ORIGINAL PAPER

k-ORDER FIBONACCI QUATERNIONS

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Abstract. *In this paper, we define and study another interesting generalization of the Fibonacci quaternions is called k-order Fibonacci quaternions. Then we obtain for $k=2$ Fibonacci quaternions, for $k=3$ Tribonacci quaternions and for $k=4$ Tetranacci quaternions. We give generating function, the summation formula and some properties about k-order Fibonacci quaternions. Also, we identify and prove the matrix representation for k-order Fibonacci quaternions. The Q_k matrix given for k-order Fibonacci numbers is defined for k-order Fibonacci quaternions and after the matrices with the k-order Fibonacci quaternions is obtained with help of auxiliary matrices, important relationships and identities are established.*

Keywords: *Fibonacci numbers; k-Order Fibonacci numbers; quaternions; Fibonacci quaternions; matrix representations.*

1. INTRODUCTION

Fibonacci numbers are defined on a interesting recurrence relation of $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = 0,\ F_1 = 1$. Fibonacci numbers and various generalizations have many interesting properties and applications to many fields of science. For more information, one can see [1-6].

Another generalization of the Fibonacci numbers is order-k Fibonacci numbers and

$$g_n^{\ i} = \sum_{j=1}^{k} g^{i}_{\ n-j}\text{, for } n > 0 \text{ and } 1 \leq i \leq k \qquad (1.1)$$

with boundary conditions for $1-k \leq n \leq 0$,

$$g_n^{\ i} = \begin{cases} 1 & , \quad i = 1-n \\ 0 & , \quad otherwise \end{cases}$$

is defined by the recurrence relation by Er in [7].

Kılıç and Tasci studied some properties about k-order Fibonacci numbers. They defined Binet formulas combinatorial representations of k-order Fibonacci numbers in [8]. Lee in [9-11] defined the generalized Binet formula for k-generalized Fibonacci sequence with a different perspective by using determinants.

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Quaternion arithmetic has been used in many fields such as computer sciences, physics, applied mathematics, differential geometry and quaternion analysis in [12].

Irish Mathematician William Rowan Hamilton first introduced the real quaternions in 1843 in [13]. The set of real quaternions can be defined as

$$H = \{ q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 : q_i \in R, i = 0,1,2,3 \}$$

as the four-dimensional vector space over R having a basis $\{e_0, e_1, e_2, e_3\}$ which satisfies the following multiplication rules:

Table 1. Multiplication Rules

$\times$	e_0	e_1	e_2	e_3
e_0	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	e_3	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

A quaternion is a hyper-complex number and is shown by the following equation:

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = \sum_{i=0}^{3} q_i e_i \in H.$$

The quaternion consists of two parts. The first part is called scalar part as $S_q = q_0 e_0$ and second is called vectoral part as $\overrightarrow{V_q} = q_1 e_1 + q_2 e_2 + q_3 e_3$. Then we can write $q = S_q + \overrightarrow{V_q}$. The conjugate of q is defined by

$$q = S_q - \overrightarrow{V_q} = q_0 e_0 - \sum_{i=1}^{3} q_i e_i \, .$$

Let q and p be two quaternions such that $q = \sum_{i=0}^{3} q_i e_i$ and $p = \sum_{i=0}^{3} p_i e_i$. The equality, addition and multiplication by scalar are defined by the following:

-Equality: $q = p$ if and only if $q_i = p_i$ for $i = 0,1,2,3$

-Addition: $q + p = \sum_{i=0}^{3} (q_i + p_i) e_i$

-Multiplication by scalar: $k.q = k.\sum_{i=0}^{3} q_i e_i = \sum_{i=0}^{3} (k q_i) e_i$

The multiplication of q and p is defined as

$$q.p = S_q S_p + S_q \overrightarrow{V_p} + \overrightarrow{V_q} S_p - \overrightarrow{V_q}.\overrightarrow{V_p} + \overrightarrow{V_q} \times \overrightarrow{V_p}$$

where

$$\overrightarrow{V_q}.\overrightarrow{V_p} = \sum_{i=1}^{3} q_i p_i$$

and

$$\vec{V_q} \times \vec{V_p} = (q_2 p_3 - q_3 p_2) e_1 - (q_1 p_3 - q_3 p_1) e_2 + (q_1 p_2 - q_2 p_1) e_3 .$$

The norm of q is defined as

$$\|q\| = N(q) = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = \sum_{i=0}^{3} q_i^2$$

F. Horadam in [14] introduced nth Fibonacci and Lucas quaternions in 1963 and examined in [15] the recurrence relations of quaternion in 1993. Also, he referred to defining Pell quaternions and generalized Pell quaternions. In [16] many interesting properties can be given about Fibonacci and Lucas quaternions. Halici in [17] examined Binet's formulas, generating functions and some properties about Fbonacci and Lucas quaternions. In [18] Cimen and Ipek introduced new kinds of sequences of quaternion number called as Pell quaternions and Pell-Lucas quaternions. Liana and Wloch in [19] defined the Jacobsthal quaternions and Jacobsthal-Lucas quaternions and gave some properties. In [20] Gamaliel C-M generalized the Tribonacci quaternions and in [21] Kecilioglu and Akkus introduced Fibonacci octanions. Polatlı, Kızılates and Kesim introduced split k-Fibonacci and k-Lucas quaternions in [22]. Tasci and Yalcin defined Fibonacci p-quaternions in [23] in 2015 and Tasci defined Padovan and Pell-quaternions in [24]. Also, Tasci generalized Jacobsthal and Jacobsthal-Lucas quaternions to k-Jacobsthal and k-Jacobsthal-Lucas quaternions in [25]. In 2017, Aydın, Koklu and Yuce defined the generalized dual Pell quaternions and gave some properties in [26].

In this paper we define and study another interesting generalization of Fibonacci quaternions is called $k-$order Fibonacci quaternions. Then we obtain for $k=2$ Fibonacci quaternions, for $k=3$ Tribonacci quaternions and for $k=4$ Tetranacci quaternions. We give generating function, the summation formula, some properties and describe the matrix representations about $k-$order Fibonacci quaternions.

2. $k-$ORDER FIBONACCI QUATERNIONS

Definition 2.1. The nth $k-$order Fibonacci quaternion $QF_n^{(k)}$ is defined

$$\begin{aligned} QF_n^{(k)} &= F_n^{(k)} e_0 + F_{n+1}^{(k)} e_1 + F_{n+2}^{(k)} e_2 + F_{n+3}^{(k)} e_3 \\ &= \sum_{i=0}^{3} F_{n+i}^{(k)} e_i \end{aligned} \tag{2.1}$$

where $F_n^{(k)}$ is nth $k-$order Fibonacci numbers.

Let $QF_n^{(k)}$ and $QM_n^{(k)}$ be two $k-$order Fibonacci quaternions such that $QF_n^{(k)} = \sum_{i=0}^{3} F_{n+i}^{(k)} e_i$ and $QM_n^{(k)} = \sum_{i=0}^{3} M_{n+i}^{(k)} e_i$. The scalar part of $k-$order Fibonacci quaternions $QF_n^{(k)}$ and $QM_n^{(k)}$ are denoted by $S_{QF_n^{(k)}} = F_n^{(k)} e_0$ and $S_{QM_n^{(k)}} = M_n^{(k)} e_0$, respectively. Also,

$\overrightarrow{V_{QF_n^{(k)}}} = \sum_{i=1}^{3} F_{n+i}^{(k)} e_i$ and $\overrightarrow{V_{QM_n^{(k)}}} = \sum_{i=1}^{3} M_{n+i}^{(k)} e_i$ are called vectorial part of $k-$order Fibonacci quaternions.

Definition 2.2. The conjugate of $QF_n^{(k)}$ is defined by

$$\overline{QF_n^{(k)}} = F_n^{(k)} e_0 - F_{n+1}^{(k)} e_1 - F_{n+2}^{(k)} e_2 - F_{n+3}^{(k)} e_3 = F_n^{(k)} e_0 - \sum_{i=1}^{3} F_{n+i}^{(k)} e_i \tag{2.2}$$

Definition 2.3. The norm of $QF_n^{(k)}$ is defined by

$$\left\| QF_n^{(k)} \right\| = N_{QF_n^{(k)}} = \left(F_n^{(k)}\right)^2 + \left(F_{n+1}^{(k)}\right)^2 + \left(F_{n+2}^{(k)}\right)^2 + \left(F_{n+3}^{(k)}\right)^2$$

Theorem 2.4. The $k-$order Fibonacci quaternions are defined by the following recurrence relation

$$QF_n^{(k)} = \sum_{j=1}^{k} QF_{n-j}^{(k)} \text{ for } n \text{ integer and } k \geq 2 \tag{2.3}$$

Proof: From (2.1), we get

$$\sum_{j=1}^{k} QF_{n-j}^{(k)} = \left(\sum_{i=0}^{k} QF_{n+i-1}^{(k)} e_i \right) + \left(\sum_{i=0}^{k} QF_{n+i-2}^{(k)} e_i \right) + \ldots + \left(\sum_{i=0}^{k} QF_{n+i-k}^{(k)} e_i \right)$$

and since from the recurrence relation of $k-$order Fibonacci numbers (1.1); we obtain (2.3)

$$QF_n^{(k)} = \sum_{j=1}^{k} QF_{n-j}^{(k)} \ .$$

Proposition 2.5. For $n > 0$ and $k \geq 2$, we have the following properties:

(i) $QF_n^{(k)} + \overline{QF_n^{(k)}} = 2F_n^{(k)}$

(ii) $\left(QF_n^{(k)}\right)^2 + QF_n^{(k)}.\overline{QF_n^{(k)}} = 2F_n^{(k)}.QF_n^{(k)}$

(iii) $QF_n^{(k)}.\overline{QF_n^{(k)}} = \left(F_n^{(k)}\right)^2 + \left(F_{n+1}^{(k)}\right)^2 + \left(F_{n+2}^{(k)}\right)^2 + \left(F_{n+3}^{(k)}\right)^2$

(iv) $QF_{n+1}^{(k)} - QF_n^{(k)} = QF_n^{(k)} - QF_{n-k}^{(k)}$

(v) $QF_{n+1}^{(k)} + QF_n^{(k)} = 3QF_n^{(k)} - QF_{n-k}^{(k)}$

Theorem 2.5. The generating function for the $k-$order Fibonacci quaternions is

$$g(t) = \sum_{n=0}^{\infty} QF_n^{(k)} t^n = \frac{QF_0^{(k)} + t\left(QF_1^{(k)} - QF_0^{(k)}\right) + t^2\left(QF_2^{(k)} - QF_1^{(k)} - QF_0^{(k)}\right)}{1 - \sum_{j=1}^{k} t^j}$$

Proof: Let $g(t)$ be the generating function of the $k-$order Fibonacci quaternions $\left\{QF_n^{(k)}\right\}$.

$$g(t)-tg(t)-...-t^k g(t)=QF_0^{(k)}+t\left(QF_1^{(k)}-QF_0^{(k)}\right)+t^2\left(QF_2^{(k)}-QF_1^{(k)}-QF_0^{(k)}\right)$$
$$+t^3\left(QF_3^{(k)}-QF_2^{(k)}-QF_1^{(k)}-QF_0^{(k)}\right)+\sum_{n=4}^{\infty}t^n\left(QF_n^{(k)}-\sum_{j=0}^{n-1}QF_j^{(k)}\right)$$

By taking $g(t)$ parenthesis we get

$$g(t)=\frac{QF_0^{(k)}+t\left(QF_1^{(k)}-QF_0^{(k)}\right)+t^2\left(QF_2^{(k)}-QF_1^{(k)}-QF_0^{(k)}\right)}{1-\sum_{j=1}^{k}t^j}.$$

Corollary 2.6. For $k=2$, we obtain the generating function of the usual Fibonacci quaternions in [17] as follows:

$$g(t)=\sum_{n=0}^{\infty}QF_n t^n=\frac{QF_0+t\left(QF_1-QF_0\right)}{1-t-t^2}.$$

Corollary 2.7. For $k=3$, we obtain the generating function of the Tribonacci quaternions as

$$g(t)=\sum_{n=0}^{\infty}QT_n t^n=\frac{QT_0+t\left(QT_1-QT_0\right)+t^2\left(QT_2-QT_1-QT_0\right)}{1-t-t^2-t^3}.$$

Theorem 2.8. The sum of the $k-$order Fibonacci quaternions is given by

$$\sum_{i=1}^{m}QF_i^{(k)}=\frac{1}{k-1}\left(QF_{k+m}^{(k)}-QF_k^{(k)}+\sum_{i=1}^{k-2}(k-i-1)\left(QF_i^{(k)}-QF_{m+i}^{(k)}\right)\right).$$

Proof: By the recurrence relation of the $k-$order Fibonacci quaternions (2.1) we have

$$QF_{n-k}^{(k)}=QF_n^{(k)}-\sum_{i=1}^{k-1}QF_{n-i}^{(k)}$$

From this equality

$$QF_1^{(k)}=QF_{k+1}^{(k)}-QF_k^{(k)}-...-QF_3^{(k)}-QF_2^{(k)}$$

$$QF_2^{(k)}=QF_{k+2}^{(k)}-QF_{k+1}^{(k)}-...-QF_4^{(k)}-QF_3^{(k)}$$

$$QF_3^{(k)}=QF_{k+3}^{(k)}-QF_{k+2}^{(k)}-...-QF_5^{(k)}-QF_4^{(k)}$$

$$\vdots$$

$$QF_{m-1}^{(k)}=QF_{k+m-1}^{(k)}-QF_{k+m-2}^{(k)}-...-QF_{m+1}^{(k)}-QF_m^{(k)}$$

$$QF_m^{(k)}=QF_{k+m}^{(k)}-QF_{k+m-1}^{(k)}-...-QF_{m+2}^{(k)}-QF_{m+1}^{(k)}$$

So, we obtain

$$\begin{aligned}\sum_{i=1}^{m} QF_i^{(k)} &= QF_{k+m}^{(k)} - QF_2^{(k)} - 2QF_3^{(k)} - 3QF_4^{(k)} \\ &-\ldots-(k-2)QF_{k-1}^{(k)} - (k-1)QF_k^{(k)} \\ &-(k-2)\sum_{i=k+1}^{m+1} QF_i^{(k)} - (k-3)QF_{m+2}^{(k)} \\ &-(k-4)QF_{m+3}^{(k)} - \ldots - 3QF_{k+m-4}^{(k)} \\ &-2QF_{k+m-3}^{(k)} - QF_{k+m-2}^{(k)}\end{aligned}$$

Adding and subtracting the following terms to above statement

$$\left(k-2\right)QF_1^{(k)} + \left(k-2\right)QF_2^{(k)} + \left(k-2\right)QF_3^{(k)} + \ldots + (k-2)QF_k^{(k)}$$

we get

$$\begin{aligned}\sum_{i=1}^{m} QF_i^{(k)} &= QF_{k+m}^{(k)} + \left(k-2\right)QF_1^{(k)} + \left(k-3\right)QF_2^{(k)} \\ &+\ldots+2QF_{k-3}^{(k)} + QF_{k-2}^{(k)} - QF_k^{(k)} - \left(k-2\right)\sum_{i=1}^{m} QF_i^{(k)} \\ &-\left(k-2\right)QF_{m+1}^{(k)} - \left(k-3\right)QF_{m+2}^{(k)} - \ldots - 3QF_{k+m-4}^{(k)} \\ &-2QF_{k+m-3}^{(k)} - QF_{k+m-2}^{(k)}\end{aligned}$$

Finally we have

$$\begin{aligned}\left(k-1\right)\sum_{i=1}^{m} QF_i^{(k)} &= QF_{k+m}^{(k)} - QF_k^{(k)} + \sum_{i=1}^{k-2}\left(k-i-1\right)QF_i^{(k)} \\ &-\sum_{i=1}^{k-2}\left(k-i-1\right)QF_{m+i}^{(k)}\end{aligned}$$

Consequently, we get

$$\sum_{i=1}^{m} QF_i^{(k)} = \frac{1}{k-1}\left(QF_{k+m}^{(k)} - QF_k^{(k)} + \sum_{i=1}^{k-2}\left(k-i-1\right)\left(QF_i^{(k)} - QF_{m+i}^{(k)}\right)\right).$$

Corollary 2.9. For $k=2$, we obtain the sum of the Fibonacci quaternions in [17] as

$$\begin{aligned}\sum_{i=1}^{m} QF_i &= QF_{m+2} - QF_2 \\ &= QF_{m+2} - \left(e_0 + 2e_1 + 3e_2 + 5e_3\right).\end{aligned}$$

Corollary 2.10. For $k=3$, we obtain the sum of the Tribonacci quaternions as

$$\sum_{i=1}^{m} QT_i = \frac{1}{2}\left(QT_{m+3} + QT_{m+1} - QT_3 - QT_1\right)$$

$$=\frac{1}{2}\left(QT_{m+3}+QT_{m+1}-\left(3e_0+5e_1+9e_2+17e_3\right)\right).$$

Now we introduce the matrices Q_k, A_k and $E_{k,n}$ that plays the role of the $Q-$matrix for Fibonacci numbers. Let Q_k, A_k and $E_{k,n}$ determine the $k\times k$ matrices defined as

$$Q_k=\begin{bmatrix}1&1&1&\cdots&1&1\\1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&\cdots&0&0\\0&0&\cdots&0&1&0\end{bmatrix}_{k\times k},\quad A_k=\begin{bmatrix}QF_{k-1}^{(k)}&QF_{k-2}^{(k)}&QF_{k-3}^{(k)}&\cdots&QF_{1}^{(k)}&QF_{0}^{(k)}\\QF_{k-2}^{(k)}&QF_{k-3}^{(k)}&QF_{k-4}^{(k)}&\cdots&QF_{0}^{(k)}&QF_{-1}^{(k)}\\QF_{k-3}^{(k)}&QF_{k-4}^{(k)}&QF_{k-5}^{(k)}&\cdots&QF_{-1}^{(k)}&QF_{-2}^{(k)}\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\QF_{1}^{(k)}&QF_{0}^{(k)}&QF_{-1}^{(k)}&\cdots&QF_{3-k}^{(k)}&QF_{2-k}^{(k)}\\QF_{0}^{(k)}&QF_{-1}^{(k)}&QF_{-2}^{(k)}&\cdots&QF_{2-k}^{(k)}&QF_{1-k}^{(k)}\end{bmatrix}_{k\times k}$$

$$E_{k,n}=\begin{bmatrix}QF_{n+k-1}^{(k)}&QF_{n+k-2}^{(k)}&QF_{n+k-3}^{(k)}&\cdots&QF_{n+1}^{(k)}&QF_{n}^{(k)}\\QF_{n+k-2}^{(k)}&QF_{n+k-3}^{(k)}&QF_{n+k-4}^{(k)}&\cdots&QF_{n}^{(k)}&QF_{n-1}^{(k)}\\QF_{n+k-3}^{(k)}&QF_{n+k-4}^{(k)}&QF_{n+k-5}^{(k)}&\cdots&QF_{n-1}^{(k)}&QF_{n-2}^{(k)}\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\QF_{n+1}^{(k)}&QF_{n}^{(k)}&QF_{n-1}^{(k)}&\cdots&QF_{n+3-k}^{(k)}&QF_{n+2-k}^{(k)}\\QF_{n}^{(k)}&QF_{n-1}^{(k)}&QF_{n-2}^{(k)}&\cdots&QF_{n+2-k}^{(k)}&QF_{n+1-k}^{(k)}\end{bmatrix}_{k\times k}$$

Now we can give the following lemma and theorem:

Lemma 2.11. Let $n\geq 1$. Then

$$E_{k,n+1}=Q_k.E_{k,n}.$$

Theorem 2.12. Let $n\geq 1$. Then

$$E_{k,n}=Q_k^n.A_k \tag{2.4}$$

Proof: We can proof by induction method on n. If $n=1$, then from the definition of the matrix $E_{k,n}$ and $k-$order Fibonacci quaternions

$$E_{k,1}=Q_k.A_k$$

Assume that the theorem holds for n

$$E_{k,n}=Q_k^n.A_k$$

Then for $n+1$ we get

$$\begin{aligned}Q_k^{n+1}.A_k&=Q_k.Q_k^n.A_k\\&=Q_k.E_{k,n}\\&=E_{k,n+1}.\end{aligned}$$

Corollary 2.13. For $k=2$, we get the matrix representation of the usual Fibonacci quaternions in [17] as follows:

$$Q_2^n.A_2=\begin{bmatrix}1&1\\1&0\end{bmatrix}^n\begin{bmatrix}QF_1&QF_0\\QF_0&QF_{-1}\end{bmatrix}$$
$$=\begin{bmatrix}QF_{n+1}&QF_n\\QF_n&QF_{n-1}\end{bmatrix}=E_{2,n}$$

Corollary 2.14. For $k=3$, we get the matrix representation of the usual Tribonacci quaternions as follows:

$$Q_3^n.A_3=\begin{bmatrix}1&1&1\\1&0&0\\0&1&0\end{bmatrix}^n\begin{bmatrix}QT_2&QT_1&QT_0\\QT_1&QT_0&QT_{-1}\\QT_0&QT_{-1}&QT_{-2}\end{bmatrix}$$
$$=\begin{bmatrix}QT_{n+2}&QT_{n+1}&QT_n\\QT_{n+1}&QT_n&QT_{n-1}\\QT_n&QT_{n-1}&QT_{n-2}\end{bmatrix}=E_{3,n}$$

Theorem 2.15. Let for $n\geq 1$ be integer. Then

$$\begin{bmatrix}1&1&1&\cdots&1&1\\1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&\cdots&0&0\\0&0&\cdots&0&1&0\end{bmatrix}^n\begin{bmatrix}QF_{k-1}^{(k)}\\QF_{k-2}^{(k)}\\QF_{k-3}^{(k)}\\ \\QF_1^{(k)}\\QF_0^{(k)}\end{bmatrix}=\begin{bmatrix}QF_{n+k-1}^{(k)}\\QF_{n+k-2}^{(k)}\\QF_{n+k-3}^{(k)}\\ \\QF_{n+1}^{(k)}\\QF_n^{(k)}\end{bmatrix}$$

Theorem 2.16. For any positive integer m and n

$$QF_{n+m}^{(k)}=F_{n+1}^{(k)}QF_m^{(k)}+\sum_{j=0}^{k-2}\left(QF_{m-(k-j-1)}^{(k)}\sum_{p=0}^{j}F_{n-p}^{(k)}\right)$$

where $F_n^{(k)}$ is the nth $k-$order Fibonacci number.

Proof: For $k\geq 2$

$$Q_k=\begin{bmatrix}1&1&1&\cdots&1&1\\1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&\cdots&0&0\\0&0&\cdots&0&1&0\end{bmatrix}$$

In (1.1),

$$Q_k^n = \begin{bmatrix} F_{n+1}^{(k)} & \cdots & F_n^{(k)} + F_{n-1}^{(k)} + F_{n-2}^{(k)} & F_n^{(k)} + F_{n-1}^{(k)} & F_n^{(k)} \\ F_n^{(k)} & \cdots & F_{n-1}^{(k)} + F_{n-2}^{(k)} + F_{n-3}^{(k)} & F_{n+1}^{(k)} + F_n^{(k)} & F_{n-1}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ F_{n-k+3}^{(k)} & \cdots & F_{n-k+2}^{(k)} + F_{n-k+1}^{(k)} + F_{n-k}^{(k)} & F_{n-k+2}^{(k)} + F_{n-k+1}^{(k)} & F_{n-k+2}^{(k)} \\ F_{n-k+2}^{(k)} & \cdots & F_{n-k+1}^{(k)} + F_{n-k}^{(k)} + F_{n-k-1}^{(k)} & F_{n-k+3}^{(k)} + F_{n-k+2}^{(k)} & F_{n-k+1}^{(k)} \end{bmatrix}$$

If we use (2.4), we get as follows

$$Q_k^{n+m} = Q_k^n Q_k^m$$

and

$$Q_k^n A_k = E_{k,n}$$

Then we have

$$E_{k,n+m} = Q_k^{n+m} A_k = Q_k^n Q_k^m A_k = Q_k^n E_{k,m}$$

If the equality of matrices is used, we get for

$$\begin{aligned} QF_{n+m}^{(k)} = F_n^{(k)} QF_{m+1-k}^{(k)} &+ \left(F_n^{(k)} + F_{n-1}^{(k)}\right) QF_{m+2-k}^{(k)} \\ &+ \left(F_n^{(k)} + F_{n-1}^{(k)} + F_{n-2}^{(k)}\right) QF_{m+3-k}^{(k)} + \ldots \\ &+ \left(F_n^{(k)} + F_{n-1}^{(k)} + \ldots + F_{n+2-k}^{(k)}\right) QF_{m-1}^{(k)} + F_{n+1}^{(k)} QF_m^{(k)} \end{aligned}$$

and the proof is complete.

Corollary 2.17: For $k = 2$, then

$$QF_{n+m} = F_{n+1} QF_m + QF_{m-1} F_n$$

where F_n is nth Fibonacci number.

Corollary 2.18. For $k = 3$, then

$$QT_{n+m} = T_{n+1} QT_m + T_n QT_{m-2} + \left(T_n + T_{n-1}\right) QT_{m-1}$$

where T_n is nth Tribonacci number.

3. CONCLUSIONS

In this paper we defined and studied another interesting generalization of Fibonacci quaternions are called $k-$order Fibonacci quaternions. Then we obtained for $k = 2$ Fiboncci quaternions, for $k = 3$ Tribonacci quaternions and for $k = 4$ Tetranacci quaternions. We gave generating function, the summation formula and some properties about $k-$order Fibonacci quaternions. Also, we identified and proved the matrix representation for $k-$order Fibonacci quaternions. The Q_k matrix given for $k-$order Fibonacci numbers was defined for $k-$order Fibonacci quaternions and after the matrices with the $k-$order Fibonacci quaternions were obtained with help of auxiliary matrices, important relationships and identities were established.

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