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On complex Leonardo numbers

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Abstract: In this study, we introduce the complex Leonardo numbers and give some of their properties including Binet formula, generating function, Cassini and d'Ocagne's identities. Also, we calculate summation formulas for complex Leonardo numbers involving complex Fibonacci and Lucas numbers.
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1 Introduction

Leonardo numbers and their some properties studied by Catarino and Borges in [5]. Also, the authors give Binet formula for Leonardo numbers and study their relationships with Fibonacci and Lucas numbers. The Leonardo sequence is very intertwined with Fibonacci and Lucas numbers, for example, one can see this bond in their generating functions. Firstly, let us give definitions for these sequences [10, 11]. Fibonacci numbers are defined as

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \tag{1}$$

where $F_0 = 0$ and $F_1 = 1$. Lucas numbers are defined as

$$L_{n+2} = L_{n+1} + L_n, \quad n \geq 0, \tag{2}$$

where $L_0 = 2$ and $L_1 = 1$. Secondly, we need to give golden (α) and silver (β) ratios which are defined as

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}. \tag{3}$$

Apart from theoretical sciences, these ratios are very popular and used in variety of different areas. Especially, architectures and industrial designers benefit from them, see [1,9]. These ratios are the roots of the characteristic equation of Fibonacci and Lucas sequences. These sequences have same characteristic equation, because of their recurrence relation. The mentioned equation is

$$C(x) = x^2 - x - 1. \tag{4}$$

Thirdly, we may recall the Binet formula which is used to get the n-th Fibonacci and the n-th Lucas number as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n. \tag{5}$$

Lastly, we mention about Fibonacci and Lucas numbers' generating functions. The generating function, is similar to the Binet formula, is used to obtain desired specific element of the sequence. Fibonacci and Lucas numbers have the same generating function with the exception of initial values.

$$F(t) = \frac{F_0 + (F_1 - F_0)t}{1 - t - t^2} \quad \text{and} \quad L(t) = \frac{L_0 + (L_1 - L_0)t}{1 - t - t^2}. \tag{6}$$

Fibonacci and Lucas numbers are studied deeply even in different algebraic structures, for instance see [4, 6–8]. As in the reference [7], Horadam defined the complex Fibonacci and complex Lucas sequences as

$$CF_{n+2} = CF_{n+1} + CF_n \quad \text{and} \quad CL_{n+2} = CL_{n+1} + CL_n \tag{7}$$

where $CF_n = F_n + iF_{n+1}$ and $CL_n = L_n + iL_{n+1}$ with $n \geq 0$.

In the next section, we give the definition of Leonardo numbers. Also, we give some connections that Leonardo numbers have with Fibonacci and Lucas numbers.

2 Leonardo numbers

Leonardo numbers are an integer sequence with following recurrence relation

$$Le_{n+2} = Le_{n+1} + Le_n + 1, \; n \geq 2 \tag{8}$$

where Le_n denote the n-th Leonardo number with initial values $Le_0 = Le_1 = 1$. Note that, every Leonardo number is odd because of the initial values and recurrence relation.

In order to show relations of the Leonardo numbers with Fibonacci and Lucas numbers, we state following proposition from [5] without proof.

Proposition 2.1. *For $n \geq 0$, the following statements hold true,*

$$Le_n = 2F_{n+1} - 1, \tag{9}$$

$$Le_n = 2\left(\frac{L_n + L_{n+2}}{5}\right) - 1, \tag{10}$$

$$Le_{n+3} = \frac{L_{n+1} + L_{n+7}}{5} - 1, \tag{11}$$

$$Le_n = L_{n+2} - F_{n+2} - 1, \tag{12}$$

where F_n and L_n are the n-th Fibonacci and the n-th Lucas numbers, respectively.

Now, let us give the Binet formula for Leonardo numbers using Equation (12).

Proposition 2.2. *For $n \geq 0$ the Binet formula for Leonardo numbers is*

$$Le_n = \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta}, \tag{13}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. Let us prove the claim using the Binet formulas of Fibonacci and Lucas numbers in Equation (12).

$$Le_n = L_{n+2} - F_{n+2} - 1$$

can be rewritten as

$$Le_n = (\alpha^{n+2} + \beta^{n+2}) - \left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right) - 1.$$

After necessary calculations, we get

$$Le_n = \frac{\alpha^{n+3} - \beta^{n+3} + \alpha\beta^{n+2} - \beta\alpha^{n+2} - \alpha + \beta - \alpha^{n+2} + \beta^{n+2}}{\alpha - \beta},$$

which is equal to

$$Le_n = \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta},$$

where we used the fact that both $\alpha^2 - \alpha\beta - \alpha$ and $\beta^2 - \alpha\beta - \beta$ are equal to 2. □

For more identities involving Leonardo numbers see [3]. Also other studies about Leonardo numbers can be listed as [2,12,13]. In the next section, we define the complex Leonardo numbers using the given background.

3 Complex Leonardo numbers

As we have given the definitions before with (7), the complex Fibonacci and Lucas numbers are defined by Horadam in [7]. Using (7) with authors' definition about Leonardo numbers in [5], we define complex Leonardo numbers as follows.

Definition 3.1. *For $n \geq 1$, n-th complex Leonardo numbers are defined by*

$$C_n = Le_n + iLe_{n+1}. \tag{14}$$

It is important to note that, we denote the n-th complex Leonardo number with C_n. Using the recurrence relation and definition of complex Leonardo numbers we get

$$C_n = (Le_{n-1} + Le_{n-2} + 1) + i(Le_n + Le_{n-1} + 1) = C_{n-1} + C_{n-2} + \varepsilon, \tag{15}$$

where $n \geq 2$ and $\varepsilon = 1 + i$.

Now let us start with the Binet formula for the complex Leonardo numbers.

Theorem 3.1. *For $n \geq 0$, the Binet formula for n-th complex Leonardo number C_n is*

$$C_n = \frac{2\underline{\alpha}\alpha^{n+1} - 2\underline{\beta}\beta^{n+1}}{\alpha - \beta} - \varepsilon. \tag{16}$$

where α and β are golden and silver ratios, $\underline{\alpha} = 1 + i\alpha$ and $\underline{\beta} = 1 + i\beta$.

Proof. Using definition of complex Leonardo numbers and the Binet of the sequence, we get

$$C_n = \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta} + i\left(\frac{\alpha(2\alpha^{n+1} - 1) - \beta(2\beta^{n+1} - 1)}{\alpha - \beta}\right).$$

Making necessary calculations we have

$$C_n = \frac{2\alpha^{n+1}(1 + i\alpha) - 2\beta^{n+1}(1 + i\beta) + (1 + i)(\beta - \alpha)}{\alpha - \beta},$$

which is equal to Equation (16) with $\underline{\alpha} = 1 + i\alpha$, $\underline{\beta} = 1 + i\beta$ and $\varepsilon = 1 + i$. □

Our Binet formula is compatible with earlier result in [2].

In the next theorem, we give generating function for complex Leonardo numbers.

Theorem 3.2. *The generating function for the complex Leonardo numbers is*

$$g(t) = \frac{C_0 - t(1 - i) + t^2(1 - i)}{1 - 2t + t^3}. \tag{17}$$

Proof. To prove this claim, we write $g(t)$ as

$$g(t) = C_0 t^0 + C_1 t^1 + C_2 t^2 + \cdots + C_n t^n + \ldots.$$

Now, calculating $2tg(t)$ and $-t^3 g(t)$ as

$$2tg(t) = \sum_{n=0}^{\infty} 2C_n t^{n+1} \quad \text{and} \quad -t^3 g(t) = -\sum_{n=0}^{\infty} C_n t^{n+3}.$$

If we use above terms, then

$$(1 - 2t + t^3)g(t) = C_0 - t(1 - i) + t^2(1 - i)$$

is obtained. After calculations we get the desired result

$$g(t) = \frac{C_0 - t(1 - i) + t^2(1 - i)}{1 - 2t + t^3}.$$

□

Binet formula and the generating function are two important equations which are used to obtain the desired element of the sequence. Therefore, these equations are always stated in the integer sequence studies. Another equation which is generally studied is the Cassini identity. In the next theorem, we give this famous identity.

Theorem 3.3. *Cassini identity for complex Leonardo numbers is*

$$C_n^2 - C_{n-1}C_{n+1} = -Le_{n-2} - Le_{n+2}Le_{n-1} - 8(-1)^{n+1} + i(Le_nLe_{n+1} - Le_{n-1}Le_{n+2}), \quad (18)$$

where Le_n is the n-th Leonardo number.

Proof. In order to prove the claim, we use definition of the n-th complex Leonardo number as

$$C_n = Le_n + iLe_{n+1}.$$

After explicitly stating the equation as

$$Le_n^2 - Le_{n-1}Le_{n+1} - (Le_{n+1}^2 - Le_nLe_{n+2}) + i(Le_nLe_{n+1} - Le_{n-1}Le_{n+2}).$$

We study real and imaginary parts separately and calculate the identity using the Binet formula for Leonardo numbers

$$\begin{aligned} LHS = &\Big(\frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta}\Big)^2 - \Big(\frac{\alpha(2\alpha^{n+1} - 1) - \beta(2\beta^{n+1} - 1)}{\alpha - \beta}\Big)^2 \\ &+ \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta}\frac{\alpha(2\alpha^{n+2} - 1) - \beta(2\beta^{n+2} - 1)}{\alpha - \beta} \\ &- \frac{\alpha(2\alpha^{n-1} - 1) - \beta(2\beta^{n-1} - 1)}{\alpha - \beta}\frac{\alpha(2\alpha^{n+1} - 1) - \beta(2\beta^{n+1} - 1)}{\alpha - \beta} \end{aligned}$$

for the real part. The imaginary part of the equation can be written using Binet formula

$$\begin{aligned} LHS = &\frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta}\frac{\alpha(2\alpha^{n+1} - 1) - \beta(2\beta^{n+1} - 1)}{\alpha - \beta} \\ &- \frac{\alpha(2\alpha^{n-1} - 1) - \beta(2\beta^{n-1} - 1)}{\alpha - \beta}\frac{\alpha(2\alpha^{n+2} - 1) - \beta(2\beta^{n+2} - 1)}{\alpha - \beta}. \end{aligned}$$

Making necessary calculations and using Leonardo number identities we get the

$$C_n^2 - C_{n-1}C_{n+1} = -Le_{n-2} - Le_{n+2}Le_{n-1} - 8(-1)^{n+1} + i(Le_nLe_{n+1} - Le_{n-1}Le_{n+2}),$$

the desired result. □

Another well-known identity for sequences is d'Ocagne's identity. Now, we give it for complex Leonardo numbers.

Theorem 3.4. *D'Ocagne's identity for complex Leonardo numbers is*

$$\begin{aligned} C_mC_{n+1} - C_{m+1}C_n = &\,2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} - Le_{n-1} \\ &- i(2(-1)^n(Le_{m-n} + 1) + Le_m - Le_n), \end{aligned} \quad (19)$$

where $m > n$ and $n > 0$.

Proof. For this proof we can utilize the Leonardo number identities instead of Binet formula. Firstly, we write the identity explicitly.

$$\begin{aligned} C_mC_{n+1} - C_{m+1}C_n = (Le_m + iLe_{m+1})(Le_{n+1} + iLe_{n+2}) \\ - (Le_{m+1} + iLe_{m+2})(Le_n + iLe_{n+1}). \end{aligned}$$

Secondly, making multiplications and using properties of complex Leonardo numbers we get following equation.

$$\begin{aligned} C_mC_{n+1} - C_{m+1}C_n = Le_mLe_{n+1} - Le_{m+1}Le_n) \\ - i(Le_mLe_{n+2} - Le_{m+2}Le_n). \end{aligned}$$

Finally, using d'Ocagne's identity for Leonardo numbers we can get

$$\begin{aligned} C_mC_{n+1} - C_{m+1}C_n = 2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} - Le_{n-1} \\ - i(2(-1)^n(Le_{m-n} + 1) + Le_m - Le_n), \end{aligned}$$

which completes the proof. □

In the next theorem, we state some summation formulas.

Theorem 3.5. *For positive integer n, we have following summation formulas for complex Leonardo numbers.*

$$\sum_{j=0}^{n} C_j = C_{n+2} - (n+2)\varepsilon - 2i. \tag{20}$$

$$\sum_{j=0}^{n} C_{2j} = C_{2n+1} - n\varepsilon - 2i. \tag{21}$$

$$\sum_{j=0}^{n} C_{2j+1} = C_{2n+2} - (n+2)\varepsilon. \tag{22}$$

Proof. Let us prove the Equation (21). Other equations may be proven accordingly. Using the definition of complex Leonardo numbers we can restate the formula as

$$\sum_{j=0}^{n} C_{2j+1} = \sum_{j=0}^{n} Le_{2j+1} + iLe_{2j+2}.$$

Applying the summation formulas for Leonardo numbers, we can rearrange the equation as

$$\sum_{j=0}^{n} C_{2j+1} = Le_{2j+2} - (n+2) + iLe_{2n+3} - i(n+2),$$

which completes the proof. □

In the next theorem, we use the complex Fibonacci CF_n, Lucas CL_n and Leonardo C_n numbers for summation formulas.

Theorem 3.6. *For $n \geq 0$, we have following summation equations*

$$\sum_{j=0}^{n} CF_j + C_j = CF_{n+1} + C_{n+2} - (n+3)\varepsilon - 2i \tag{23}$$

and

$$\sum_{j=0}^{n} CL_j + C_j = CL_{n+1} + C_{n+2} - (n+3)\varepsilon - 4i, \tag{24}$$

where CF_j and CL_j are the j-th complex Fibonacci and complex Lucas number, respectively.

Proof. Let us prove only the Equation (23). For non-zero integer n,

$$\sum_{j=0}^{n} CL_j + C_j = \sum_{j=0}^{n} CL_j + \sum_{j=0}^{n} C_j.$$
$$\sum_{j=0}^{n} CL_j + C_j = \sum_{j=0}^{n} L_j + iL_{j+1} + \sum_{j=0}^{n} Le_j + iLe_{j+1}.$$

Using summation formulas in [5], we get following explicit form

$$\sum_{j=0}^{n} CL_j + C_j = CL_{n+1} + C_{n+2} - (n+3)\varepsilon - 4i,$$

which ends the proof. □

4 Conclusion

In this paper, the complex Leonardo numbers are introduced, including Binet formula, generating function, Cassini and d'Ocagne's identities. Furthermore, summation formulas involving complex Fibonacci and Lucas numbers are presented. More identities can be obtained in the future studies.

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