

SYMMETRIC BI- (σ, τ) DERIVATIONS OF PRIME AND SEMI PRIME GAMMA RINGS

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ABSTRACT. The purpose of this paper is to define the symmetric bi- (σ, τ) derivations on prime and semi prime Gamma rings and to prove some results concerning symmetric bi- (σ, τ) derivations on prime and semi prime Gamma rings.

1. Introduction

The gamma rings are defined in [2] as follows. Let M and Γ be additive abelian groups. If for $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the following conditions are satisfied.

- i) $a\beta b \in M$
- ii) $(a + b)\alpha c = a\alpha c + b\alpha c$
- iii) $a(\alpha + \beta)b = a\alpha b + a\beta b, a\alpha(b + c) = a\alpha b + a\alpha c$
- iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$

then M is called a Γ -ring. (in the sense of [1]). Every ring is a Γ -ring. A right (left) ideal of a Γ -ring M is an additive subgroup of U of M such that $U\Gamma M \subset U(M\Gamma U \subset U)$. If is both right and left ideal, then we say U is an ideal of M . If the following condition holds for a gamma ring M , then M is called a prime gamma ring [1]

$$a\Gamma M\Gamma b = 0 \Rightarrow a = 0 \text{ or } b = 0 \quad a, b \in M.$$

Throughout this paper all gamma rings will be associative. We shall denote by $Z(R)$ the center of gamma ring. We shall write $[x, y]_\gamma = x\gamma y - y\gamma x$ for all $x, y \in M, \gamma \in \Gamma$. An additive mapping $D : M \rightarrow M$ is called a derivation if $D(x\gamma y) = D(x)\gamma y + x\gamma D(y)$ hold for all $x, y \in M, \gamma \in \Gamma$.

A derivation D is inner if there exists $a \in M$ such that $D(x) = [a, x]_\gamma$ hold for all $x \in M, \gamma \in \Gamma$. A mapping $B(., .) : MxM \rightarrow M$ is said to be

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symmetric if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in M$. A mapping $f : M \rightarrow M$ defined by $f(x) = B(x, x)$ where $B(., .) : M \times M \rightarrow M$ is a symmetric mapping, is called the trace of B . It is obvious that, in case $B(., .) : M \times M \rightarrow M$ is a symmetric mapping which is also bi-additive, then trace of B satisfies the relation $f(x + y) = f(x) + f(y) + 2B(x, y)$, $x, y \in M$. We shall use also the fact that trace of symmetric bi-additive mapping is an even function. Let σ, τ be gamma ring-automorphisms of M . $D(., .) : M \times M \rightarrow M$ is called a symmetric bi- (σ, τ) derivation, if

$$D(x\gamma y, z) = D(x, z)\gamma\sigma(y) + \tau(x)\gamma D(y, z)$$

is fulfilled for all $x, y, z \in M, \gamma \in \Gamma, \sigma, \tau$ automorphisms. Obviously, in this case also the relation

$$D(x, y\gamma z) = D(x, y)\gamma\sigma(z) + \tau(y)\gamma D(x, z)$$

holds for all $x, y, z \in M, \gamma \in \Gamma, \sigma, \tau$ automorphisms.

2. The results

The following lemmas are useful for us.

LEMMA 1. Let $D : M \rightarrow M$ be a derivation where M is a prime gamma ring. Let U be a nonzero right ideal of M . Suppose either

(i) $a\gamma D(x) = 0, \quad x \in U, \gamma \in \Gamma.$

(ii) $D(x)\gamma x = 0, \quad x \in U, \gamma \in \Gamma$ holds. In both cases we have $a = 0$ or $D = 0$.

Proof. (i) Writing $x\gamma y$ instead of x in $a\gamma D(x) = 0$ for all $x \in R$ we get

$$\begin{aligned} 0 &= a\gamma D(x\gamma y) \\ &= a\gamma(D(x)\gamma y + x\gamma D(y)) \\ &= a\gamma D(x)\gamma y + a\gamma x\gamma D(y) \end{aligned}$$

for all $x, y \in R$. Since $a\gamma x\gamma D(y) = 0$ then $a\gamma R\gamma D(x) = 0 \Rightarrow a = 0$ or $D = 0$.

(ii) Writing $x\gamma y$ instead of x in $D(x)\gamma a = 0 \quad x \in U, \gamma \in \Gamma$ then we get

$$\begin{aligned} D(x\gamma y)\gamma a &= (D(x)\gamma y + x\gamma D(y))\gamma a \\ &= D(x)\gamma y\gamma a + x\gamma D(y)\gamma a \\ &= D(x)\gamma y\gamma a. \end{aligned}$$

From this equality we get $D(x)\gamma R\gamma a = 0 \Rightarrow a = 0$ or $D = 0$ this ends the proof. \square

LEMMA 2. Let M be a prime gamma ring of characteristic not two and let U a nonzero ideal of M . Let $a, b \in M$ be fixed elements. If $a\gamma u\beta b + b\gamma u\beta a = 0$ is fulfilled for all $u \in U, \alpha, \beta \in \Gamma$ then either $a = 0$ or $b = 0$.

THEOREM 1. Let M be a prime gamma ring of characteristic not two and U be nonzero ideal of M . Suppose there exist symmetric bi- (σ, τ) derivations $D_1(.,.) : MxM \rightarrow M$ and $D_2(.,.) : MxM \rightarrow M$ such that $D_1(d_2(u), u) = 0$ holds for all $u \in U$ where d_2 denotes the trace of D_2 . In this case $D_1 = 0$ or $D_2 = 0$.

Proof. By linearization of the relation

$$(2.1) \quad D_1(d_2(u), u) = 0$$

one obtains

$$D_1(d_2(u) + d_2(v) + 2D_2(u, v), u + v) = 0$$

whence,

$$D_1(d_2(v), u) + 2D_1(D_2(u, v), u) + D_1(d_2(u), v) + 2D_1(D_2(u, v), v) = 0$$

according to (2.1). Substituting in the equation above u by $-u$ we obtain by comparing this new equation with the equation above that

$$(2.2) \quad D_1(d_2(u), v) + 2D_1(D_2(u, v), u) = 0$$

is fulfilled for all pairs $u, v \in U$. Let us replace in (2.2) v by $u\alpha v, \alpha \in \Gamma$.

Then

$$\begin{aligned} 0 &= D_1(d_2(u), u\alpha v) + 2D_1(D_2(u, u\alpha v), u) \\ &= \tau(u)\alpha D_1(d_2(u), v) + 2\tau(d_2(u))\alpha D_1(\sigma(v), u) \\ &\quad + 2D_1(\tau(u), u)\alpha\sigma(D_2(u, v)) + 2\tau^2(u)\alpha D_1(D_2(u, v), u). \end{aligned}$$

In the above calculation we used (2.1) and (2.2). Thus we have

$$(2.3) \quad 0 = d_2(u)\alpha D_1(u, v) + d_1(u)\alpha D_2(u, v) \quad u, v \in U, \alpha \in \Gamma.$$

Let us write in (2.3) $v\beta u$ instead of v We have

$$\begin{aligned} 0 &= d_2(u)\alpha D_1(v\beta u, u) + d_1(u)\alpha D_2(v\beta u, u) \\ &= (d_2(u)\alpha D_1(v, u)\beta\sigma(u) + d_2(u)\alpha\tau(v)\beta d_1(u) \\ &\quad + d_1(u)\alpha D_2(v, u))\beta\sigma(u) + d_1(u)\alpha\tau(v)\beta d_2(u) \\ &= d_2(u)\alpha\tau(v)\beta d_1(u) + d_1(u)\alpha\tau(v)\beta d_2(u). \end{aligned}$$

Thus we have

$$(2.4) \quad d_2(u)\alpha\tau(v)\beta d_1(u) + d_1(u)\alpha\tau(v)\beta d_2(u) = 0 \quad u, v \in U, \alpha, \beta \in \Gamma.$$

Let us assume that d_1 and d_2 are both different from zero. In other words there exist elements $u_1, u_2 \in U$ such that $d_1(u_1) \neq 0$ and $d_2(u_2) \neq 0$. From (2.4) and Lemma 2 it follows that $d_1(u_2) = d_2(u_1) = 0$. Since $d_1(u_2) = 0$, the relation (2.3) reduces to $d_2(u_2)\alpha D_1(u_2, v) = 0$. Using this relation and Lemma 1, we obtain that $D_1(u_2, v) = 0$ holds for all $v \in U$ since $d_2(u_2) \neq 0$. (recall that a mapping $v \rightarrow D_1(u_2, v)$ is a derivation). In particular we have $D_1(u_2, u_1) = 0$. Similarly we obtain that $d_1(u_1 + u_2) = d_1(u_1) + d_2(u_2) + 2D_1(u_1, u_2) = d_1(u_1) \neq 0$. Similarly one obtains $d_2(v) \neq 0$. But $d_1(v)$ and $d_2(v)$ cannot be both different from zero according to (2.4) and Lemma 2. Therefore we have proved that either $d_1 = 0$ or $d_2 = 0$ which is actually the assertion of the theorem. \square

In case $D_1 = D_2$ Theorem 2 can be proved for semi-prime gamma rings.

THEOREM 2. *Let M be a 2-torsion free semi-prime gamma ring and U be nonzero ideal of M . Suppose there exists such a symmetric bi- (σ, τ) derivation $D(., .) : M \times M \rightarrow M$ that $D(d(u), u) = 0$ holds for all $u \in U$. Where d denotes the trace of D . In this case we have $D = 0$.*

Proof. In this case (2.4) reduces to $d(u)\alpha\tau(v)\beta d(u) = 0, u, v \in U, \alpha, \beta \in \Gamma$ which implies that $d(u) = 0$ for all $u \in U$, by [9, Lemma 2(iii)], and semi-primeness of M . \square

THEOREM 3. *Let M be prime gamma ring of characteristic not two and three. Let $\tau(U) \subset U, U$ be nonzero ideal of M and $\sigma\tau = \tau\sigma$ and $\sigma = \tau$. Let U be nonzero ideal of M . Let $D_1(., .) : M \times M \rightarrow M$ and $D_2(., .) : M \times M \rightarrow M$ be symmetric bi- (σ, τ) derivations. Suppose further that there exists a symmetric bi-additive mapping $B(., .) : M \times M \rightarrow M$ such that $d_1(d_2(u)) = f(u)$ holds for all $u \in U$, where d_1 and d_2 are the traces of D_1 and D_2 respectively, and f is the trace of B . Then either $D_1 = 0$ or $D_2 = 0$.*

Proof. The linearization of the relation

$$(2.5) \quad d_1(d_2(u)) = f(u)$$

gives us

$$d_1(d_2(u) + d_2(v) + 2D_2(u, v)) = f(u) + f(v) + 2B(u, v)$$

and

$$\begin{aligned} & d_1(d_2(u)) + d_1(d_2(v)) + 4d_1(D_2(u, v)) + 2D_1(d_2(u), d_2(v)) \\ & + 4D_1(d_2(u), D_2(u, v)) + 4D_1(d_2(v), D_2(u, v)) \\ = & f(u) + f(v) + 2B(u, v). \end{aligned}$$

Using (2.5) we arrive at

$$\begin{aligned} & 2d_1(D_2(u, v)) + D_1(d_2(u), d_2(v)) \\ & + 2D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v)) = B(u, v). \end{aligned}$$

Substituting in the equation above u by $-u$ we obtain by comparing this new equation with the equation above

$$(2.6) \quad 2D_1(d_2(v), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v)) = B(u, v)$$

holds for all $u, v \in U$. Let us replace in (2.6) u by $2u$. We have

$$(2.7) \quad 8D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(u), D_2(u, v)) = B(u, v).$$

By comparing (2.6) and (2.7) we obtain

$$6D_1(d_2(u), D_2(u, v)) = 0$$

which has to

$$(2.8) \quad D_1(d_2(u), D_2(u, v)) = 0.$$

Since we have assumed that M is not only of characteristic different from two but also of characteristic different from three. From (2.8) it follows that both terms on the left side of relation (2.6) are zero, which means that $B = 0$. Hence (2.5) reduces to

$$(2.9) \quad d_1(d_2(u)) = 0 \quad u \in U.$$

Let in (2.8) v be $v\alpha u, \alpha \in \Gamma$. We have

$$\begin{aligned} 0 & = D_1(d_2(u), D_2(u, v\alpha u)) \\ & = D_1(d_2(u), D_2(u, v))\alpha\sigma^2(u) + \tau(D_2(u, v))\alpha D_1(d_2(u), \sigma(u)) \\ & \quad + D_1(d_2(u), \tau(v))\alpha\sigma(d_2(u)) + \tau^2(v)\alpha d_1(d_2(u)) \end{aligned}$$

which leads to

$$(2.10) \quad D_1(d_2(u), \tau(v))\alpha\tau(d_2(u)) + \tau(D_2(u, v))\alpha D_1(d_2(u), \sigma(u)) = 0$$

according to (2.8) and (2.9). Let us replace in (2.10) v , by $u\beta v, \beta \in \Gamma$. We have

$$\begin{aligned} 0 &= D_1(d_2(u), \tau(u\beta v))\alpha\tau(d_2(u)) + \tau(D_2(u, u\beta v))\alpha D_1(d_2(u), \sigma(u)) \\ &= D_1(d_2(u), \tau(u))\beta\sigma(\tau(v))\alpha\sigma(d_2(u)) \\ &\quad + \tau^2(u)\beta D_1(d_2(u), \tau(v))\alpha\sigma(d_2(u)) \\ &\quad + \tau(d_2(u))\beta\tau(\sigma(v))\alpha D_1(d_2(u), \sigma(u)) \\ &\quad + \tau^2(u)\beta\tau(D_2(u, v))\alpha D_1(d_2(u), \sigma(u)). \end{aligned}$$

Now, by (2.10) we arrive finally at

$$(2.11) \quad D_1(d_2(u), \tau(u))\beta\sigma(\tau(v))\alpha d_2(u) + \tau(d_2(u))\beta\sigma(v)\alpha D_1(d_2(u), \sigma(u)) = 0 \quad u, v \in U \quad \alpha, \beta \in \Gamma.$$

From the relation above one can conclude that

$$D_1(d_2(u), u) = 0$$

is fulfilled for all $u \in U$. Namely, if

$$D_1(d_2(u), u) \neq 0$$

for some $u \in U$, then $d_2(u) = 0$ according to lemma1 (ii) and lemma 2, contrary to the assumption

$$D_1(d_2(u), u) \neq 0.$$

Therefore, since

$$D_1(d_2(u), u) = 0$$

for all $u \in U$, the proof of the theorem is complete since all the requirements of Theorem 3 are fulfilled. \square

THEOREM 4. *Let M be a semi-prime gamma ring which is 2-torsion and 3-torsion free. Let $d(U) \subset U, \tau(U) \subset U, \sigma(U) \subset U$, be nonzero ideal of M , $\sigma\tau = \tau\sigma$ and $\sigma = \tau$. Let $D(.,.) : M \times M \rightarrow M$ be a symmetric bi- (σ, τ) derivation and a symmetric bi-additive mapping, respectively. Suppose that $d(d(u)) = f(u)$ holds for all $u \in U$, where d is the trace of D and f is the trace of B . In this case we have $D = 0$.*

Proof. Obviously, we can use the beginning of the proof of Theorem 3. In this case relations (2.8) and (2.9) can be written in the from.

$$(2.12) \quad D(d(u), D(u, v)) = 0 \quad u, v \in U$$

and

$$(2.13) \quad d(d(u)) = 0 \quad u \in U$$

Let us write in (2.12) $w \in U, \alpha \in \Gamma, v\alpha w$ instead of v . We have

$$\begin{aligned} 0 &= D(d(u), D(u, v\alpha w)) \\ &= D(d(u), D(u, v)\alpha\sigma^2(w)) + \tau(D(u, v)\alpha D(d(u), \sigma(w))) \\ &\quad + D(d(u), \tau(v))\alpha\sigma(D(u, w)) + \tau^2(v)\alpha D(d(u), D(u, w)) \end{aligned}$$

Hence by (2.12) we have

$$\tau(D(u, v))\alpha D(d(u), \sigma(w)) + D(d(u), \tau(v))\alpha\sigma(D(u, w)) = 0$$

and, in particular for $w = d(u)$ we obtain

$$(2.14) \quad D(d(u), \tau(v))\Gamma D(u, d(u)) = 0 \quad u, v \in U$$

according to (2.13). Replace in (2.14) v by $u\beta v, \beta \in \Gamma$. We have

$$\begin{aligned} 0 &= D(d(u), \tau(u\beta v))\alpha\sigma(D(u, d(u))) \\ &= D(d(u), \tau(u))\beta\sigma(\tau(v))\alpha\sigma(D(u, d(u))) \\ &\quad + \tau(u)\beta D(d(u), \tau(v))\beta\sigma(\tau(v))\alpha\sigma(D(u, d(u))). \end{aligned}$$

Hence by $\sigma\tau = \tau\sigma, \sigma = \tau, \tau(U) \subset U$, we have

$$D(d(u), u)\Gamma U\Gamma D(d(u), u) = 0 \quad u \in U$$

according to [9, lemma 2 (iii)] and finally

$$D(d(u), u) = 0 \quad \text{for all } u \in U$$

since we have assumed that M is semi-prime gamma ring. Now Theorem 4 completes the proof. \square

EXAMPLE 1. Here is an example of symmetric bi- (σ, τ) derivation:

Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\}$ be a prime ring.

EXAMPLE 2. $d : R \longrightarrow R \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix} \sigma : R \longrightarrow R$
 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = I : R \longrightarrow R \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$

Then it is easy to show that d is a symmetric bi- (σ, τ) derivation.

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