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NUMERICAL SOLUTION OF THE CONFORMABLE FRACTIONAL DIFFUSION EQUATION

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Abstract. In this paper, a numerical approach for solving space-time fractional diffusion equation with variable coefficients is proposed. The fractional derivatives are described in the conformable sense. The numerical approach is based on shifted Chebyshev polynomials of the second kind. The space-time fractional diffusion equation with variable coefficients is reduced to a system of ordinary differential equations by using the properties of Chebyshev polynomials. The finite difference method is applied to solve this system of equations. Numerical results are provided to verify the accuracy and efficiency of the proposed approach.

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Keywords: Space-time fractional diffusion equation, Shifted Chebyshev polynomials of the second kind, Conformable fractional derivative, Finite difference method

1. INTRODUCTION

Fractional differential equations have been the focus of many studies due to their applications in various fields of science and engineering (see, for example, [2, 10, 15–17]). The main physical purpose of investigating fractional diffusion equations is to describe phenomena of anomalous diffusion in transport processes. Fractional diffusion equations have been also used in modeling turbulent flow [4], groundwater contaminant transport [3] and chaotic dynamics of classical conservative systems [25].

Recently, some different numerical methods have been proposed for solving the fractional diffusion equation. The space fractional diffusion equation which is described in Caputo sense with variable coefficients has been investigated in [1, 7, 8, 18, 22]. In [22], space fractional diffusion equation has been solved by using shifted Chebyshev polynomials of the second kind with finite difference method, respectively. In [1, 8], it has been solved by using shifted Chebyshev polynomials of first kind with finite difference method. In [18], a numerical scheme based on the shifted Legendre tau method to solve the space fractional diffusion equation has been studied. In [7], a numerical approximation for the space fractional diffusion equation via splines has been suggested.

The space fractional diffusion equations which are described in Riemann-Liouville sense have been studied in [5, 12, 13]. In [12], the authors have been used the finite difference method based on a modified Grünwald approximation. In [5], compact finite difference scheme has been used to the space fractional diffusion equation with constant coefficient. In [13], the finite difference approximation obtained from the Grünwald-Letnikov formulation has been applied to the space fractional diffusion equation with variable coefficients. In [21], the Crank-Nicolson finite difference method has been applied to the time fractional diffusion equations with constant coefficients. Generally, fractional diffusion equations have been studied for derivatives in Caputo sense or Riemann-Liouville sense in the literature.

Nowadays, solutions of conformable partial differential equations have been investigated. Conformable time fractional and space-time fractional partial differential equations have been studied by using extended reduced conformable differential transform method and fractional differential transform method in [20]-[23], respectively. Residual power series method has been applied to high-order linear conformable partial differential equations in [6]. Numerical solution of the conformable space-time fractional wave equation has been obtained by using Shifted Chebyshev polynomials and finite difference method in [24]. Furthermore, there are also studies on traveling wave solutions of the conformable partial differential equations (see, for example, [14, 19]). But, the methods used to find the traveling wave solutions can not be applied to the conformable partial differential equations with variable coefficients.

In this paper, we consider conformable space-time fractional diffusion equation with variable coefficient. Fundamental goal of this work is to obtain an analytical approximate solution in terms of the shifted Chebyshev polynomials of the second kind of the space-time fractional diffusion equation with the initial and boundary conditions. Consider the one-dimensional space-time fractional diffusion equation of the form:

$$\begin{gathered} T_t^{\mu}u(x,t) = f(x,t)T_x^{\alpha}u(x,t) + g(x,t),\ 0 < x < L,\ 0 < t \leq T, \\ 0 < \mu \leq 1, 1 < \alpha \leq 2, \end{gathered} \tag{1.1}$$

with initial condition

$$u(x,0) = f_1(x),\ 0 < x < L, \tag{1.2}$$

and the boundary conditions:

$$u(0,t) = h_1(t),\ 0 < t \leq T, \tag{1.3}$$

$$u(1,t) = h_2(t),\ 0 < t \leq T \tag{1.4}$$

where x is a space variable, t is a time variable; the parameters μ and α refer to the order of conformable fractional derivative with respect to time variable and space variable, respectively. The functions $f(x,t)$, $g(x,t)$, $f_1(x)$, $h_1(t)$ and $h_2(t)$ are given functions. In case of $\mu = 1$, $\alpha = 2$, Eq. (1.1) is the classical diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = f(x,t)\frac{\partial^2 u(x,t)}{\partial t^2} + g(x,t).$$

The organization of this paper is as follows: In Section 2, definition and properties of the conformable fractional derivative are presented. In the Section 3, properties of Chebyshev polynomials of the second kind are given. In Section 4, the conformable fractional derivative is written by using shifted Chebyshev polynomials of the second kind. In Section 5, numerical scheme is given to obtain an analytical approximate solution in terms of the shifted Chebyshev polynomials of the second kind of the problem (1.1)-(1.4). In Section 6, numerical results are given to clarify the method. Conclusion is presented in Section 7. Note that numerical results have been computed by using the Matlab programming.

2. Description of conformal fractional derivative and its properties

For a function $f : (0,\infty) \to R$, the conformal fractional derivative of f of order $0 < \alpha < 1$ in variable t is defined as (see, for example, [9])

$$T_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

Some important properties of the the conformal fractional derivative are as follows:

Theorem 1. *(see, for example, [9]) Let $\alpha \in (0,1]$ and f, g be α differentiable at a point $t > 0$. Then*

$$T_t^\alpha (af + bg)(t) = aT_t^\alpha f(t) + bT_t^\alpha g(t), \qquad \forall a,b \in \mathbb{R}, \tag{2.1}$$

$$T_t^\alpha (fg)(t) = f(t)T_t^\alpha g(t) + g(t)T_t^\alpha f(t), \tag{2.2}$$

$$T_t^\alpha (t^p) = pt^{p-\alpha}, \qquad \forall p \in \mathbb{R}. \tag{2.3}$$

If, in addition, f is differentiable, then

$$T_t^\alpha (f(t)) = t^{1-\alpha} f'(t). \tag{2.4}$$

The fractional derivative starting from b of a function $f : [b,\infty) \to \mathbb{R}$, of order α, where $f^{(n)}(t)$ exists, is defined by

$$T_t^\alpha (f(t)) = T_t^\beta (f^{(n)}(t)), \; n < \alpha \le n+1, \; \beta = \alpha - n. \tag{2.5}$$

3. Some properties of Chebyshev polynomials of the second kind

The Chebyshev polynomials $U_n(x)$ of the second kind are orthogonal polynomials of degree n in x defined on the $[-1,1]$ (see, for example,[11])

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where $x = \cos\theta$ and $\theta \in [0,\pi]$. Also, these polynomials $U_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner product

$$< U_n(x), U_m(x) >= \int_{-1}^{1} \sqrt{1-x^2}\, U_n(x) U_m(x) dx = \left\{ \begin{array}{ll} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{array} \right\},$$

where $\sqrt{1-x^2}$ is weight function corresponding to $U_n(x)$. The polynomials $U_n(x)$ may be generated by using the recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \; n = 2,3,...,$$

with $U_0(x) = 1$, $U_1(x) = 2x$.

Shifted Chebyshev polynomials of the second kind $U_n^*(x)$ of degree n in x on $[0,1]$ is given by $U_n^*(x) = U_n(2x-1)$. These polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$< U_n^*(x), U_m^*(x) >= \int_{0}^{1} \sqrt{x-x^2}\, U_n^*(x) U_m^*(x) dx = \left\{ \begin{array}{ll} 0, & n \neq m, \\ \frac{\pi}{8}, & n = m. \end{array} \right\},$$

where $\sqrt{x-x^2}$ is weight function corresponding to $U_n^*(x)$. The polynomials $U_n^*(x)$ may be generated by using the recurrence relations

$$U_n^*(x) = 2(2x-1)U_{n-1}^*(x) - U_{n-2}^*(x), \; n = 2,3,...,$$

with $U_0^*(x) = 1$, $U_1^*(x) = 4x-2$.

The analytical form of the shifted Chebyshev polynomials of the second kind $U_n^*(x)$ of degree n is given by

$$U_n^*(x) = \sum_{k=0}^{N} (-1)^k 2^{2n-2k} \frac{\Gamma(2n-k+2)x^{n-k}}{2n-2k+2}. \tag{3.1}$$

The function $y(x)$ which belongs to the space of square integrable in $[0,1]$ may be expressed in terms of shifted Chebyshev polynomials of the second kind as

$$y(x) = \sum_{i=0}^{\infty} b_i U_i^*(x),$$

where the coefficients b_i are given by

$$b_i = \frac{8}{\pi}\int_0^1 y(x)\sqrt{x-x^2}U_i^*(x)dx. \tag{3.2}$$

Consider only the first $(N+1)-$ terms of shifted Chebyshev polynomials of the second kind, so we can write

$$y_N(x) = \sum_{i=0}^{N} b_i U_i^*(x). \tag{3.3}$$

4. Evaluation of the conformable fractional derivative using shifted Chebyshev polynomials of the second kind

In this section, we derive an approximate formula of the conformable fractional derivative of $y_N(x)$. By using the linearity of the conformable fractional differentiation given in Eq. (2.1) and by using definition of approximated function $y_N(x)$ as in Eq. (3.3) we have:

$$T_x^{\alpha} y_N(x) = \sum_{i=0}^{N} b_i T_x^{\alpha} U_i^*(x),\ \ \alpha > 0. \tag{4.1}$$

Moreover, from the properties of linearity of the conformable derivative in addition to Eqs. (2.3) and (2.5) we get:

$$T_x^{\alpha} U_i^*(x) = 0,\ \ i = 0,1,\dots,n,\ \ n<\alpha\leq n+1. \tag{4.2}$$

Also we have for $n<\alpha\leq n+1$

$$T_x^{\alpha} U_i^*(x) = \sum_{k=0}^{i-(n+1)} (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k-n)} x^{i-k-\alpha}. \tag{4.3}$$

By combinations Eqs. (4.1), (4.2) and (4.3) we obtain for $n<\alpha\leq n+1$

$$T_x^{\alpha} y_N(x) = \sum_{i=n+1}^{N}\sum_{k=0}^{i-(n+1)} b_i (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k-n)} x^{i-k-\alpha},$$

which can be rewritten as the form:

$$T_x^{\alpha} y_N(x) = \sum_{i=n+1}^{N}\sum_{k=0}^{i-(n+1)} b_i A_{i,k}^{\alpha} x^{i-k-\alpha},\ \ \ n<\alpha\leq n+1, \tag{4.4}$$

where

$$A_{i,k}^{\alpha} = (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k-n)}. \tag{4.5}$$

Test example: Consider $y(x) = x^2$ with $N = 3$ and $\alpha = 1.5$. Using Eqs. (1) and (2.5) we obtain:

$$T_x^{1.5} x^2 = T_x^{0.5} 2x = 2x^{0.5}.$$

Then, using the proposed method we obtain:

$$T_x^{1.5}x^2 = \sum_{i=2}^{3}\sum_{k=0}^{i-2} b_i A_{i,k}^{1.5} x^{i-k-1.5}, \tag{4.6}$$

where

$$A_{2,0}^{1.5} = 32,\ A_{3,0}^{1.5} = 384,\ A_{3,1}^{1.5} = -\frac{192}{\Gamma(2).}$$

Substituting the constants $b_2 = \frac{1}{16}$ and $b_3 = 0$ from Eq. (3.2) into Eq. (4.6) we get:

$$T_x^{1.5}x^2 = 2x^{0.5}.$$

5. Numerical scheme

In this section, we apply Chebyshev collocation method to problem (1.1)-(1.4). Assume that the solution of the problem (1.1)-(1.4) can be written as

$$u_N(x,t) = \sum_{i=0}^{N} a_i(t) U_i^*(x). \tag{5.1}$$

From (1.1), (1), (4.4) and (4.5) we obtain

$$\sum_{i=0}^{N} t^{1-\mu}\frac{da_i(t)}{dt}U_i^*(x) = f(x,t)\sum_{i=0}^{N} a_i(t) T_x^{\alpha} U_i^*(x) + g(x,t).$$

$$\sum_{i=0}^{N} t^{1-\mu}\frac{da_i(t)}{dt}U_i^*(x) = f(x,t)\sum_{i=n+1}^{N}\ \sum_{k=0}^{i-(n+1)} a_i(t) A_{i,k}^{\alpha} x^{i-k-\alpha} + g(x,t). \tag{5.2}$$

Now, we collocate Eq. (5.2) at $N-n$ points x_p as follows:

$$\sum_{i=0}^{N} t^{1-\mu}\frac{da_i(t)}{dt}U_i^*(x_p) = f(x_p,t)\sum_{i=n+1}^{N}\ \sum_{k=0}^{i-(n+1)} a_i(t) A_{i,k}^{\alpha} x_p^{i-k-\alpha} + g(x_p,t). \tag{5.3}$$

We use the roots of shifted Chebyshev polynomials of the second kinds $U_{N-n}^*(x)$ to suitable the collocation points.

The boundary conditions (1.3) and (1.4) can be written as

$$\sum_{i=0}^{N} (-1)^i (i+1) a_i(t) = h_1(t),\ \sum_{i=0}^{N} (i+1) a_i(t) = h_2(t). \tag{5.4}$$

Since $1 < \alpha \le 2$ in the problem (1.1), we can take $n = 1$. Let us take $0 < t_k \le T$, $\Delta t = \frac{T}{M}$, $t_k = k\Delta t$, $k = 0,1,\dots,M$, $a_i(t_k) = a_i^k$, $g(x_p,t_k) = g_p^k$, $h_1(t_k) = h_1^k$ and $h_2(t_k) = h_2^k$. Applying the finite difference method to the system (5.3) and considering (5.4) we have the following system

$$(A-B)V^j = CV^{j-1} + D,\ \ j = 1,\dots,M, \tag{5.5}$$

where

$$D=\left(\Delta t g_1^j,\Delta t g_2^j,\dots,\Delta t g_{N-1}^j,h_1^j,h_2^j\right)^*_{(N+1)\times 1},\quad V^j=\left(a_0^j,a_1^j,\dots,a_N^j\right)^*_{(N+1)\times 1},$$

$$A=\begin{pmatrix} t_j^{1-\mu}U_0^*(x_1) & t_j^{1-\mu}U_1^*(x_1) & \dots & t_j^{1-\mu}U_N^*(x_1)\\ . & . & \dots & .\\ t_j^{1-\mu}U_0^*(x_{N-1}) & t_j^{1-\mu}U_1^*(x_{N-1}) & \dots & t_j^{1-\mu}U_N^*(x_{N-1})\\ 1 & -2 & \dots & (-1)^N(N+1)\\ 1 & 2 & \dots & (N+1)\end{pmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{pmatrix} 0 & 0 & F_2^j(x_1) & \dots & F_N^j(x_1)\\ . & . & . & \dots & .\\ 0 & 0 & F_2^j(x_{N-1}) & \dots & F_N^j(x_{N-1})\\ 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 0\end{pmatrix}_{(N+1)\times(N+1)},$$

$$F_i^j(x_p)=\Delta t f(x_p,t_j)\sum_{k=0}^{i-2}A_{i,k}^{\alpha}x_p^{i-k-\alpha},\qquad i=2,\dots,N,\, p=1,\dots,N-1,$$

$$C=t_j^{1-\mu}\begin{pmatrix} U_0^*(x_1) & U_1^*(x_1) & \dots & U_N^*(x_1)\\ . & . & \dots & .\\ U_0^*(x_{N-1}) & U_1^*(x_{N-1}) & \dots & U_N^*(x_{N-1})\\ 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\end{pmatrix}_{(N+1)\times(N+1)}$$

and $*$ denotes transposition of the matrix. To compute V^j, $j=1,2,\dots M$, in the system (5.5), we need to obtain initial case V^0. Substituting Eqs. (3.2) and (5.1) into initial condition (1.2) we can compute the constants $a_i(t_0)=a_i^0$ in the initial case at $j=1$.

6. Applications

Example 1. Let us consider the following space-time fractional diffusion equation with initial and boundary conditions

$$T_t^{0.4}u(x,t)=\frac{x^{1.7}t^{0.6}}{2}T_x^{1.7}u(x,t)+2x^3t^{0.6}\exp(t),\ 0<x<1,\ t>0, \tag{6.1}$$

$$u(x,0)=x^2-x^3,\ 0<x<1, \tag{6.2}$$

$$u(0,t)=0,\ u(1,t)=0,\ 0<t\le T. \tag{6.3}$$

Note that the exact solution to this problem is $u_e(x,t)=(x^2-x^3)\exp(t)$. We apply the suggested method for $N=3$ and approximate to the solution $u(x,t)$ as follows:

$$u_3(x,t)=\sum_{i=0}^{3}a_i(t)U_i^*(x). \tag{6.4}$$

By using the obtained matrix equation (5.5) for $N=3$ with the initial data $V^0 = (3/32, 1/32, -1/32, -1/64)^*$, we obtain the unknown coefficients $a_i(t), i=0,1,2,3$, for $t=0.25;\ 0.5;\ 0.75;\ 1;\ 5$ which are given in Table 1. In Table 2, the absolute errors between the exact solution $u_e(x,t)$ and the approximate solution $u_3(x,t)$, at $t=0.25; 0.5; 0.75; 1; 5$. It is clear from Table 2 that the obtained numerical solution by using our approach is in very good agreement with the exact solution even for small value $N=3$. Furthermore, the computation is completed in a very short time since a few terms of the series solution (i.e. $N=3$) are used. Fig.1 shows 3D plot of the obtained solution (6.4) for $0<x<1$ and $0<t<1$.

TABLE 1. The obtained coefficients $a_i (i=0,1,2,3)$ for different values t in Example 1.

t	a_0	a_1	a_2	a_3
0.25	0.120377383756128	0.040125794456509	−0.040125794585376	−0.020062897228255
0.5	0.154567620926466	0.051522539911611	−0.051522540308822	−0.025761269955805
0.75	0.198468754317411	0.066156250704570	−0.066156251439137	−0.033078125352285
1	0.254838925339978	0.084946307306756	−0.084946308446659	−0.042473153653378
5	13.913734969534085	4.637911222166822	−4.637911656511361	−2.318955611083411

TABLE 2. The absolute errors between the exact and approximate solutions in Example 1.

x	$t=0.25$	$t=0.5$	$t=0.75$	$t=1$	$t=5$
0	10^{-17}	8.10^{-17}	0	2.10^{-17}	3.10^{-15}
0.1	2.10^{-9}	6.10^{-10}	10^{-9}	10^{-9}	6.10^{-7}
0.2	5.10^{-9}	10^{-9}	2.10^{-9}	3.10^{-9}	10^{-6}
0.3	8.10^{-9}	10^{-9}	2.10^{-9}	4.10^{-9}	10^{-6}
0.4	10^{-9}	2.10^{-9}	3.10^{-9}	4.10^{-9}	10^{-6}
0.5	10^{-9}	2.10^{-9}	3.10^{-9}	5.10^{-9}	10^{-6}
0.6	10^{-9}	2.10^{-9}	3.10^{-9}	5.10^{-9}	10^{-6}
0.7	10^{-9}	2.10^{-9}	3.10^{-9}	4.10^{-9}	10^{-6}
0.8	10^{-9}	10^{-9}	2.10^{-9}	3.10^{-9}	10^{-6}
0.9	6.10^{-10}	10^{-9}	10^{-9}	2.10^{-9}	6.10^{-7}
1	10^{-17}	8.10^{-17}	2.10^{-17}	0	0

Example 2. Consider the following space-time fractional diffusion problem

$$T_t^{0.3} u(x,t) = \frac{x^{0.8}}{2} T_x^{1.8} u(x,t) - (x^2+1) \sin\left(\frac{t^{0.3}}{0.3}\right) - x \cos\left(\frac{t^{0.3}}{0.3}\right) \tag{6.5}$$

$$0<x<1,\ t>0,$$

$$u(x,0) = x^2+1,\ 0<x<1, \tag{6.6}$$

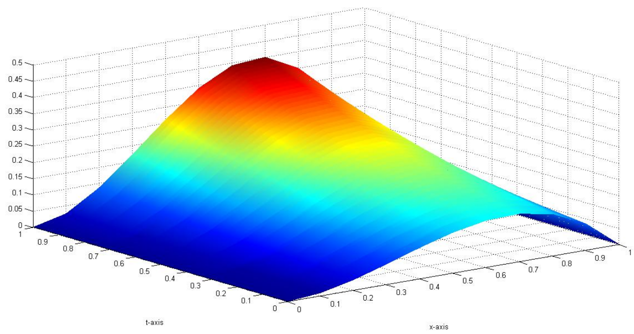

FIGURE 1. The behavior of the approximate solution of the problem (6.1)-(6.3).

$$u(0,t) = \cos\left(\frac{t^{0.3}}{0.3}\right), \; u(1,t) = 2\cos\left(\frac{t^{0.3}}{0.3}\right), \; 0 < t \leq T. \tag{6.7}$$

The exact solution is $u_e(x,t) = (x^2+1)\cos(\frac{t^{0.3}}{0.3})$. By using the obtained matrix equation (5.5) for $N = 2$ with the initial data $V^0 = (21/16, 1/4, 1/16)^*$, we obtain the unknown coefficients $a_i(t), i = 0,1,2$. In Table 3, the absolute errors between the exact solution $u_e(x,t)$ and the approximate solution $u_2(x,t)$, at $t = 0.25; 0.5; 0.75; 1; 5$. From the results of Table 3, it is obvious that the presented method gives high accuracy even for the small value N. Fig. 2 shows 3D plot of the solution of (6.5)-(6.7) for $0 < x < 1$ and $0 < t < 15$.

TABLE 3. The absolute errors between the exact and approximate solutions in Example 2.

x	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1$	$t = 5$
0	0	0	0	0	0
0.1	10^{-6}	5.10^{-7}	2.10^{-7}	2.10^{-16}	4.10^{-9}
0.2	2.10^{-6}	10^{-6}	5.10^{-7}	10^{-7}	7.10^{-9}
0.3	3.10^{-6}	10^{-6}	6.10^{-7}	3.10^{-7}	9.10^{-9}
0.4	4.10^{-6}	10^{-6}	7.10^{-7}	4.10^{-7}	10^{-9}
0.5	4.10^{-6}	10^{-6}	8.10^{-7}	4.10^{-7}	10^{-8}
0.6	4.10^{-6}	10^{-6}	7.10^{-7}	4.10^{-7}	10^{-8}
0.7	3.10^{-6}	10^{-6}	6.10^{-7}	4.10^{-7}	9.10^{-9}
0.8	2.10^{-6}	10^{-6}	5.10^{-7}	4.10^{-7}	7.10^{-9}
0.9	10^{-6}	5.10^{-7}	2.10^{-7}	3.10^{-7}	4.10^{-9}
1	0	0	0	2.10^{-16}	0

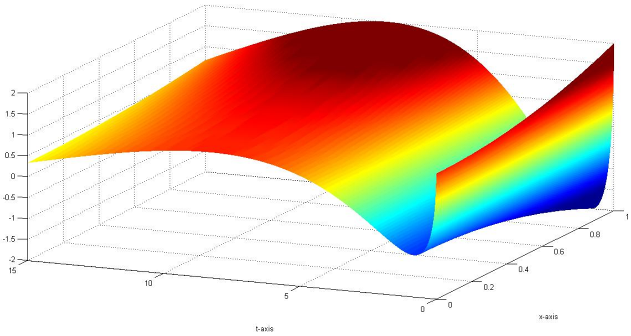

FIGURE 2. The behavior of the approximate solution of the problem (6.5)-(6.7).

7. CONCLUSION

In this paper, using the shifted Chebyshev polynomials of the second kind and its properties together with the Chebyshev collocation method, the space-time fractional diffusion equation with variable coefficients is reduced to a system of ordinary differential equations which is solved by the finite difference method. The fractional derivatives are considered in the conformable sense. The numerical results obtained by the proposed technique are compared with the exact solution to illustrate validity and applicability of the proposed technique. From the numerical results, it is obvious that the proposed method gives high accuracy and efficiency even for small values N. Furthermore, accuracy of the method can be increased by adding new terms to the series (5.1).

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