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POSITIVE SOLUTIONS FOR HIGHER-ORDER MULTI-POINT FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we consider the higher-order multi-point fractional boundary value problem. We establish the criteria for the existence of at least one and three positive solutions for higher order nonlinear m-point fractional boundary value problem by using the Krasnosel'skii fixed point theorem and the Legget-Williams fixed point theorem, respectively.

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1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, [1, 2, 6, 7, 13, 17–21].

Among all the researches on the theory of the fractional differential equations, the study of the boundary value problems for fractional differential equations recently has attracted a great deal of attention from many researchers. Some results have been obtained on the existence of positive solutions of the boundary value problems for some specific fractional differential equations [3–5, 8–11, 15, 22, 23].

In [16], Nyamoradi and Javidi were interested in the fractional order multi-point boundary value problem

$$\begin{cases} D_{0^+}^{\sigma}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \ t \in [0,1], \, 1 < \sigma \le 2, \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ au(0) - bu'(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \displaystyle\sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where $D_{0^+}^{\sigma}$ is the standard Riemann-Liouville fractional derivative of order σ. Some existence results for at least one positive solutions were established by using Krasnosel'skii fixed point theorem.

In this paper, we study the existence of positive solutions to multi-point boundary value problem (BVP) for higher order fractional differential equations:

$$\begin{cases} -D_{0^+}^{\eta-2}(u''(t))+f(t,u(t))=0,\ t\in[0,1], \\ u''(0)=u'''(0)=...=u^{(n-2)}(0)=0,\ u'''(1)=0, \\ \alpha u(0)-\beta u'(0)=\sum\limits_{p=1}^{m-2}a_p u'(\xi_p), \\ \gamma u(1)+\delta u'(1)=\sum\limits_{p=1}^{m-2}b_p u'(\xi_p), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\eta-2}$ is the Riemann-Liouville fractional derivative of order $\eta-2$. Throughout the paper, we suppose that $m,n\geq 3$ and $n-1<\eta\leq n$, where $n,m\in\mathbb{N}$ and $\beta>\alpha>1$, γ, $\delta>0$, a_p, $b_p\geq 0$ are given constants and $0<\xi_1<\ldots<\xi_{m-2}<1$. We assume that $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous.

We have organized the paper as follows. First, we provide some definitions and preliminary lemmas which are key tools for our main results. Second, we obtained the existence of at least one positive solution for the BVP (1.1) by using the Krasnosel'skii fixed point theorem. Finally, we use the Legget-Williams fixed-point theorem to show that the existence of at least three positive solutions to the BVP (1.1).

We assume that the following conditions are satisfied:

(H1) If $m\geq 3$, then $\gamma\sum\limits_{k=1}^{m-2}a_k\geq\alpha\sum\limits_{k=1}^{m-2}b_k$ and
if $m>3$, then $\alpha\delta>\gamma\sum\limits_{k=1}^{j-1}a_k\geq\alpha\sum\limits_{k=1}^{j-1}b_k>\beta\gamma$, where $2\leq j\leq m-2$.

(H2) $\alpha\delta>\alpha\sum\limits_{p=1}^{m-2}b_p+\gamma\sum\limits_{p=1}^{m-2}a_p$.

2. Preliminaries

To state the main results of this paper, we will need the following lemmas and we present some notation.

Definition 1. The Riemann-Liouville fractional derivative of order $\alpha>0$ for a function $u:(0,\infty)\to\mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}u(s)ds,$$

where $n=[\alpha]+1$.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 1 ([13]). *The equality $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0,1)$.*

Lemma 2 ([13]). *Let $\alpha > 0$. Then the differential equation*

$$D_{0^+}^{\alpha} u = 0 \tag{2.1}$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1,\ldots,n$, where $n-1 < \alpha \le n$.

Lemma 3 ([13]). *Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0^+}^{\alpha} u \in L(0,1)$:*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \tag{2.2}$$

$c_i \in \mathbb{R}$, $i = 1,\ldots,n$, where $n-1 < \alpha \le n$.

If $-u''(t) = y(t)$ and $\eta - 2 = \sigma$, then the problem

$$\begin{cases} -D_{0^+}^{\eta-2}(u''(t)) + f(t,u(t)) = 0, \ t \in [0,1], \\ u''(0) = u'''(0) = \ldots = u^{(n-2)}(0) = 0, \ u'''(1) = 0 \end{cases} \tag{2.3}$$

is turned into the problem

$$\begin{cases} D_{0^+}^{\sigma} y(t) + f(t,u(t)) = 0, \ t \in [0,1], \\ y(0) = y'(0) = \ldots = y^{(n-4)}(0) = 0, \ y'(1) = 0. \end{cases} \tag{2.4}$$

Lemma 4. *The boundary value problem (2.4) has a unique solution*

$$y(t) = \int_0^1 H(t,s) f(s,u(s))ds, \tag{2.5}$$

where

$$H(t,s) = \begin{cases} \dfrac{(1-s)^{\sigma-2} t^{\sigma-1}}{\Gamma(\sigma)}, & t \le s, \\ \\ \dfrac{(1-s)^{\sigma-2} t^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)}, & t \ge s. \end{cases} \tag{2.6}$$

Proof. According to Lemma 3, we obtain

$$y(t) = -\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} f(s,u(s))ds + c_1 t^{\sigma-1} + c_2 t^{\sigma-2} + \ldots + c_{n-2} t^{\sigma-n+2}. \quad (2.7)$$

By boundary conditions of (2.4) we get $c_2 = c_3 = \ldots = c_{n-2} = 0$ and

$$c_1 = \frac{1}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-2} f(s,u(s))ds.$$

Thus, the unique solution of problem (2.4) is

$$\begin{aligned} y(t) &= \int_0^t \frac{(1-s)^{\sigma-2}t^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s,u(s))ds + \int_t^1 \frac{(1-s)^{\sigma-2}t^{\sigma-1}}{\Gamma(\sigma)} f(s,u(s))ds \\ &= \int_0^1 H(t,s) f(s,u(s))ds. \end{aligned}$$

The proof is complete. □

Lemma 5. *If* (H1) *and* (H2) *hold and*

$$K := \alpha\gamma + \alpha\delta - \alpha\sum_{p=1}^{m-2} b_p + \gamma\beta + \gamma\sum_{p=1}^{m-2} a_p,$$

then for $y \in C[0,1]$*, the boundary value problem*

$$\begin{cases} -u''(t) = y(t),\ t \in [0,1], \\ \alpha u(0) - \beta u'(0) = \sum_{p=1}^{m-2} a_p u'(\xi_p), \\ \gamma u(1) + \delta u'(1) = \sum_{p=1}^{m-2} b_p u'(\xi_p) \end{cases} \quad (2.8)$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s)ds, \quad (2.9)$$

where

$$G(t,s)=\frac{1}{K}\begin{cases}(\alpha s+\beta)(\gamma(1-t)+\delta-\sum\limits_{p=1}^{m-2}b_p),0\le s\le\xi_1,\,t\ge s\\(\gamma(1-s)+\delta-\sum\limits_{p=1}^{m-2}b_p)(\alpha t+\beta)+\gamma\sum\limits_{p=1}^{m-2}a_p(t-s),\,0\le s\le\xi_1,\,t\le s\\(\alpha s+\beta+\sum\limits_{k=1}^{j-1}a_k)(\gamma(1-t)+\delta-\sum\limits_{p=j}^{m-2}b_p)+\sum\limits_{k=1}^{j-1}b_k(\alpha(t-s)+\sum\limits_{p=j}^{m-2}a_p),\\ \qquad\xi_{j-1}<s\le\xi_j,\,t\ge s,\,2\le j\le m-2\\(\gamma(1-s)+\delta-\sum\limits_{p=j}^{m-2}b_p)(\alpha t+\beta+\sum\limits_{k=1}^{j-1}a_k)+\sum\limits_{p=j}^{m-2}a_p(\gamma(t-s)+\sum\limits_{k=1}^{j-1}b_k),\\ \qquad\xi_{j-1}<s\le\xi_j,\,t\le s,\,2\le j\le m-2\\(\alpha s+\beta+\sum\limits_{k=1}^{m-2}a_k)(\gamma(1-t)+\delta)+\alpha\sum\limits_{k=1}^{m-2}b_k(t-s),\,\xi_{m-2}<s\le 1,\,t\ge s\\(\gamma(1-s)+\delta)(\alpha t+\beta+\sum\limits_{k=1}^{m-2}a_k),\,\xi_{m-2}<s\le 1,\,t\le s.\end{cases}\tag{2.10}$$

Proof. A direct calculation gives that if $y\in C[0,1]$, then the boundary value problem (2.8) has the unique solution

$$\begin{aligned}u(t)=&-\int_0^t(t-s)y(s)\Delta s+\frac{t}{K}\left\{\alpha\int_0^1(\gamma(1-s)+\delta)y(s)\Delta s+\sum_{p=1}^{m-2}(\gamma a_p-\alpha b_p)\int_0^{\xi_p}y(s)\Delta s\right\}\\&+\frac{1}{K}\left\{(\beta+\sum_{p=1}^{m-2}a_p)\int_0^1(\gamma(1-s)+\delta)y(s)\Delta s+(-(\beta+\sum_{p=1}^{m-2}a_p))\sum_{p=1}^{m-2}b_p\int_0^{\xi_p}y(s)\Delta s\right.\\&+\left.\left(\frac{\gamma(\beta+\sum_{p=1}^{m-2}a_p)}{\alpha}-\frac{K}{\alpha}\right)\sum_{p=1}^{m-2}a_p\int_0^{\xi_p}y(s)\Delta s\right\}.\end{aligned}$$

Hence, we obtain (2.9). □

Lemma 6. *The Green's function $G(t,s)$ in (2.10) satisfies*

$$0<G(t,s)\le G(s,s)$$

for $(t,s)\in[0,1]\times[0,1]$.

Proof. From (H1), (H2) and (2.10), $G(t,s) > 0$. Now, we will show that $G(t,s) \le G(s,s)$.

(i) Let $s \in [0,\xi_1]$ and $t \ge s$. Since $G(t,s)$ is decreasing in t, we get $G(t,s) \le G(s,s)$.
(ii) Let $s \in [0,\xi_1]$ and $t \le s$. Since $G(t,s)$ is increasing in t, we have $G(t,s) \le G(s,s)$.
(iii) Take $s \in (\xi_{j-1},\xi_j]$, $2 \le j \le m-2$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. So we obtain $G(t,s) \le G(s,s)$.
(iv) Take $s \in (\xi_{j-1},\xi_j]$, $2 \le j \le m-2$ and $t \le s$. Since $G(t,s)$ is increasing in t, we get $G(t,s) \le G(s,s)$.
(v) Let $s \in (\xi_{m-2},\ 1]$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. So we have $G(t,s) \le G(s,s)$.
(vi) Let $s \in (\xi_{m-2},\ 1]$ and $t \le s$. Since $G(t,s)$ is increasing in t, we obtain $G(t,s) \le G(s,s)$.

□

Lemma 7. *Green's function $G(t,s)$ in (2.10) satisfies*

$$\min_{t\in[0,1]} G(t,s) \ge z\|G(.,s)\|,$$

with

$$z = \min\{z_1, z_2, z_3, z_4, z_5, z_6\}, \tag{2.11}$$

where

$$z_1 = \frac{\delta - \sum\limits_{p=1}^{m-2} b_p}{\gamma+\delta-\sum\limits_{p=1}^{m-2} b_p}, \quad z_2 = \frac{(\alpha\delta - \alpha\sum\limits_{p=1}^{m-2} b_p - \gamma\sum\limits_{p=1}^{m-2} a_p)}{(\gamma+\delta-\sum\limits_{p=1}^{m-2} b_p)(\alpha+\beta)}, \quad z_3 = \frac{\delta - \sum\limits_{p=j}^{m-2} b_p}{\gamma+\delta-\sum\limits_{p=j}^{m-2} b_p},$$

$$z_4 = \frac{\frac{\beta}{\alpha}-1}{\alpha+\beta+\sum\limits_{k=1}^{j-1} a_k}, \quad z_5 = \frac{\delta}{\gamma+\delta}, \quad z_6 = \frac{\frac{\beta}{\alpha}-1}{\alpha+\beta+\sum\limits_{k=1}^{m-2} a_k}$$

and $\|.\|$ is defined by $\|x\| = \max\limits_{t\in[0,1]} |x(t)|$.

Proof. (i) Take $s \in [0,\xi_1]$ and $t \ge s$. Since $G(t,s)$ is decreasing in t and $0 < z_1 < 1$, we get $\min\limits_{t\in[0,1]} G(t,s) = G(1,s)$ and $\min\limits_{t\in[0,1]} G(t,s) \ge z_1 G(s,s) = z_1\|G(.,s)\|$.
(ii) Take $s \in [0,\xi_1]$ and $t \le s$. Since $G(t,s)$ is increasing in t and $0 < z_2 < 1$, we have $\min\limits_{t\in[0,1]} G(t,s) = G(0,s)$ and $\min\limits_{t\in[0,1]} G(t,s) \ge z_2 G(s,s) = z_2\|G(.,s)\|$.

(iii) Let $s \in (\xi_{j-1}, \xi_j]$, $2 \le j \le m-2$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. It is clear that $0 < z_3 < 1$. So $\min_{t\in[0,1]} G(t,s) = G(1,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_3 G(s,s) = z_3\|G(.,s)\|$.

(iv) Let $s \in (\xi_{j-1}, \xi_j]$, $2 \le j \le m-2$ and $t \le s$. Since $G(t,s)$ is increasing in t and $0 < z_4 < 1$, we have $\min_{t\in[0,1]} G(t,s) = G(0,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_4 G(s,s) = z_4\|G(.,s)\|$.

(v) Take $s \in (\xi_{m-2},\ 1]$ and $t \ge s$. From (H1), $G(t,s)$ is decreasing in t. It is clear that $0 < z_5 < 1$. So we get $\min_{t\in[0,1]} G(t,s) = G(1,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_5 G(s,s) = z_5\|G(.,s)\|$.

(vi) Take $s \in (\xi_{m-2},\ 1]$ and $t \le s$. Since $G(t,s)$ is increasing in t and $0 < z_6 < 1$, we have $\min_{t\in[0,1]} G(t,s) = G(0,s)$ and $\min_{t\in[0,1]} G(t,s) \ge z_6 G(s,s) = z_6\|G(.,s)\|$.

Thus $\min_{t\in[0,1]} G(t,s) \ge z\|G(.,s)\|$, where $z = \min\{z_1, z_2, z_3, z_4, z_5, z_6\}$. □

Lemma 8. *For $t,s \in [0,1]$, we have $0 \le H(t,s) \le H(1,s)$.*

Proof. From (2.6), we obtain $0 \le H(t,s)$. Now we will show that $H(t,s) \le H(1,s)$.

(i) Let $t \le s$. $H(t,s) = \frac{(1-s)^{\sigma-2}t^{\sigma-1}}{\Gamma(\sigma)} \le \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma)} = H(1,s)$.

(ii) Let $t \ge s$. Since $H(t,s)$ is increasing in t, we have $H(t,s) \le H(1,s)$.

□

Lemma 9. $\min_{t\in[\xi_{m-2},1]} H(t,s) \ge k^{\sigma-1}H(1,s)$ *for $0 \le t,s \le 1$, where $k \in (0,\xi_{m-2})$ is a constant.*

Proof. (i) Take $t \le s$. Since $H(t,s)$ is an increasing function, we get

$$\min_{t\in[\xi_{m-2},1]} H(t,s) = \frac{(1-s)^{\sigma-2}\xi_{m-2}^{\sigma-1}}{\Gamma(\sigma)} \ge \frac{(1-s)^{\sigma-2}(k)^{\sigma-1}}{\Gamma(\sigma)} = k^{\sigma-1}H(1,s).$$

(ii) For $s \le t$, we have

$$\begin{aligned}
\min_{t\in[\xi_{m-2},1]} H(t,s) &= \frac{(1-s)^{\sigma-2}\xi_{m-2}^{\sigma-1} - (\xi_{m-2}-s)^{\sigma-1}}{\Gamma(\sigma)} \\
&> \frac{(1-s)^{\sigma-2}\xi_{m-2}^{\sigma-1} - (\xi_{m-2}-\xi_{m-2}s)^{\sigma-1}}{\Gamma(\sigma)} \\
&= \frac{\xi_{m-2}^{\sigma-1}((1-s)^{\sigma-2} - (1-s)^{\sigma-1})}{\Gamma(\sigma)} \\
&= \xi_{m-2}^{\sigma-1}H(1,s) \\
&> k^{\sigma-1}H(1,s).
\end{aligned}$$

Thus $\min_{t\in[\xi_{m-2},1]} H(t,s) \ge k^{\sigma-1}H(1,s)$. □

From Lemma 4 and Lemma 5, we know that $u(t)$ is a solution of the problem (1.1) if and only if

$$u(t) = \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds. \tag{2.12}$$

Let E denote the Banach space $C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \geq 0, \min_{t\in[0,1]} u(t) \geq z\|u\|\} \tag{2.13}$$

where z is given in (2.11).

We can define the operator $A : P \to E$ by

$$Au(t) = \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds, \tag{2.14}$$

where $u \in P$. Therefore solving (2.12) in P is equivalent to finding fixed points of the operator A.

Lemma 10. *If the conditions* (H1) *and* (H2) *hold, then $AP \subset P$.*

Proof. If $u \in P$, then $Au(t) \geq 0$ on $[0,1]$ by using Lemma 6 and Lemma 8. On the other hand, we have

$$\begin{aligned}
\min_{t\in[0,1]} Au(t) &= \int_0^1 \min_{t\in[0,1]} G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq z \int_0^1 \max_{t\in[0,1]} G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&= z\|Au\|,
\end{aligned}$$

by Lemma 7. Thus $Au \in P$ and therefore $AP \subset P$. □

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove main theorems.

Theorem 1 (Krasnosel'skii Fixed Point Theorem, [12]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$ *or*
(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2 (Leggett-Williams Fixed Point Theorem, [14])**.** *Let P be a cone in the real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\},$$

$$P(\psi, a, b) := \{x \in P : a \le \psi(x),\ \|x\| \le b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \le \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \le r$ such that the following conditions hold,

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \varnothing$ *and* $\psi(Au) > q$ *for all* $u \in P(\psi, q, l)$,

(ii) $\|Au\| < p$ *for* $\|u\| \le p$,

(iii) $\psi(Au) > q$ *for* $u \in P(\psi, q, r)$ *with* $\|Au\| > l$,

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. Main Results

For convenience, we introduce the following notations. Let

$$M = \int_0^1 H(1, \tau) d\tau, \tag{3.1}$$

$$L = \int_0^1 G(s, s) ds, \tag{3.2}$$

$$I = \int_{\xi_{m-2}}^1 G(s, s) ds. \tag{3.3}$$

Now, we will give the sufficient conditions to have at least one positive solution for the BVP (1.1). Krasnosel'skii fixed point theorem will be used to prove the next theorem.

Theorem 3. *Suppose* (H1) *and* (H2) *hold. In addition, let there exist numbers $0 < r < R < \infty$ such that the function f satisfies the following conditions:*

(i) $f(t, u) < \frac{1}{LM} u(t)$ *for* $(t, u) \in [0, 1] \times [0, r]$,

(ii) $f(t, u) > \frac{1}{k^{\sigma-1} z^2 IM} u(t)$ *for* $(t, u) \in [0, 1] \times [R, \infty)$.

Then the BVP (1.1) has at least one positive solution.

Proof. Define the open bounded subsets of E by $\Omega_1 = \{u \in P : \|u\| < r\}$ and $\Omega_2 = \left\{ u \in P : \|u\| < \frac{R}{z} \right\}$. It is easy to check that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is completely continuous operator.

If $u \in P \cap \partial\Omega_1$, then $\|u\| = r$. Therefore, by using the hypothesis (i), Lemma 6 and Lemma 8,

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\ &< \frac{1}{LM} \int_0^1 G(t,s) \int_0^1 H(s,\tau) u(\tau) d\tau ds \\ &\leq \frac{1}{LM} \|u\| \int_0^1 G(t,s) \int_0^1 H(s,\tau) d\tau ds \\ &\leq \frac{1}{LM} \|u\| \int_0^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds \\ &= \|u\|, \end{aligned}$$

for all $t \in [0,1]$. Thus $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$. On the other hand, $u \in P \cap \partial\Omega_2$ implies

$$u(t) \geq z\|u\| = R,$$

for $t \in [0,1]$ and

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\ &> \frac{1}{k^{\sigma-1} z^2 IM} \int_0^1 G(t,s) \int_0^1 H(s,\tau) u(\tau) d\tau ds \\ &\geq \frac{1}{k^{\sigma-1} z^2 IM} z\|u\| \int_0^1 G(t,s) \int_0^1 H(s,\tau) d\tau ds \\ &\geq \frac{1}{k^{\sigma-1} z^2 IM} z\|u\| z k^{\sigma-1} \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds \\ &= \|u\|, \end{aligned}$$

from (ii), Lemma 7 and Lemma 9. Consequently, $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$.

By the first part of Theorem 1, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \leq \|u\| \leq \frac{R}{z}$. Therefore BVP (1.1) has at least one positive solution. $\square$

Now we will use the Legget-Williams fixed point theorem to prove the next theorem.

Theorem 4. *Assume that* (H1) *and* (H2) *hold. Suppose that there exist numbers* $0 < p < q < \frac{q}{z} \leq r$ *such that the function* f *satisfies the following conditions:*

(i) $f(t,u) \leq \frac{r}{ML}$ *for* $(t,u) \in [0,1] \times [0,r]$,
(ii) $f(t,u) < \frac{p}{ML}$ *for* $(t,u) \in [0,1] \times [0,p]$,
(iii) $f(t,u) > \frac{q}{k^{\sigma-1}zIM}$ *for* $(t,u) \in [0,1] \times [q, \frac{q}{z}]$,

where z*,* M*,* L *and* I *are as in (2.11), (3.1), (3.2) and (3.3), respectively and* k *is defined in Lemma 9. Then the BVP (1.1) has at least three positive solutions* u_1*,* u_2 *and* u_3 *satisfying*

$$\max_{t\in[0,1]} u_1(t) < p, \ \min_{t\in[0,1]} u_2(t) > q$$
$$\max_{t\in[0,1]} u_3(t) > p \text{ with } \min_{t\in[0,1]} u_3(t) < q.$$

Proof. Define the nonnegative, continuous, concave functional $\psi : P \to [0,\infty)$ to be $\psi(y) = \min\limits_{t\in[0,1]} u(t)$ and the cone P as in (2.13). For all $u \in P$, we have $\psi(u) \leq \|u\|$. Now we show that $A : \overline{P_r} \to \overline{P_r}$ is completely continuous operator. If $u \in \overline{P_r}$, then $0 \leq u(t) \leq r$ for all $t \in [0,1]$. We get,

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]} |Au(t)| \\
&= \int_0^1 \max_{t\in[0,1]} G(t,s) \int_0^1 H(s,\tau) f(\tau,u(\tau)) d\tau ds \\
&\leq \int_0^1 G(s,s) \int_0^1 H(1,\tau) f(\tau,u(\tau)) d\tau ds \\
&\leq \frac{r}{ML} \int_0^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds \\
&\leq r
\end{aligned}$$

by hypothesis (i), Lemma 6 and Lemma 8. Thus $A : \overline{P_r} \to \overline{P_r}$. It easy to check that $A : \overline{P_r} \to \overline{P_r}$ is completely continuous.

Since $z < 1$, $u(t) = \frac{q}{z} \in P(\psi, q, \frac{q}{z})$ and $\psi(\frac{q}{z}) > q$. Then, we have

$$\{u \in P(\psi, q, \frac{q}{z}) : \psi(u) > q\} \neq \varnothing.$$

On the other hand, for all $u \in P(\psi, q, \frac{q}{z})$, we have $q \leq u(t) \leq \frac{q}{z}$ for $t \in [0,1]$. Using assumption (iii), Lemma 7 and Lemma 9, we find

$$\begin{aligned}
\psi(Au) &= \min_{t\in[0,1]} \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq z \int_0^1 G(s,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq z \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq k^{\sigma-1} z \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(1,\tau) f(\tau, u(\tau)) d\tau ds \\
&\geq k^{\sigma-1} z \frac{q}{k^{\sigma-1} z I M} \int_{\xi_{m-2}}^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds = q.
\end{aligned}$$

Thus condition (i) of Theorem 2 holds.

For $\|u\| < p$, we have $0 \leq u(t) \leq p$ for $t \in [0,1]$. Then from assumption (ii), Lemma 6 and Lemma 8, we obtain

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) \int_0^1 H(s,\tau) f(\tau, u(\tau)) d\tau ds \\
&\leq \int_0^1 G(s,s) \int_0^1 H(1,\tau) f(\tau, u(\tau)) d\tau ds \\
&< \frac{p}{ML} \int_0^1 G(s,s) \int_0^1 H(1,\tau) d\tau ds = p.
\end{aligned}$$

It follows that condition (ii) of Theorem 2 is satisfied.

Finally, we will check condition (iii) of Theorem 2. We suppose that $u \in P(\psi, q, r)$ with $\|Au\| > \frac{q}{z}$. Then we obtain

$$\psi(Au) = \min_{t\in[0,1]} Au(t) \geq z\|Au\| > q. \tag{3.4}$$

Since all conditions of the Legget-Williams fixed point theorem are satisfied, the BVP (1.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,1]} u_1(t) < p, \quad \min_{t\in[0,1]} u_2(t) > q$$

$$\max_{t\in[0,1]} u_3(t) > p \text{ with } \min_{t\in[0,1]} u_3(t) < q.$$

□

Example 1. Taking $n=5$, $m=3$, $\xi_1=\frac{1}{2}$, $\alpha=\gamma=\delta=3$, $\beta=4$, $a_1=b_1=1$, $k=\frac{1}{4}$ and $\sigma=\frac{5}{2}$, we consider the following boundary value problem:

$$\begin{cases} -D_{0_+}^{\frac{5}{2}}(u''(t)) = \dfrac{100000u^2}{u^2+1}, \ t\in[0,1], \\ u''(0)=u'''(0)=0,\, u'''(1)=0, \\ 3u(0)-4u'(0)=u'(\dfrac{1}{2}), \\ 3u(1)+3y'(1)=u'(\dfrac{1}{2}). \end{cases} \tag{3.5}$$

Then we get $K=30$, $M=\frac{16}{45\sqrt{\pi}}$, $L=\frac{47}{60}$, $I=0,45$, $z=\frac{3}{35}$ and $k^{\sigma-1}=k^{\frac{3}{2}}=0.125$.

If we take $p=7.10^{-7}, q=10$ and $r=130000$, then $0<p<q<\frac{q}{z}<r$ and all the conditions in Theorem 4 are fulfilled. Hence, by Theorem 4, the BVP (3.5) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\max_{t\in[0,1]} u_1(t) < p, \ \min_{t\in[0,1]} u_2(t) > q$$
$$\max_{t\in[0,1]} u_3(t) > p \text{ with } \min_{t\in[0,1]} u_3(t) < q.$$

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