(t) \leq Cf^{**}(t),\ t \in (0,\infty)$$

holds, (see [4], Chapter 3, Theorem 3.8) where f^* is the non-increasing rearrangement of f such that $f^*(t) := \inf\{\lambda > 0 : d_f(\lambda) \leq 0\}$, $d_f(\lambda)$ denotes the distribution function of f given by $d_f(\lambda) := |\{x \in (0,\infty) : |f(x)| > \lambda\}|$ for all $t > 0$, and $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)ds$.

The Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ are a new class of functions and introduced by Mingione in [18] as follows.

Definition 1.1 : Let $1 \leq p < \infty$, $0 < q < \infty$, $0 \leq \lambda \leq n$, and $f \in \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$. Then the Lorentz-Morrey space $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ iff

$$\| f \|_{\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \| f\chi_{B(x,r)} \|_{L_{p,q}(\mathbb{R}^n)} < \infty.$$

Mingione [18], studied the boundedness of the restricted fractional maximal operator M_{α,B_0} in the restricted Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(B)$, where B_0 is a given ball and B is any other ball contained in B_0 and containing x. The author derived a general non-linear version, extending *a priori* estimates and *regularity* results for possibly degenerate non-linear elliptic problems to the various spaces of Lorentz and Lorentz-Morrey type considered in [1, 3, 18] and [22]. In [22], Ragusa studied some embeddings between these spaces. Note that the spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ and $L_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n)$ defined by Mingione and Ragusa respectively, coincide, thus $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n) = L_{p,q;\lambda\frac{q}{p}}(\mathbb{R}^n)$. The local variant of Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ replacing by $B(0,r)$ instead of $B(x,r)$, so called the local Morrey-Lorentz spaces $\mathcal{L}^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ are introduced and the basic properties of these spaces are given in [2]. Recently, in [3, 11] and [12], the authors studied the boundedness of some classical operators of harmonic analysis in these spaces.

In this paper, first, we give some basic properties of Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$. Furthermore, we get the sharp rearrangement inequalities which we use while proving our results. Next, in section 3, we obtain the boundedness conditions for the maximal function Mf in the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ and we get the necessary and sufficient conditions for boundedness of the fractional maximal operator M_α in the spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$. Finally, in section 4, we apply these results to the Bochner-Riesz operator B_r^δ and the Schrödinger-type operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$ in the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$, respectively, where the nonnegative potential V belongs to the reverse Hölder class $B_\infty(\mathbb{R}^n)$.

Throughout the paper, we denote by c and C for positive constants, independent of appropriate parameters and not necessary the same at each occurrence. If $p\in[1,\infty]$, the conjugate number p' is defined by $\frac{1}{p}+\frac{1}{p'}=1.$ Finally, for non-negative expressions A_1, A_2 we use the symbol $A_1\approx A_2$ to express that $cA_1\le A_2\le CA_1$ for some positive constants c and C independent of the variables in the expressions A_1 and A_2.

2. Preliminaries

The Lorentz space $L_{p,q}(\mathbb{R}^n)$ is the collection of all measurable functions of f on $\mathbb{R}^n$ such the quantity

$$\|f\|_{L_{p,q}(\mathbb{R}^n)}:=\begin{cases}\left(\int_0^\infty\left(t^{\frac{1}{p}}f^*(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}, & 0<q<\infty, 0<q<\infty\\ \sup\limits_{t>0}t^{\frac{1}{p}}f^*(t), & 0<p\le\infty, q=\infty,\end{cases}$$

is finite. If $1<p\le\infty, 1\le q\le\infty,$ then

$$\|f\|_{Lp,q(\mathbb{R}^n)}\le\|f\|^*_{Lp,q(\mathbb{R}^n)}\le\frac{p}{p-1}\|f\|_{Lp,q(\mathbb{R}^n)}.$$

For more detail useful references about Lorentz spaces considered in [4].

We denote by $L_{p,\lambda}(\mathbb{R}^n)$ Morrey space given in [19]; $0\le\lambda\le n, 1\le p\le\infty, f\in L_{p,\lambda},$ if $f\in L_p^{loc}(\mathbb{R}^n)$ and

$$\| f \|_{L_{p,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \| f \|_{L_p(B(x,r))} < \infty.$$

Morrey spaces appeared to be useful in the study of local behavior properties of the solutions of second order elliptic PDEs. For more information about Morrey-type spaces see [5, 6, 10, 13] and [14].

The Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ are a very natural generalization of the Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ and Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$.

Remark 2.1 : As a consequence by Lemma 2.2 (*ii*), if $q = p$ then $\mathcal{L}_{p,p,\lambda}(\mathbb{R}^n) \equiv L_{p,\lambda}(\mathbb{R}^n)$, if $\lambda = 0$ then $\mathcal{L}_{p,q,0}(\mathbb{R}^n) \equiv L_{p,q}(\mathbb{R}^n)$, and $\lambda = n, p = q$, then $\mathcal{L}_{p,p,n}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n) \equiv \theta$, where θ the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.2 : [4], [8], [21]

(i) *Let* $0 < p < \infty$ *, then* $\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty \left(f^*(t)\right)^p dt$ *holds.*

(ii) *For any* $t > 0$, $\sup_{|E|=t} \int_E |f(x)| dx = \int_0^t f^*(s) ds$.

(iii) *For any* $t > 0$, $(f+g)^*(t) \le f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right)$ *holds.*

Lemma 2.3 : *Let* $0 \le \alpha < n$. *Then there exist a positive constant C, depending on* α *and n such that*

$$\sup_{t>0} t^{1-\frac{\alpha}{n}} (M_\alpha f \chi_{B(x,t)})^*(t) \le C \int_{\mathbb{R}^n} | f(x) | \, dx \tag{2.1}$$

and

$$\sup_{t>0} (M_\alpha f \chi_{B(x,t)})^*(t) \le C \sup_{t>0} t^{\frac{\alpha}{n}} f^*(t). \tag{2.2}$$

Proof : The estimate (2.1) follows from (Theorem 1.1, in [7]). For the estimate (2.2), for every $B(x,r) \subset \mathbb{R}^n$, we get

$$\sup_{r>0} | B(x,r) |^{\frac{\alpha}{n}-1} \int_{B(x,r)} | f(y) | \, dy \le | B(x,r) |^{\frac{\alpha}{n}-1} \int_0^{|B(x,r)|} t^{\frac{\alpha}{n}} f^*(t) t^{-\frac{\alpha}{n}} dt$$
$$\le \frac{n}{n-\alpha} \sup_{t>0} t^{\frac{\alpha}{n}} f^*(t).$$

Hence the proof is completed. □

Lemma 2.4 : *Let* $0 \le \alpha < n$. *Then there exist a positive constant C, depending only on n and* α, *such that*

$$(M_\alpha f\chi_{B(x,t)})^*(t) \le C \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau),\ t>0 \tag{2.3}$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n)$. *Inequality (2.3) is sharp in the sense that for all* $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$ *there exists a function* f *on* $\mathbb{R}^n$ *such that* $f^* = \varphi$ *a.e. on* $(0,\infty)$ *and*

$$(M_\alpha f\chi_{B(x,t)})^*(t) \ge c \sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau),\ t>0, \tag{2.4}$$

where $\mathcal{M}^+(0,\infty;\downarrow)$ *is the set of all non-negative and non-increasing measurable functions on* $(0,\infty)$ *and c is a positive constant which depends only on n and* α.

Proof : To prove the inequality (2.3), we may suppose that

$$\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau) < \infty,$$

otherwise there is nothing to prove. Then by Lemma 2.2 (i)

$$\int_{\mathbb{R}^n} \left| f\chi_{B(x,r)}(x)\right| dx = \int_0^t (f\chi_{B(x,r)})^*(s)ds$$

holds for all $E \subset \mathbb{R}^n$ with $|E| \le t$. In particular, if we put

$$E = \left\{x : |f(x)| > (f\chi_{B(x,r)})^*(t)\right\}$$

then $|E| \le t$ and so $f \in L_1(E)$. Then the function

$$g_t(x) = \max\left\{|f(x)| - (f\chi_{B(x,r)})^*(t), 0\right\} sgn\ f(x),$$

belongs to $L_1(\mathbb{R}^n)$. Also the function

$$h_t(x) = \min\left\{|f(x)|, (f\chi_{B(x,r)})^*(t)\right\} sgn\ f(x),$$

holds

$$(h_t)^*(\tau) = \min\left\{(f\chi_{B(x,r)})^*(\tau), (f\chi_{B(x,r)})^*(t)\right\}, \tau \in (0,\infty).$$

Thus

$$\begin{aligned}\sup_{\tau>0} \tau^{\frac{\alpha}{n}} (h_t)^*(\tau) &= \max\left\{ \sup_{0<\tau<t} t^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^*(t), \sup_{t\le\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^*(\tau) \right\} \\ &= \sup_{t\le\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^*(\tau) \le \sup_{t\le\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(t)\end{aligned} \tag{2.5}$$

which together with the inequality (2.4) implies that $h_t \in WL_{\frac{n}{\alpha}}$. Furthermore, since $f = h_t + g_t$ and

$$(g_t)^*(\tau) = \chi_{[0,t)}(\tau)\left((f\chi_{B(x,r)})^*(\tau) - (f\chi_{B(x,r)})^*(t)\right), \tau \in (0,\infty). \tag{2.6}$$

By using Lemma 2.2 (iv), Lemma 2.3, the inequalities (2.5) and (2.6), we get

$$\begin{aligned}(M_\alpha f)^*(t) &\le (M_\alpha g_t)^*\left(\frac{t}{2}\right) + \left(M_\alpha h_t\right)^*\left(\frac{t}{2}\right) \\ &\lesssim \left(\left(\frac{t}{2}\right)^{\frac{\alpha}{n}-1} \int_{\mathbb{R}^n} g_t(y)dy + \sup_{\tau>0} \tau^{\frac{\alpha}{n}} (h_t)^*(\tau) \right) \\ &\lesssim t^{\frac{\alpha}{n}-1} \int_0^t \left((f\chi_{B(x,r)})^*(\tau) - (f\chi_{B(x,r)})^*(t)\right) d\tau \\ &\quad + \sup_{0<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau) \\ &\lesssim \sup_{0<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau)\end{aligned}$$

and the inequality (2.3) follows. Furthermore, the inequality (2.4) exist for all $t \in (0,\infty)$. Let $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$, where $\mathcal{M}^+(0,\infty;\downarrow)$ is the set of all non-negative and non-increasing measurable functions on $(0,\infty)$. Putting $f(x) = \varphi(\omega_n |x|^n)$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, $\omega_n = |B(0,r)|$; and $y \in B(x,r)$, we have $(f\chi_{B(x,r)})^* = \varphi(0,\infty)$. Moreover, denote by $B(x,|y|)$ the ball with centered x and having radius $|y|$. Then, for $|y| > |x|$,

$$\begin{aligned}(M_\alpha f)^*(t) &= \sup_{r>0} |B(x,r)|^{\frac{\alpha}{n}-1} \int_{B(x,r)} |f(y)|\,dy \\ &\ge |B(x,|y|)|^{\frac{\alpha}{n}-1} \int_{B(x,|y|)} |f(y)|\,dy\end{aligned}$$

$$= c\left(\omega_n(|y|^n)^{\frac{\alpha}{n}-1}\int_0^{\omega_n|y|^n}(f\chi_{B(x,r)})^*(\tau)d\tau\right)$$
$$= cH(\omega_n|y|^n),$$

where H is the Hardy operator given in [23] defined as $H(t) = t^{\frac{\alpha}{n}-1}\int_0^t \varphi(\tau)d\tau, t \in (0,\infty)$. Consequently,

$$(M_\alpha f)(x) \geq c \sup_{\tau > \omega_n|x|^n} H(\tau),$$

thus the inequality (2.4) follows on taking rearrrangements. Hence the proof is completed. □

3. Main Results

In this section, we characterize the boundedness conditions of maximal operators M and prove the necessary and sufficient conditions for the fractional maximal oparator M_α in the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ by using the obtained sharp rearrangement estimates.

Theorem 3.1 : *Let $1 < p < \infty, 1 \leq q < \infty$ and $0 \leq \lambda \leq n$ and for all $f \in \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$, then maximal operator M is bounded in the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$.*

Proof : Let $1 < p < \infty, 1 \leq q < \infty$. Then by using definition of the spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$, Lemma 2.2 (ii) and Lemma 2.3, we get

$$\|Mf\|_{\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)} = \sup_{r>0} r^{-\frac{\lambda}{p}}\left\|t^{\frac{1}{p}-\frac{1}{q}}(Mf)^*(t)\right\|_{L_q(0,\infty)}$$
$$= \sup_{r>0} r^{-\frac{\lambda}{p}}\left(\int_0^\infty\left(t^{\frac{1}{p}}(Mf)^*(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\approx \sup_{r>0} r^{-\frac{\lambda}{p}}\left(\int_0^\infty\left(t^{\frac{1}{p}}(f\chi_{B(x,r)})^{**}(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\leq \sup_{r>0} r^{-\frac{\lambda}{p}}\|f\|^*_{L_{p,q}(B(x,r))}$$
$$\leq \frac{p}{p-1}\|f\|_{\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)}.$$

Hence the maximal operator M is bounded on the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$. □

Theorem 3.2 : *Let* $0 \le \alpha < n$. *Then the following statements are equivalent:*

(i) *If* $1 < p \le q < \infty, 1 \le u \le s \le \infty, 1 < p < \frac{n-\lambda}{\alpha}, 0 < \lambda < n$, *then the fractional maximal operator* M_α *is bounded from Lorentz-Morrey space* $\mathcal{L}_{p,u,\lambda}(\mathbb{R}^n)$ *to another one* $\mathcal{L}_{q,s,\lambda}(\mathbb{R}^n)$ *such that*

$$\|M_\alpha f\|_{\mathcal{L}_{q,s;\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}_{p,u;\lambda}(\mathbb{R}^n)}.$$

(ii) *For all* $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$ *there exists a positive constant C such that*

$$\begin{aligned} &\sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left(\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}-1} \int_0^\tau \varphi(\sigma)d\sigma\right)^s t^{\frac{s}{q}-1} dt\right]^{\frac{1}{s}} \\ &\le C \sup_{r>0} r^{-\frac{\lambda}{p}} \left[\int_0^\infty \varphi^p(t) t^{\frac{u}{p}-1} dt\right]^{\frac{1}{u}}. \end{aligned} \tag{3.1}$$

(iii) $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.

Proof : (i) $\Leftrightarrow$ (ii).

(i) Assume that the fractional maximal operator M_α is bounded from $\mathcal{L}_{p,u,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,s,\lambda}(\mathbb{R}^n)$. Then $\|M_\alpha f\|_{\mathcal{L}_{q,s,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}_{p,u,\lambda}(\mathbb{R}^n)}$ holds.

For every $\varphi = (f\chi_{B(x,r)})^*(t) \in \mathcal{M}^+(0,\infty;\downarrow)$, $(f\chi_{B(x,r)})^* = \varphi$ a.e. on $(0,\infty)$, and from Lemma 2.4

$$\begin{aligned} &\sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left(\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}-1} \int_0^\tau (f\chi_{B(x,r)})^*(\sigma)d\sigma\right)^s t^{\frac{s}{q}-1} dt\right]^{\frac{1}{s}} \\ &= \sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left(\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau)\right)^s t^{\frac{s}{q}-1} dt\right]^{\frac{1}{s}} \\ &\lesssim \sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left((M_\alpha f)^*(t)\right)^s t^{\frac{s}{q}-1} dt\right]^{\frac{1}{s}} \end{aligned}$$

$$\lesssim \sup_{r>0} r^{-\frac{\lambda}{p}} \left[\int_0^\infty \left((f\chi_{B(x,r)})^*(t) \right)^u t^{\frac{u}{p}-1} dt \right]^{\frac{1}{u}}$$

holds.

(ii) Conversely, for every $\varphi = (f\chi_{B(x,r)})^*(t) \in \mathcal{M}^+(0,\infty;\downarrow)$, $(f\chi_{B(x,r)})^* = \varphi$ a.e. on $(0,\infty)$, and from Lemma 2.4

$$\begin{aligned} \|M_\alpha f\|_{\mathcal{L}_{q,s,\lambda}(\mathbb{R}^n)} &= \sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left((M_\alpha f)^*(t) \right)^s t^{\frac{s}{q}-1} dt \right]^{\frac{1}{s}} \\ &\lesssim \sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left(\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}} (f\chi_{B(x,r)})^{**}(\tau) \right)^s t^{\frac{s}{q}-1} dt \right]^{\frac{1}{s}} \\ &= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left[\int_0^\infty \left(\sup_{t<\tau<\infty} \tau^{\frac{\alpha}{n}-1} \int_0^\tau (f\chi_{B(x,r)})^*(\sigma) d\sigma \right)^s t^{\frac{s}{q}-1} dt \right]^{\frac{1}{s}} \\ &\lesssim \sup_{r>0} r^{-\frac{\lambda}{p}} \left[\int_0^\infty \left((f\chi_{B(x,r)})^*(t) \right)^p t^{\frac{u}{p}-1} dt \right]^{\frac{1}{u}} \\ &= \|f\|_{\mathcal{L}_{p,u,\lambda}(\mathbb{R}^n)} \end{aligned}$$

holds.

$(ii) \Leftrightarrow (iii)$ The equivalence of (ii) and (iii) follows from the same proof method in [20]. Hence the proof is completed. □

4. Some Applications

4.1 *The estimate of Bochner-Riesz operator in the spaces* $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$

Let $\delta > (n-1)/2$, $B_r^\delta(f)^\wedge(\xi) = (1-r^2|\xi|^2)_+^\delta \hat{f}(\xi)$, and $B_r^\delta(x) = r^{-n} B^\delta(x/r)$ for $r > 0$. The maximal Bochner-Riesz operator is defined by (see [16] and [17])

$$B_{\delta,*}(f)(x) = \sup_{r>0} |B_r^\delta(f)(x)|.$$

It is clear that (see [9])

$$B_{\delta,*}(f)(x) \lesssim Mf(x). \tag{4.1}$$

Since the maximal operator M is bounded on the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$, then from Theorem 3.1 we get the following statement.

Theorem 4.1 : *Let $1 < p < \infty, 1 \le q < \infty$ and $0 \le \lambda \le n$, and there exist a positive constant C independent of f and for all $f \in \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$. Then the Bochner-Riesz operator B_r^δ is bounded on the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$.*

Proof : The idea of proofs of Theorem 4.1 is based on the inequality (4.1) in which the maximal Bochner-Riesz operator $B_{\delta,*}$ dominated by the operator M. Hence, the proof is step by step the same as in the proof of Theorem 3.1. □

For the case $\lambda = 0$, from Theorem 4.1 we get the following statement.

Corollary 4.2 : *Let $1 < p < \infty, 1 \le q < \infty$ and $0 \le \lambda \le n$. Then the Bochner-Riesz operator B_r^δ is bounded on the Lorentz spaces $L_{p,q}(\mathbb{R}^n)$.*

4.2 *The estimates of Schrödinger-type operators* $\boldsymbol{V^\gamma(-\Delta+V)^{-\beta}}$ *and* $\boldsymbol{V^\gamma\nabla(-\Delta+V)^{-\beta}}$ *in the spaces* $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$

When V is a non-negative polynomial, Zhong ([26]) proved that the operators $V^k(-\Delta+V)^{-k}$ and $V^{k-1/2}\nabla(-\Delta+V)^{-k}$, $k \in \mathrm{N}$, are bounded on $L_p(\mathbb{R}^n)$, $1 < p \le \infty$. Shen [24] studied the Schrödinger operator $-\Delta+V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the $L_p(\mathbb{R}^n)$ boundedness of the operators $(-\Delta+V)^{i\gamma}$, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$.

We give the boundedness of the Schrödinger-type operators

$$\mathcal{T}_1 = V^\gamma(-\Delta+V)^{-\beta},\ 0 \le \gamma \le \beta \le 1,$$

and

$$\mathcal{T}_2 = V^\gamma\nabla(-\Delta+V)^{-\beta},\ 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, \beta - \gamma \ge \frac{1}{2}$$

from the Lorentz-Morrey spaces $\mathcal{L}_{p,u;\lambda}(\mathbb{R}^n)$ to another one $\mathcal{L}_{q,s;\lambda}(\mathbb{R}^n)$. Note that the operators $V(-\Delta+V)^{-1}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ in [15] are the special case of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

It is worth pointing out that we need to establish pointwise estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [15]. Then we prove the boundedness of the Schrödinger-type operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$ in the

Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ by using boundedness of the fractional maximal operators M_α in these spaces.

The following two pointwise estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ are proved in [25] with the potential $V \in B_\infty$.

Theorem A : [25] *Suppose that* $V \in B_\infty$ *and* $0 \le \gamma \le \beta \le 1$. *Then for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\left|\mathcal{T}_1 f(x)\right| \lesssim M_\alpha f(x),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem B : [25] *Suppose that* $V \in B_\infty$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Then for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\left|\mathcal{T}_2 f(x)\right| \lesssim M_\alpha f(x),$$

where $\alpha = 2(\beta - \gamma) - 1$.

From Theorem 3.2 and by using Theorems A and B we get the following two statements, respectively.

Theorem 4.3 : *Let* $V \in B_\infty$, $0 \le \gamma \le \beta \le 1$. *Then the following statements are equivalent:*

(i) *If* $1 < p \le q < \infty, 1 \le u \le s \le \infty, 1 < p < \frac{n-\lambda}{2(\beta-\gamma)}, 0 < \lambda < n$. *Then the Schrödinger-type operator* $\mathcal{T}_1$ *is bounded from* $\mathcal{L}_{p,u;\lambda}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,s;\lambda}(\mathbb{R}^n)$, *such that there is a positive constant C the inequality*

$$\| \mathcal{T}_1 f \|_{\mathcal{L}_{q,s;\lambda}(\mathbb{R}^n)} \le C \| f \|_{\mathcal{L}_{p,u;\lambda}(\mathbb{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$.

(ii) *The inequality (3.1) holds for all* $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$.

(iii) $\frac{1}{p} - \frac{1}{q} = \frac{2(\beta-\gamma)}{n-\lambda}$.

Theorem 4.4 : *Let* $V \in B_\infty$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1, \beta - \gamma \ge \frac{1}{2}$. *Then the following statements are equivalent:*

(i) *If* $1 < p \le q < \infty, 1 \le u \le s \le \infty, 1 < p < \frac{n-\lambda}{2(\beta-\gamma)-1}, 0 < \lambda < n$. *Then the Schrödinger-type operator* $\mathcal{T}_2$ *is bounded from* $\mathcal{L}_{p,u;\lambda}(\mathbb{R}^n)$ *to* $\mathcal{L}_{q,s;\lambda}(\mathbb{R}^n)$, *such that there is a positive constant C the inequality*

$$\| \mathcal{T}_2 f \|_{\mathcal{L}_{q,s;\lambda}(\mathbb{R}^n)} \leq C \| f \|_{\mathcal{L}_{p,u;\lambda}(\mathbb{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$.

(ii) *The inequality (3.1) holds for all* $\varphi \in \mathcal{M}^+(0,\infty;\downarrow)$.

(iii) $\frac{1}{p} - \frac{1}{q} = \frac{2(\beta-\gamma)-1}{n-\lambda}$.

Proof : The idea of proofs of Theorem 4.3 and Theorem 4.4 are based on the Theorem A and Theorem B in which the Schrödinger-type operators $\mathcal{T}_1$ and $\mathcal{T}_2$ dominated by the operator M_α, respectively. Hence, the proofs are step by step the same as in the proof of Theorem 3.2. □

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