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Research Article

Constructing the Fuzzy Hyperbola and Its Applications in Analytical Fuzzy Plane Geometry

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In this paper, we studied about a detailed analysis of fuzzy hyperbola. In the previous studies, some methods for fuzzy parabola are discussed (Ghosh and Chakraborty, 2019). To define the fuzzy hyperbola, it is necessary to modify the method applied for the fuzzy parabola. To obtain a conic, it is necessary to know at least five points on this curve. First of all, in this study, we examined how to detect these five fuzzy points. We have discussed in detail the impact of points in this examination on finding fuzzy membership degrees and determining the curve. We show the use of the algorithm for calculating the coefficients in the conic equation on the examples. We make detailed drawings of all the fuzzy hyperbolas found and depicted the geometric location of fuzzy points with different membership degrees on the graph. As can be seen from the figures in our study, the importance of membership degrees in fuzzy space is that it causes us to find different numbers of hyperbola curves for the five points we study with. In addition, finding the membership of a given point to the fuzzy hyperbola is possible by solving nonlinear equations under different angular approaches. This examination is shown in detail in this study, and the results in the examples are evaluated by geometric comments. The systems formed by the fuzzy hyperbola curves are found to have different areas of use, as presented in the Conclusion. Some of usage areas of fuzzy hyperbola are radar systems, scanning devices, photosynthesis, heat, and CO_2 distribution of plants.

1. Introduction

In the case of crisp sets, a given object x may belong to a set A or not belong to this set, and these two options are denoted by $x \in A$ or $x \notin A$, A classic set may be described by the characteristic function (χ_A) that takes two values: first is 1 (x belongs to a set A), and second is 0 (x does not belong to set A)

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{1}$$

Fuzzy sets are introduced and described using membership functions by Zadeh in 1965 [1]. As opposed to a crisp set, if $\overline{A}$ is a fuzzy set, we write its membership function as $\mu(x|\overline{A})$, and $\mu(x|\overline{A})$ is in $[0,1]$ for all x.

Some basic definitions of fuzzy logic are given by Zadeh in study [2]. Likewise, basic properties for fuzzy plane geometry have been given by Buckley and Eslami in the study [3] may be the first to analyze fuzzy sets.

Fuzzy points and the fuzzy distance between fuzzy points were defined by Buckley and Eslami in [3]. And they showed it is a (weak) fuzzy metric and fuzzy point, fuzzy line segment, fuzzy distance, and the angle between two fuzzy segments, and same and inverse points are defined by Ghosh and Chakraborty [4].

Gebray and Reddy dealt with properties of fuzzy set, fuzzy metric, fuzzy field, fuzzy set field, and fuzzy magnitude [5].

A fuzzy line passing through several fuzzy points whose cores are collinear is reviewed by Ghosh and Chakraborty in the study [6].

It has been studied about geometric reasoning with extended objects and transferred to geography [7].

Rosenfeld presented fuzzy geometry and fuzzy topology of image subsets [8].

The fuzzy triangle as the intersection of three fuzzy half-planes and its geometric forms was discussed by Rosenfeld in the study [9].

Zimmermann dealt with types of fuzzy sets, fuzzy measures, fuzzy functions, and applications of fuzzy set theory and gave basic definitions and theorems about fuzzy sets [10].

Ashraf et al. approached the ordering of the fuzzy numbers in the decision and fuzzy distance [11].

It is known that in analytical geometry to get a conic also a hyperbola, we need five points. In this perspective, Ghosh and Chakraborty introduced a fuzzy parabola in the study [12].

When all these studies are examined, it is seen that only fuzzy circle and fuzzy parabola curves are studied from the conics. No study has got been done to construct a fuzzy hyperbola. Fuzzy systems are used in the planning of technological structures developing in the field of engineering nowadays. On the other hand, hyperbola curves are very common curves in the wave distribution of radars, in the most suitable matching of points in scanning machines, and even in the carbon dioxide distribution of plants. While we aimed to analyze how the fuzzy hyperbola could be defined, calculated, and graphed mathematically, we thought that it would be useful to work on combining the applications mentioned above. We examine and prove these calculations with the evaluation of previous studies.

2. Preliminaries

In this section, we will mention the needed mathematical specialties for fuzzy plane geometry.

We will draw "a bar" over capital letters to denote a fuzzy subset of R^n, *i.e* $\overline{A}, \overline{B}, \overline{X}, \overline{Y}, \ldots$, and we will write membership of fuzzy set $\overline{A}$ as $\mu(x|\overline{A}), x \in R^n$ and $\mu(R^n)$ is in $[0,1]$.

Definition 1. (Fuzzy Set) Let X be a nonempty set and $\mu\colon X \longrightarrow [0,1]$ be a mapping. $\overline{A} = \{(\alpha, \mu(\alpha))\colon \alpha \in X\}$ is called a fuzzy set in X with membership function μ [5].

Definition 2. For a fuzzy set $\overline{A}$ of R^n, its $\alpha-$ cut is denoted by $\overline{A}(\alpha)$, and it is defined by

$$\overline{A}(\alpha) = \begin{cases} \{x, \mu(x|\overline{A}) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \text{Clouse}\{x, \mu(x|\overline{A}) > 0\}, & \text{if } \alpha = 0. \end{cases} \tag{2}$$

The support of the fuzzy set $\overline{A}$ is defined as positive membership degree namely $\mu(x|\overline{A}) > 0$. These points that have been positive membership degrees also denoted by $\overline{A}(0) = \{x, \mu(x|\overline{A}) > 0\}$, and additionally, the set $\overline{A}(0)$ defined as the base of fuzzy set $\overline{A}$. On the other hand, the core of fuzzy set $\overline{A}$ is denoted by $\{x, \mu(x|\overline{A}) = 1\}$. When the core of fuzzy set is nonempty and $\alpha-$cuts are convex, it can be said that the fuzzy set is convex and normal [4].

Definition 3. (Fuzzy points): A fuzzy point at (a,b) in R^2, written as $\overline{P}(a,b)$ is defined by its membership function:

(i) $\mu((x,y)|\overline{P}(a,b))$ is upper semicontinuous

(ii) $\mu((x,y)|\overline{P}(a,b)) = 1$ if and only if $(x,y) = (a,b)$

(iii) $\overline{P}(a,b)(\alpha)$ is a compact, convex subset of R^2 of all α in $[0,1]$

The notations $\overline{P}_1(a,b), \overline{P}_2(a,b), \overline{P}_3(a,b), \ldots$ or $\overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots$ are used to represent fuzzy points [3].

Definition 4. (Same points with respect to fuzzy points): Let take two points (x_1, y_1) and (x_2, y_2). Such that (x_1, y_1) is support of fuzzy point $\overline{P}(a,b)$ and similarly (x_2, y_2) is support of fuzzy point $\overline{P}(c,d)$. Let L_1 is a line joining (x_1, y_1) and (a,b). There exists a fuzzy number $\overline{r}_1$ for the fuzzy point $\overline{P}(a,b)$, along the line L_1. The membership function for $\overline{r}_1$ can be written as follows:

$$\mu((x,y)|\overline{r}_1) = \begin{cases} \mu((x,y)|\overline{P}(a,b)), & \text{for } (x,y) \text{ in } L_1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Similarly, along a line, L_2 say, joining (x_2, y_2) and (c,d), there exists a fuzzy number, $\overline{r}_2$ say, on the support of $\overline{P}(c,d)$. Now, (x_1, y_1), (x_2, y_2) are said to be same points with respect to $\overline{P}(a,b)$ and $\overline{P}(c,d)$ if

(i) (x_1, y_1) and (x_2, y_2) are same points with respect to $\overline{r}_1$ and $\overline{r}_2$

(ii) L_1, L_2 have equal angle with line joining (a,b) and (c,d) [4]

3. Fuzzy Hyperbola

In this section, we will develop a method for obtaining a fuzzy hyperbola. As it is known, the general conic equation has the following form:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{4}$$

If we divide both sides of this equation by $a\,(a \neq 0)$, this equation takes the following form:

$$x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0. \tag{5}$$

Thus, the number of unknown coefficients, in the equation containing six terms, becomes five. Then, the common solution of the five equations, will be obtained by substituting five different pairs for x and y, will be sufficient to find these unknowns. So, five points in the plane are enough to write a conic equation. Namely, five points on the plane will denote a single conic.

In this study, since we will define a hyperbola in fuzzy plane geometry, first of all, these five points must be points in the fuzzy space that ensure the necessary properties. It will also be seen that the hyperbola in fuzzy space is formed by different crisp hyperbolic curves. Their combination will form the fuzzy hyperbola. The hyperbola curve passing through the core of five fuzzy points will be called a crisp hyperbola and is denoted by CH. However, since these points are fuzzy points, their membership degrees may change. Differences in membership degrees affect the drawing of the resulting hyperbola curves. Therefore, we will calculate five different coefficients in the conic equation (4). Calculation of these coefficients is possible with five determinants. For this reason, five different curves emerge for the fuzzy hyperbola that we want to reach in our study. Therefore, in terms of the importance of the fuzzy membership degree, the definite hyperbola CH with membership degree one is taken. The other four curves are hyperbola, and the combination of all of them gives the fuzzy hyperbola and is denoted by FH. The system formed by these curves can also be considered as a curvilinear system or distribution in mathematical applications.

Now, we will denote a method to create a fuzzy hyperbola in a fuzzy plane by taking five fuzzy points. These points will be the same points which we gave in Definition 4 in the preliminaries section.

Necessary explanations and proofs are presented below.

Now, we get five fuzzy points whose cores lie on a crisp hyperbola CH, and these points are denoted by $\overline{H}_i(a_i, b_i)$, $i = 1, 2, \ldots, 5$. First, to create a fuzzy hyperbola that passing through these points, we construct a fuzzy hyperbolic segment, and we will denote as $\overline{FH}_{1,\ldots,5}$ briefly, joining $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_5$. Subsequently, we extend the four sides of the hyperbolic segment $\overline{FH}_{1,\ldots,5}$ to infinity to construct the fuzzy hyperbola. We will denote fuzzy hyperbola as $\overline{FH}$, briefly. While constructing conic equations in fuzzy space, it is sufficient to examine only finite segments in closed curves such as ellipses and circles. But for curves with points at infinity, such as parabola and hyperbola, it is not sufficient to study only with finite segments. For this reason, based on the method presented for finite segments, semi-infinite segments are defined with the idea that it can start from a point and extend to infinity, that is, the image set must have a point at infinity. First of all, a finite segment is created by selecting at least two points on each hyperbola curve as shown in Figure 1. Then, as proved in Proposition 1, the semi-infinite segments are developed so that they remain parallel to the crisp hyperbola. Thus, as a result, fuzzy hyperbola $\overline{FH}$ is obtained. We will express this more clearly below.

We get four fuzzy points with a core on CH which are an infinitely far from the side of $\overline{H}_1(a_1, b_1)$, $\overline{H}_3(a_3, b_3)$, $\overline{H}_4(a_4, b_4)$, and $\overline{H}_5(a_5, b_5)$. Let these four fuzzy points be $\overline{H}_{1\infty}, \overline{H}_{3\infty}, \overline{H}_{4\infty}$, and $\overline{H}_{5\infty}$, respectively, and $\overline{FH}_{1\infty}$, $\overline{FH}_{3\infty}, \overline{FH}_{4\infty}$, and $\overline{FH}_{5\infty}$ be the semi-infinite fuzzy hyperbolic segments combining $\overline{H}_1$ with $\overline{H}_{1\infty}, \overline{H}_3$ with $\overline{H}_{3\infty}, \overline{H}_4$ with $\overline{H}_{4\infty}$, and $\overline{H}_5$ with $\overline{H}_{5\infty}$. Thereafter, we can define the fuzzy hyperbola $\overline{FH}$ with the combinations of the hyperbolic segment $\overline{FH}_{1,\ldots,5}$ and the semi-infinite segments,

$$\overline{FH} = \overline{FH}_{1\infty} \cup \overline{FH}_{3\infty} \cup \overline{FH}_{1,\ldots,5} \cup \overline{FH}_{4\infty} \cup \overline{FH}_{5\infty}. \tag{6}$$

3.1. Construction of $\overline{FH}_{1,\ldots,5}$. In this section, we construct the segment $\overline{FH}_{1,\ldots,5}$ for the fuzzy hyperbola. This segment is defined as

$$\overline{FH}_{1,\ldots,5} = \bigvee_{\alpha\in[0,1]} \left\{ \begin{array}{c} FH_\alpha\text{: Where } FH_\alpha \text{ is a crisp hyperbola that passes} \\ \text{through five same points an } \overline{H}_i(a_i, b_i), \\ i = 1, 2, \ldots, 5 \text{ with membership degree } \alpha \end{array} \right\}. \tag{7}$$

By the aid of the membership function, the hyperbolic segment $\overline{FH}_{1,\ldots,5}$ can be defined by

$$\mu\left((x, y)|\overline{FH}_{1,\ldots,5}\right) = \sup \left\{ \begin{array}{c} \alpha\text{: Where } (x, y) \text{ lies on } FH_\alpha \text{ that} \\ \text{passes through five same points on} \\ \overline{H}_i, i = 1, 2, \ldots, 5 \text{ with membership degree } \alpha \end{array} \right\}. \tag{8}$$

This definition shows that the fuzzy hyperbolic-segment $\overline{FH}_{1,\ldots,5}$ is a collection of crisp points with different degrees of membership. The fuzzy hyperbola is the union of all crisp hyperbolas passing through five same points on the supports of $\overline{H}_i, i = 1, 2, \ldots, 5$.

3.2. Construction of $\overline{FH}_{1\infty}, \overline{FH}_{3\infty}$ and $\overline{FH}_{4\infty}, \overline{FH}_{5\infty}$. In this section, we create semi-infinitive fuzzy hyperbola segment $\overline{FH}_{1\infty}, \overline{FH}_{3\infty}, \overline{FH}_{4\infty}$, and $\overline{FH}_{5\infty}$. But for this, fuzzy points $\overline{H}_{1\infty}$, $\overline{H}_{3\infty}, \overline{H}_{4\infty}$, and $\overline{H}_{5\infty}$ must be obtained first. We present these fuzzy points in the following Proposition 1, and also, the meaning of the phrase "infinitely far" is given. The shape and position of these fuzzy points $\overline{H}_{i\infty}$, $(i = 1, 3, 4, 5)$ are not important for the construction of semi-infinity fuzzy segments $\overline{FH}_{i\infty}$, $(i = 1, 3, 4, 5)$. There are two things to note here as follows:

(i) The support of fuzzy points $\overline{H}_{i\infty}$, $(i = 1, 3, 4, 5)$ should be taken as a compact set

(ii) The cores of fuzzy points $\overline{H}_{i\infty}$, $(i = 1, 3, 4, 5)$ should lie on the core hyperbola CH

Fuzzy hyperbola segments $\overline{FH}_{i\infty}$, $(i = 1, 3, 4, 5)$ can be formed by using fuzzy same points $\overline{H}_{i\infty}$, $(i = 1, 3, 4, 5)$ on the hyperbola, and their fuzzy points extended to infinity. The important detail here is connecting the same points of $\overline{H}_1$ with $\overline{H}_{1\infty}$, $\overline{H}_3$ with $\overline{H}_{3\infty}$, $\overline{H}_4$ with $\overline{H}_{4\infty}$, $\overline{H}_5$ with $\overline{H}_{5\infty}$ segments with these connecting same points give us the fuzzy hyperbola. Another feature we will draw attention to in

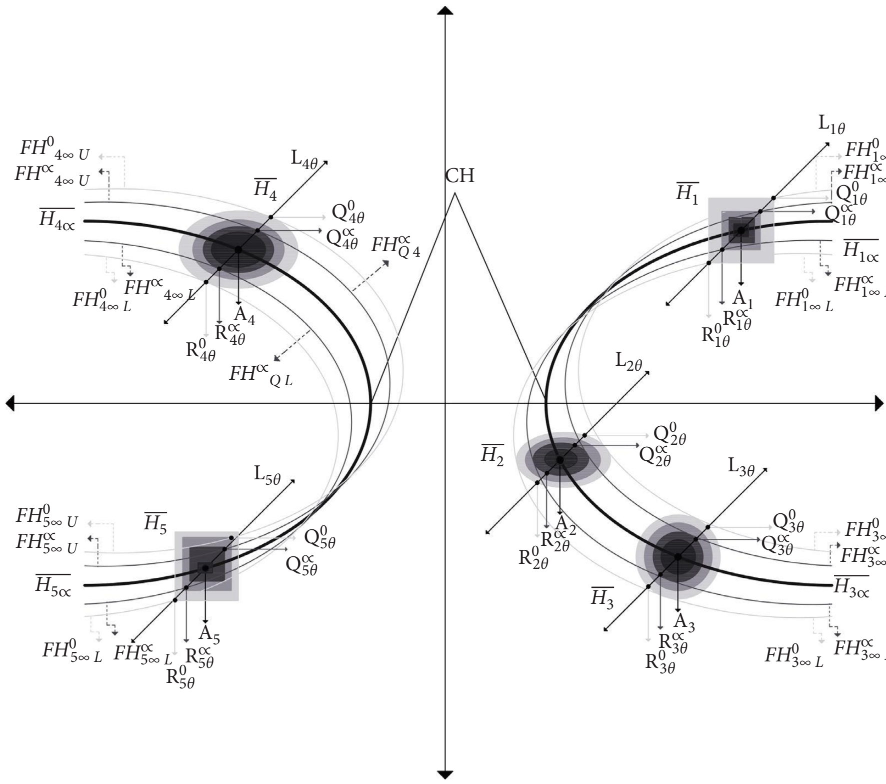

Figure 1: Construction of fuzzy hyperbola in the method.

Proposition 1 is that these fuzzy hyperbola segments should be parallel to the core hyperbola $\overline{FH}(1)$. We will show later that a core fuzzy hyperbola $\overline{FH}(1)$ and a crisp fuzzy hyperbola CH are equivalent. We can say the membership degrees of the hyperbolic segments used here will be different. According to this, semi-infinity crisp hyperbolic segments should be parallel to the CH.

Proposition 1. *A fuzzy hyperbola $\overline{FH}$ formed by the determined fuzzy points, while the semi-infinite crisp hyperbolic segments in $\overline{FH}_{i\infty}$, $(i = 1, 3, 4, 5)$ must be parallel to the core hyperbola $\overline{FH}(1)$.*

Proof. Let us prove with $FH_{1\infty}$ that any of the semi-infinite crisp hyperbolic segments. Suppose that there exists a semi-infinite crisp hyperbolic segment $FH_{1\infty}$ that is not parallel to $\overline{FH}(1)$.

As per to the $\overline{FH}_{1\infty}(0)$, it is union of the semi-infinite hyperbolic segments combining same points of the fuzzy points $\overline{H}_1$ and $\overline{H}_{1\infty}$. Hence, corresponding to $FH_{1\infty}$, there must be two same points $(x_1, y_1) \in \overline{H}_1(0)$ and $(x_{1\infty}, y_{1\infty}) \in \overline{H}_{1\infty}(0)$ which are two extremities of $FH_{1\infty}$.

Because $FH_{1\infty}$ is not parallel to $\overline{FH}(1)$, there are two cases. First case is the intersection $FH_{1\infty}$ with $\overline{FH}(1)$, and the second case is the distance between the point $(x_{1\infty}, y_{1\infty})$ and the semi-infinite hyperbolic segment $FH_{1\infty}$ must be infinitely wide.

The first case evidently clearly implies that $(x_1, y_1) \in \overline{H}_1(0)$ and $(x_{1\infty}, y_{1\infty}) \in \overline{H}_{1\infty}(0)$ are not same points because, in this case, (x_1, y_1) and $(x_{1\infty}, y_{1\infty})$ lie on two different sides of $\overline{FH}(1)$. Therefore, it lies on two different sides of the line combining core points of $\overline{H}_1$ and $\overline{H}_{1\infty}$.

The second case implies that $\overline{H}_{1\infty}(0)$ which support of $\overline{H}_{1\infty}$ is unlimited, and therefore, $\overline{H}_{1\infty}$ cannot be a fuzzy point.

Therefore, it is impossible in both cases. So, a contradiction occurs.

Then, $FH_{1\infty}$ must be parallel to $\overline{FH}(1)$. Hence, whole the semi-infinite hyperbolic segments in support of $\overline{FH}_{1\infty}$ must be parallel to core hyperbola $\overline{FH}(1)$. As with $\overline{FH}_{1\infty}(0)$, supports of $\overline{FH}_{3\infty}, \overline{FH}_{4\infty}$, and $\overline{FH}_{5\infty}$ must be parallel to core hyperbola $\overline{FH}(1)$. We can now write $\overline{FH}_{1\infty}, \overline{FH}_{3\infty}, \overline{FH}_{4\infty}$, and $\overline{FH}_{5\infty}$ using the following formulations:

$$\overline{FH}_{1\infty} = \vee\left\{\begin{array}{c} FH_{1\infty}(x,y)\text{: where } (x,y)\in\overline{H}_1(0) \text{ and } FH_{1\infty}(x,y) \text{ is a semi} \\ \text{infinite hyperbolic segment parallel to } \overline{FH}(1) \text{ with membership} \\ \text{degree } (x,y) \text{ on } \overline{H}_1 \end{array}\right\},$$

$$\overline{FH}_{3\infty} = \vee\left\{\begin{array}{c} FH_{3\infty}(x,y)\text{: where } (x,y)\in\overline{H}_3(0) \text{ and } FH_{3\infty}(x,y) \text{ is a semi} \\ \text{infinite hyperbolic segment parallel to } \overline{FH}(1) \text{ with membership} \\ \text{degree } (x,y) \text{ on } \overline{H}_3 \end{array}\right\},$$

$$\overline{FH}_{4\infty} = \vee\left\{\begin{array}{c} FH_{4\infty}(x,y)\text{: where } (x,y)\in\overline{H}_4(0) \text{ and } FH_{4\infty}(x,y) \text{ is a semi} \\ \text{infinite hyperbolic segment parallel to } \overline{FH}(1) \text{ with membership} \\ \text{degree } (x,y) \text{ on } \overline{H}_4 \end{array}\right\}, \tag{9}$$

$$\overline{FH}_{5\infty} = \vee\left\{\begin{array}{c} FH_{5\infty}(x,y)\text{: where } (x,y)\in\overline{H}_5(0) \text{ and } FH_{5\infty}(x,y) \text{ is a semi} \\ \text{infinite hyperbolic segment parallel to } \overline{FH}(1) \text{ with membership} \\ \text{degree } (x,y) \text{ on } \overline{H}_5 \end{array}\right\}.$$

Let now depict the fuzzy hyperbola in detail with the membership degrees of the given fuzzy points by making geometric comments.

In Figure 1, $\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4$, and $\overline{H}_5$ are five fuzzy points. The regions inside the square with center at A_1, ellipse with center A_2, circle with center A_3, ellipse with center at A_4, and square with center at A_5 are the supports of the points $\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4$, and $\overline{H}_5$, respectively. The gray color transitions that inside the supports of the fuzzy points indicate different α-cuts. That is, the intensity of the gray levels on the support of the fuzzy points expresses the change of membership degrees. The regions that depicted in dark gray in the graph are formed by points with high membership degrees. Light gray regions on the graph are obtained as the membership degrees approach 0. Namely, the membership degrees of the centers of circle, squares, and ellipses are "1;" then, it decreases gradually to "0" on the periphery of the support of $\overline{H}_i$ for each $i = 1, 2, 3, 4, 5$.

In Figure 1, $L_{i\theta}$'s are five lines that passes through A_i, $(i = 1, 2, 3, 4, 5)$. The $L_{i\theta}$'s here have an angle θ with the positive x-axis. Since $\overline{H}_i(A_i)(\alpha)$, the α-cut of a fuzzy point is convex, and A_i is an interior point of $\overline{H}_i(A_i)(\alpha)$, and the line $L_{i\theta}$ must intersect with the boundary of $\overline{H}_i(A_i)(\alpha)$ at two points which we will call $Q^\alpha_{i\theta}$ and $R^\alpha_{i\theta}$. Each $Q^\alpha_{1\theta}, Q^\alpha_{2\theta}, Q^\alpha_{3\theta}, Q^\alpha_{4\theta}$, and $Q^\alpha_{5\theta}$ with respect to the angles θ form a set of five same points with membership degree α. Likewise, $R^\alpha_{1\theta}, R^\alpha_{2\theta}$, $R^\alpha_{3\theta}, R^\alpha_{4\theta}$, and $R^\alpha_{5\theta}$ are the set of five same points with the same membership degree α.

Let $FH^\alpha_{\theta U}$ is the hyperbola passing through $Q^\alpha_{i\theta}$'s and $FH^\alpha_{\theta L}$ be the hyperbola passing through the five points $R^\alpha_{i\theta}$'s in Figure 1, $i = 1, 2, 3, 4, 5$. Because of all the points, $Q^\alpha_{i\theta}$ and $R^\alpha_{i\theta}$ have "α" membership degree, and we have assigned a membership degree of α to the hyperbola $FH^\alpha_{\theta U}$ and $FH^\alpha_{\theta L}$ on $\overline{FH}$, $i = 1, 2, 3, 4, 5$.

To get various hyperbolas as $FH^\alpha_{\theta U}$ and $FH^\alpha_{\theta L}$, we will change the angle θ in $[0, 2\pi]$ and membership degree α in $[0, 1]$. By definition, the fuzzy hyperbolic segment $\overline{FH}_{1,\ldots,5}$ is the combination of whole the hyperbolas $FH^\alpha_{\theta U}$ and $FH^\alpha_{\theta L}$ with membership degree α.

Namely, we say

$$\overline{FH}_{1,\ldots,5} = \bigvee_{\substack{\theta\in[0,2\pi] \\ \alpha\in[0,1]}} \{FH^\alpha_{\theta U}, FH^\alpha_{\theta L}\}. \tag{10}$$

We take a hyperbola FH in $\overline{FH}(0)$. Let define the membership degree of on hyperbola FH in $\overline{FH}$ with

$$\mu(FH|\overline{FH}) = \min_{(x,y)\in FH} \mu((x,y)|\overline{FH}). \tag{11}$$

Let us show how to obtain the membership degree of FH which is any hyperbola in $\overline{FH}$ with Theorem 1. □

Theorem 1. *Suppose that FH be any hyperbola in $\overline{FH}$ and five same points $(x_i, y_i) \in \overline{H}_i(0)$ with $\mu((x_i, y_i)|\overline{FH}) = \alpha$ for all $i = 1, 2, 3, 4, 5$ exist such that FH is the hyperbola that passes through the five (x_i, y_i)'s and $\mu(FH|\overline{FH}) = \alpha$.*

Proof. We examine the proof in two different cases that (i) $\mu(FH|\overline{FH}) \nless \alpha$ and (ii) $\mu(FH|\overline{FH}) \ngtr \alpha$.

(i) By contrast, let assume that $\mu(FH|\overline{FH}) < \alpha$. In that case, by the definition of $\mu(FH|\overline{FH})$, there exist (x_0, y_0) in $\overline{FH}$ such that $(x_0, y_0) \in FH$ and $\mu((x_0, y_0)|\overline{FH}) < \alpha$. Let say $\mu((x_0, y_0)|\overline{FH}) = \beta$. Since $(x_0, y_0) \in FH$ and FH is a hyperbola which is combining the five same points, let the membership degree of these same points must be α.

$$\mu((x_0, y_0)\overline{FH}) = \sup\left\{\begin{array}{c}\psi\text{: where } (x, y) \text{ lies on the hyperbola} \\ \text{which is combining the five same points} \\ \text{with membership degree } \psi\end{array}\right\} \geq \alpha. \tag{12}$$

But this contradicts our acceptance $\beta < \alpha$. So, $\mu(FH|\overline{FH}) \nless \alpha$.

(ii) Since $\mu(FH|\overline{FH}) = \min\left\{\begin{array}{c}\alpha\text{: where } (x, y) \text{ lies on} \\ FH \text{ and } \mu(FH|\overline{FH}) = \alpha\end{array}\right\}$ and all the points (x_i, y_i), lie on FH, it is clear that $\mu(FH|\overline{FH}) \ngtr \alpha$, $i = 1, 2, 3, 4, 5$.

Therefore, $\mu(FH|\overline{FH}) = \alpha$ is obtained.

We depict the complete fuzzy hyperbola $\overline{FH}$ in Figure 2. The region between the curves f_0L and f_0U is the support of the $\overline{FH}$. CH is the core hyperbola $CH \equiv \overline{FH}(1)$ which passing through the five core points A_i of the fuzzy points $\overline{H}_i$.

Let's mention the line perpendicular to $CH \equiv \overline{FH}(1)$ that we take as the $C\ D$ line in Figure 2. Along the $C\ D$, there exists a LR type fuzzy number that we denoted by $(F/G/H)_{LR}$. If we explain LR type fuzzy number like this, L and R are reference functions L and R: $[0, +\infty) \longrightarrow [0, 1]$ that does not decrease and satisfies two conditions $L(x) = L(-x)$ and $L(0) = 1$. Where α and β are positive and $\overline{A}$ is a fuzzy number, $\mu(x|\overline{A})$ can be written as follows:

$$\mu(x|\overline{A}) = \begin{cases} L\left(\dfrac{m - x}{\alpha}\right), & \text{if } x \leq m, \\ \\ R\left(\dfrac{x - m}{\beta}\right), & \text{if } x \geq m. \end{cases} \tag{13}$$

The notation $(m - \alpha/m/m + \beta)_{LR}$ is used to represent an LR-type fuzzy number. That is, all fuzzy hyperbolas can be visualized as a three-dimensional figure (a subset of $(x, y) \times [0, 1]$) whose cross section across $\overline{FH}$ is a fuzzy number such as $(F/G/H)_{LR}$.

Let $(F/G/H)_{LR}$ is a fuzzy number on fuzzy hyperbola $\overline{FH}$ and take a convex region on $\overline{FH}(0)$ such that, except F and H, all points on the line segment $[FH]$ are inner of the convex region. When we take a fuzzy point $\overline{H}$ such that the membership function is $\mu((x,y)|\overline{H}=\mu((x,\ y)|(F/G/H)_{LR}$, if $(x, y) \in [FH]$, $\mu((x, y)|\overline{H}) \leq \mu((x, y)|\overline{FH})$. Only at G, $\mu((x, y)|\overline{H}) = 1$. Membership degree decreases gradually to "0" that approach F or H. □

3.3. Construction of Membership Function. To get the membership function first, we have to consider the middle hyperbolic segment $\overline{FH}_{1\ldots5}$ and the semi-infinite tails $\overline{FH}_{i\infty}$, $i = 1, 2, 3, 4, 5$. The membership degree $\mu((x, y)|\overline{FH}_{1,\ldots,5})$ might not always be simple to evaluate. Also, we consider the membership degree and the definition of fuzzy hyperbola in the following form:

$$\begin{aligned} \mu((x, y)|\overline{FH}_{1\ldots5}) &= \sup\left\{\begin{array}{c}\alpha\text{: where } (x, y) \text{ lies in a hyperbola that} \\ \text{passes through five same points in } \overline{H}_i, \\ i = 1, 2, 3, 4, 5 \text{ with membership degree } \alpha\end{array}\right\}, \\ \mu((x, y)|\overline{FH}_{i\infty}) &= \sup\left\{\begin{array}{c}\alpha\text{: where } (x, y) \text{ lies on a crisp hyperbola } FH_\alpha \text{ that} \\ \text{passes through a point on } \overline{H}_i(0) \\ \text{with membership degree } \alpha, i = 1, 2, 3, 4, 5\end{array}\right\}. \end{aligned} \tag{14}$$

Also, obtaining the closed form of the membership function of $\overline{FH}$ is a really difficult task. As in the definition of $\mu((x, y)|\overline{FH}_{1,\ldots,5})$, to obtain $\mu((x, y)|\overline{FH})$, we must get five same points with membership degree $\alpha \in [0, 1]$. Thereafter, all possible values of α are identified for which (x, y) lies on the hyperbola that joins five same points with membership degrees. To obtain α membership degree, the nonlinear equation systems that obtained by calculating $5 * 5$ determinants as can be seen in the next section are solved. From the solution of the equation, there may be real values between 0 and 1. The supremum of all these real α values is the membership degree of $\mu((x, y)|\overline{FH}_{1,\ldots,5})$. We refer the hyperbola whose membership degree is supremum.

Now, we obtain a systematic procedure to identify the membership degree of the point (x_0, y_0) in a fuzzy hyperbola $\overline{FH}$ which passes through five fuzzy points $\overline{H}_i$, $i = 1, 2, 3, 4, 5$

We show the expansion of the same points on $\overline{H}_i$'s as $(x^\alpha_{i\theta}, y^\alpha_{i\theta})$, $i = 1, 2, 3, 4, 5$ $(0 \leq \theta \leq 2\pi,\ \alpha \in [0, 1])$.

As a result, we will have to examine the existence of solution of nonlinear equations by giving various values to θ and determining the α membership degrees according to the angle θ.

Let the angle $\theta = \theta_0\ (0 \leq \theta \leq 2\pi)$ and S_{θ_0}'s are the sets of membership degrees that can be compatible with respect to the various angle θ_0.

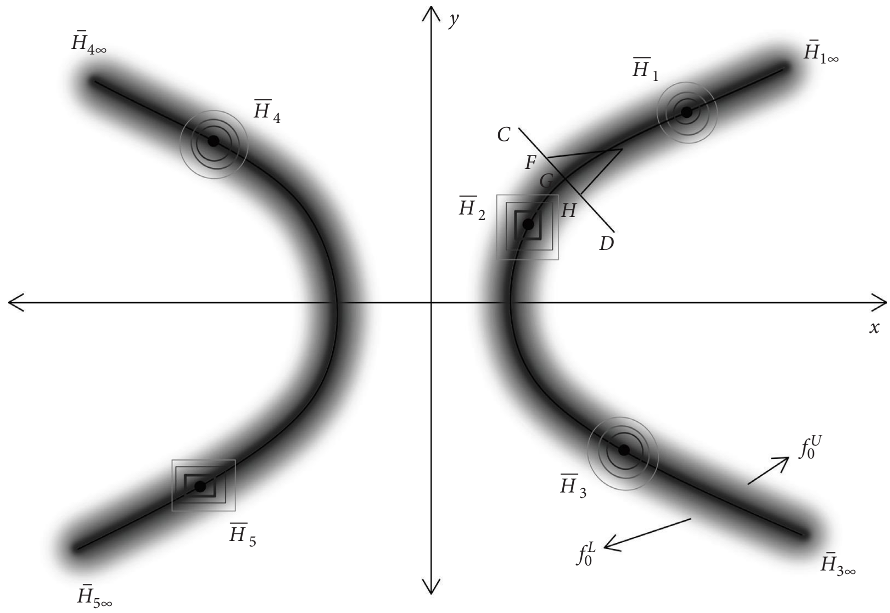

Figure 2: Fuzzy hyperbola (towards completing the fuzzy hyperbola in Figure 1).

We assume that the supremum of the set S_{θ_0} as s_{θ_0}. It can be seen from the given examples that nonlinear equation systems may not have a solution for some θ_0. Fuzzy hyperbola $\overline{FH}$ is obtained by determining and giving appropriate values.

Then, the membership degree of (x_0, y_0) in the $\overline{FH}$ fuzzy hyperbola is given by

$$\mu\left((x_0, y_0)|\overline{FH}\right) = \sup_{\theta} s_{\theta_0}. \tag{15}$$

The explanation of this part is given also in this section where the membership function is explained.

Let us give the application of the procedure with the following examples.

Example 1. Let $\overline{H}_1(1,0), \overline{H}_2(2, 3\sqrt{3}), \overline{H}_3(-3, 6\sqrt{2}), \overline{H}_4(-4/3, -\sqrt{7})$, and $\overline{H}_5(5/2, -3\sqrt{21}/2)$ be five fuzzy points. Let us get the fuzzy hyperbola that passes through these points.

The equation of the core hyperbola through the points is

$$\left\{(x, y)\colon\ x^2 - \frac{y^2}{9} = 1\right\}. \tag{16}$$

In this example, we take care of that the points are in different regions on the curve. The membership functions of these selected five fuzzy points are cones with circular and elliptical bases similar to the display of the points in Figure 1:

$$\begin{aligned}
&\left\{(x, y)\colon\ (x-1)^2 + y^2 \le 1\right\} \text{(circular)},\\
&\left\{(x, y)\colon\ (x-2)^2 + 4\left(y - 3\sqrt{3}\right)^2 \le 1\right\} \text{(eliptical)},\\
&\left\{(x, y)\colon\ (x+3)^2 + \left(y - 6\sqrt{2}\right)^2 \le 1\right\} \text{(circular)},\\
&\left\{(x, y)\colon\ 4\left(x + \frac{4}{3}\right)^2 + \left(y + \sqrt{7}\right)^2 \le 1\right\} \text{(eliptical)},\\
&\left\{(x, y)\colon\ \left(x - \frac{5}{2}\right)^2 + \left(y + \frac{3\sqrt{21}}{2}\right)^2 \le 1\right\} \text{(circular)}.
\end{aligned} \tag{17}$$

The vertices of the membership functions are $\overline{H}_1(1,0), \overline{H}_2(2, 3\sqrt{3}), \overline{H}_3(-3, 6\sqrt{2}), \overline{H}_4(-4/3, -\sqrt{7})$, and $\overline{H}_5(5/2, -3\sqrt{21}/2)$, respectively.

Now, for $\alpha \in [0,1]$, we may find the same points with membership degree α on $\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4$, and $\overline{H}_5$ as follows:

$$
\begin{aligned}
Q_{1\theta}^{\alpha}&: \left(x_{1\theta}^{\alpha}, y_{1\theta}^{\alpha}\right) = (1 + (1-\alpha)\cos\,\theta, (1-\alpha)\sin\,\theta),\\
Q_{2\theta}^{\alpha} &= \left(x_{2\theta}^{\alpha}, y_{2\theta}^{\alpha}\right) = \left(2 + (1-\alpha)\cdot\frac{\cos\,\theta}{\sqrt{1+3\,\sin^2\,\theta}}, 3\sqrt{3} + (1-\alpha)\cdot\frac{\sin\,\theta}{\sqrt{1+3\,\sin^2\,\theta}}\right),\\
Q_{3\theta}^{\alpha} &= \left(x_{3\theta}^{\alpha}, y_{3\theta}^{\alpha}\right) = (-3 + (1-\alpha)\cos\,\theta, 6\sqrt{2} + (1-\alpha)\sin\,\theta),\\
Q_{4\theta}^{\alpha} &= \left(x_{4\theta}^{\alpha}, y_{4\theta}^{\alpha}\right) = \left(-\frac{4}{3} + (1-\alpha)\cdot\frac{\cos\,\theta}{\sqrt{1+8\,\cos^2\,\theta}}, -\sqrt{7} + (1-\alpha)\cdot\frac{\sin\,\theta}{\sqrt{1+8\,\cos^2\,\theta}}\right),\\
Q_{5\theta}^{\alpha} &= \left(x_{5\theta}^{\alpha}, y_{5\theta}^{\alpha}\right) = \left(\frac{5}{2} + (1-\alpha)\cos\,\theta, -\frac{3\sqrt{21}}{2} + (1-\alpha)\sin\,\theta\right).
\end{aligned}
\tag{18}
$$

The hyperbola H_{θ}^{α} that passes through $Q_{1\theta}^{\alpha}, Q_{2\theta}^{\alpha}, Q_{3\theta}^{\alpha}, Q_{4\theta}^{\alpha}$, and $Q_{5\theta}^{\alpha}$ in (18) can be determinant by the following equation:

$$a_{\theta}^{\alpha}x^2 + 2h_{\theta}^{\alpha}xy + b_{\theta}^{\alpha}y^2 + 2g_{\theta}^{\alpha}x + 2f_{\theta}^{\alpha}y + c_{\theta}^{\alpha} = 0, \tag{19}$$

with $h_{\theta}^{\alpha 2} > a_{\theta}^{\alpha}\cdot b_{\theta}^{\alpha}$ where

$$
a_{\theta}^{\alpha} = \frac{2h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}
-x_{1\theta}^{\alpha}y_{1\theta}^{\alpha} & y_{1\theta}^{\alpha^2} & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1\\
-x_{2\theta}^{\alpha}y_{2\theta}^{\alpha} & y_{2\theta}^{\alpha^2} & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1\\
-x_{3\theta}^{\alpha}y_{3\theta}^{\alpha} & y_{3\theta}^{\alpha^2} & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1\\
-x_{4\theta}^{\alpha}y_{4\theta}^{\alpha} & y_{4\theta}^{\alpha^2} & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1\\
-x_{5\theta}^{\alpha}y_{1\theta}^{\alpha} & y_{5\theta}^{\alpha^2} & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1
\end{vmatrix},
$$

$$
b_{\theta}^{\alpha} = \frac{2h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}
x_{1\theta}^{\alpha^2} & -x_{1\theta}^{\alpha}y_{1\theta}^{\alpha} & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1\\
x_{2\theta}^{\alpha^2} & -x_{2\theta}^{\alpha}y_{2\theta}^{\alpha} & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1\\
x_{3\theta}^{\alpha^2} & -x_{3\theta}^{\alpha}y_{3\theta}^{\alpha} & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1\\
x_{4\theta}^{\alpha^2} & -x_{4\theta}^{\alpha}y_{4\theta}^{\alpha} & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1\\
x_{5\theta}^{\alpha^2} & -x_{5\theta}^{\alpha}y_{1\theta}^{\alpha} & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1
\end{vmatrix},
$$

$$
g_{\theta}^{\alpha} = \frac{h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}
x_{1\theta}^{\alpha^2} & y_{1\theta}^{\alpha^2} & -x_{1\theta}^{\alpha}y_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1\\
x_{2\theta}^{\alpha^2} & y_{2\theta}^{\alpha^2} & -x_{2\theta}^{\alpha}y_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1\\
x_{3\theta}^{\alpha^2} & y_{3\theta}^{\alpha^2} & -x_{3\theta}^{\alpha}y_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1\\
x_{4\theta}^{\alpha^2} & y_{4\theta}^{\alpha^2} & -x_{4\theta}^{\alpha}y_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1\\
x_{5\theta}^{\alpha^2} & y_{5\theta}^{\alpha^2} & -x_{5\theta}^{\alpha}y_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1
\end{vmatrix},
$$

$$
f_{\theta}^{\alpha} = \frac{h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}
x_{1\theta}^{\alpha^2} & y_{1\theta}^{\alpha^2} & x_{1\theta}^{\alpha} & -x_{1\theta}^{\alpha}y_{1\theta}^{\alpha} & 1\\
x_{2\theta}^{\alpha^2} & y_{2\theta}^{\alpha^2} & x_{2\theta}^{\alpha} & -x_{2\theta}^{\alpha}y_{2\theta}^{\alpha} & 1\\
x_{3\theta}^{\alpha^2} & y_{3\theta}^{\alpha^2} & x_{3\theta}^{\alpha} & -x_{3\theta}^{\alpha}y_{3\theta}^{\alpha} & 1\\
x_{4\theta}^{\alpha^2} & y_{4\theta}^{\alpha^2} & x_{4\theta}^{\alpha} & -x_{4\theta}^{\alpha}y_{4\theta}^{\alpha} & 1\\
x_{5\theta}^{\alpha^2} & y_{5\theta}^{\alpha^2} & x_{5\theta}^{\alpha} & -x_{5\theta}^{\alpha}y_{5\theta}^{\alpha} & 1
\end{vmatrix},
$$

$$
c_{\theta}^{\alpha} = \frac{2h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}
x_{1\theta}^{\alpha^2} & y_{1\theta}^{\alpha^2} & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & -x_{1\theta}^{\alpha}y_{1\theta}^{\alpha}\\
x_{2\theta}^{\alpha^2} & y_{2\theta}^{\alpha^2} & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & -x_{2\theta}^{\alpha}y_{2\theta}^{\alpha}\\
x_{3\theta}^{\alpha^2} & y_{3\theta}^{\alpha^2} & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & -x_{3\theta}^{\alpha}y_{3\theta}^{\alpha}\\
x_{4\theta}^{\alpha^2} & y_{4\theta}^{\alpha^2} & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & -x_{4\theta}^{\alpha}y_{4\theta}^{\alpha}\\
x_{5\theta}^{\alpha^2} & y_{5\theta}^{\alpha^2} & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & -x_{5\theta}^{\alpha}y_{5\theta}^{\alpha}
\end{vmatrix},
$$

$$
k_{\theta}^{\alpha} = \begin{vmatrix}
x_{1\theta}^{\alpha^2} & y_{1\theta}^{\alpha^2} & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1\\
x_{2\theta}^{\alpha^2} & y_{2\theta}^{\alpha^2} & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1\\
x_{3\theta}^{\alpha^2} & y_{3\theta}^{\alpha^2} & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1\\
x_{4\theta}^{\alpha^2} & y_{4\theta}^{\alpha^2} & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1\\
x_{5\theta}^{\alpha^2} & y_{5\theta}^{\alpha^2} & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1
\end{vmatrix}. \tag{20}
$$

These determinants are composed by writing column

$$
\begin{bmatrix}
-x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha}\\
-x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha}\\
-x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha}\\
-x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha}\\
-x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha}
\end{bmatrix}, \tag{21}
$$

instead of columns in determinant

$$\begin{vmatrix} x_{1\theta}^{\alpha^2} & y_{1\theta}^{\alpha^2} & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1 \\ x_{2\theta}^{\alpha^2} & y_{2\theta}^{\alpha^2} & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1 \\ x_{3\theta}^{\alpha^2} & y_{3\theta}^{\alpha^2} & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1 \\ x_{4\theta}^{\alpha^2} & y_{4\theta}^{\alpha^2} & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1 \\ x_{5\theta}^{\alpha^2} & y_{5\theta}^{\alpha^2} & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1 \end{vmatrix}. \tag{22}$$

Let A, B, C, F, G, and K be the determinant values used to find the values of $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$, and k_θ^α, respectively. So,

$$\begin{aligned} a_\theta^\alpha &= \frac{2h_\theta^\alpha}{k_\theta^\alpha}\cdot A, \\ b_\theta^\alpha &= \frac{2h_\theta^\alpha}{k_\theta^\alpha}\cdot B, \\ c_\theta^\alpha &= \frac{2h_\theta^\alpha}{k_\theta^\alpha}\cdot C, \\ f_\theta^\alpha &= \frac{h_\theta^\alpha}{k_\theta^\alpha}\cdot F, \\ g_\theta^\alpha &= \frac{h_\theta^\alpha}{k_\theta^\alpha}\cdot G, \\ k_\theta^\alpha &= K, \end{aligned} \tag{23}$$

are obtained.

The fuzzy hyperbola $\overline{FH}$ that passes through $\overline{H}_i$'s, $i = 1, 2, 3, 4, 5$ is the union of all possible hyperbola H_θ^α's that lies between $Q_{1\theta}^\alpha$ and $Q_{5\theta}^\alpha$'s.

That is,

$$\overline{FH}_{1,\ldots,5} = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \left\{(x, y) = a_\theta^\alpha x^2 + 2h_\theta^\alpha xy + b_\theta^\alpha y^2 + 2g_\theta^\alpha x + 2f_\theta^\alpha y + c_\theta^\alpha = 0\right\}. \tag{24}$$

The infinite tails $\overline{FH}_{2\infty}, \overline{FH}_{3\infty}, \overline{FH}_{4\infty}$, and $\overline{FH}_{5\infty}$ are determined by the following membership functions:

$$\begin{aligned} \mu\left((x, y)|\overline{FH}_{2\infty}\right) &= \begin{cases} \alpha, & 9\left(x^2 - x_{2\theta}^{\alpha^2}\right) = y^2 - y_{2\theta}^{\alpha^2}, \theta = 16.1^\circ, \\ 0, & \text{elsewhere,} \end{cases} \\ \mu\left((x, y)|\overline{FH}_{3\infty}\right) &= \begin{cases} \alpha, & 9\left(x^2 - x_{3\theta}^{\alpha^2}\right) = y^2 - y_{3\theta}^{\alpha^2}, \theta = 19.4^\circ, \\ 0, & \text{elsewhere,} \end{cases} \\ \mu\left((x, y)|\overline{FH}_{4\infty}\right) &= \begin{cases} \alpha, & 9\left(x^2 - x_{4\theta}^{\alpha^2}\right) = y^2 - y_{4\theta}^{\alpha^2}, \theta = 26.7^\circ, \\ 0, & \text{elsewhere,} \end{cases} \\ \mu\left((x, y)|\overline{FH}_{5\infty}\right) &= \begin{cases} \alpha, & 9\left(x^2 - x_{5\theta}^{\alpha^2}\right) = y^2 - y_{5\theta}^{\alpha^2}, \theta = 16.98^\circ, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{25}$$

The angles $\theta = 16.1^\circ, 19.4^\circ, 26.7^\circ$, and 16.98° are, respectively, the angles of the normals to the core hyperbola at $(2, 3\sqrt{3})$, $(-3, 6\sqrt{2})$, $(-4/3, -\sqrt{7})$, and $(5/2, -3\sqrt{21}/2)$ with the positive x-axis.

Now, we choose a point on the fuzzy hyperbola $\overline{FH}$ satisfying (19). This point can be taken as $(1.2, 0.4)$. We can show below how to calculate the membership degree of this selected point. It can also be seen from the figures that a fuzzy hyperbola passing through five points is formed as a combination of curves. In order to use the most suitable curve, the membership degrees must be known. The value of membership degrees is important for fuzzy hyperbolas.

Now, we get the equation as follows:

$$a_\theta^\alpha (1.2)^2 + 2h_\theta^\alpha (1.2)(0.4) + b_\theta^\alpha (0.4)^2 + 2g_\theta^\alpha (1.2) + 2f_\theta^\alpha (0.4) + c_\theta^\alpha = 0, \tag{26}$$

which simplifies to

$$1,44a_\theta^\alpha + 0,96h_\theta^\alpha + 0,16b_\theta^\alpha + 2,4g_\theta^\alpha + 0,8f_\theta^\alpha + c_\theta^\alpha = 0. \tag{27}$$

Now, let us examine the angular values that L lines make with the x-axis.

First, we admit that $\theta_0 = 45^\circ$ in equation (27). We calculate above determinant values for this angle and will find k_θ^α.

$$k_\theta^\alpha = K,$$

$$K = \begin{vmatrix} [1+(1-\alpha)\cos 45^\circ]^2 & [(1-\alpha)\sin 45^\circ]^2 & 1+(1-\alpha)\cos 45^\circ & (1-\alpha)\sin 45^\circ & 1 \\ \left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & \left[3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] & 2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 1 \\ [-3+(1-\alpha)\cos 45^\circ]^2 & [6\sqrt{2}+(1-\alpha)\sin 45^\circ]^2 & -3+(1-\alpha)\cos 45^\circ & 6\sqrt{2}+(1-\alpha)\sin 45^\circ & 1 \\ \left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & \left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & -\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & -\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & 1 \\ \left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]^2 & \left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right]^2 & \left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right] & -\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ & 1 \end{vmatrix}, \tag{28}$$

$$K = 2.337\alpha^3 - 73.417\alpha^2 - 657.057\alpha - 585.977.$$

Then, we calculate $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha$, and g_θ^α.

For the value of A,

$$\begin{vmatrix} -[1+(1-\alpha)\cos 45^\circ][(1-\alpha)\cos 45^\circ] & [(1-\alpha)\sin 45^\circ]^2 & 1+(1-\alpha)\cos 45^\circ & (1-\alpha)\sin 45^\circ & 1 \\ -\left[2+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]\left[3\sqrt{3}+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] & \left[3\sqrt{3}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] & 2+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 3\sqrt{3}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 1 \\ -[-3+(1-\alpha)\cos 45^\circ][6\sqrt{2}+(1-\alpha)\sin 45^\circ] & [6\sqrt{2}+(1-\alpha)\sin 45^\circ]^2 & -3+(1-\alpha)\cos 45^\circ & 6\sqrt{2}+(1-\alpha)\sin 45^\circ & 1 \\ -\left[-\frac{4}{3}+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]\left[-\sqrt{7}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right] & \left[-\sqrt{7}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & -\frac{4}{3}+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & -\sqrt{7}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & 1 \\ -\left[\frac{5}{2}+(1-\alpha)\cos 45^\circ\right]\left[-\frac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right] & \left[-\frac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right]^2 & \left[\frac{5}{2}+(1-\alpha)\cos 45^\circ\right] & -\frac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ & 1 \end{vmatrix}$$
$$= -1.168\alpha^3 + 859.293\alpha^2 - 1587.371\alpha - 71752.51. \tag{29}$$

For the value of B,

$$\begin{vmatrix} [1+(1-\alpha)\cos 45^\circ]^2 & -[1+(1-\alpha)\cos 45^\circ][(1-\alpha)\cos 45^\circ] & 1+(1-\alpha)\cos 45^\circ & (1-\alpha)\sin 45^\circ & 1 \\ \left[2+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & -\left[2+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]\left[3\sqrt{3}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] & 2+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 3\sqrt{3}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 1 \\ [-3+(1-\alpha)\cos 45^\circ]^2 & -[-3+(1-\alpha)\cos 45^\circ][6\sqrt{2}+(1-\alpha)\sin 45^\circ] & -3+(1-\alpha)\cos 45^\circ & 6\sqrt{2}+(1-\alpha)\sin 45^\circ & 1 \\ \left[-\frac{4}{3}+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & -\left[-\frac{4}{3}+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]\left[-\sqrt{7}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right] & -\frac{4}{3}+\frac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & -\sqrt{7}+\frac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & 1 \\ \left[\frac{5}{2}+(1-\alpha)\cos 45^\circ\right]^2 & -\left[\frac{5}{2}+(1-\alpha)\cos 45^\circ\right]\left[-\frac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right] & \left[\frac{5}{2}+(1-\alpha)\cos 45^\circ\right] & -\frac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ & 1 \end{vmatrix}$$
$$= -1.168\alpha^3 - 121.751\alpha^2 + 643.496\alpha + 7532.952. \tag{30}$$

For the value of G,

$$\begin{vmatrix} [1+(1-\alpha)\cos 45^\circ]^2 & [(1-\alpha)\sin 45^\circ]^2 & -[1+(1-\alpha)\cos 45^\circ][(1-\alpha)\cos 45^\circ] & (1-\alpha)\sin 45^\circ & 1 \\ \left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & \left[3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & -\left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]\left[3\sqrt{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] & 3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 1 \\ [-3+(1-\alpha)\cos 45^\circ]^2 & [6\sqrt{2}+(1-\alpha)\sin 45^\circ]^2 & -[-3+(1-\alpha)\cos 45^\circ][6\sqrt{2}+(1-\alpha)\sin 45^\circ] & 6\sqrt{2}+(1-\alpha)\sin 45^\circ & 1 \\ \left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & \left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & -\left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]\left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right] & -\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{1+8\cos^2 45^\circ} & 1 \\ \left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]^2 & \left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right]^2 & -\left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]\left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right] & -\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ & 1 \end{vmatrix}$$

$$= 1033.751\alpha^3 - 2942.633\alpha^2 - 86043.569\alpha + 87952.451. \tag{31}$$

For the value of F,

$$\begin{vmatrix} [1+(1-\alpha)\cos 45^\circ]^2 & [(1-\alpha)\sin 45^\circ]^2 & 1+(1-\alpha)\cos 45^\circ & -[1+(1-\alpha)\cos 45^\circ][(1-\alpha)\sin 45^\circ] & 1 \\ \left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & \left[3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & 2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & -\left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]\left[3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] & 1 \\ [-3+(1-\alpha)\cos 45^\circ]^2 & [6\sqrt{2}+(1-\alpha)\sin 45^\circ]^2 & -3+(1-\alpha)\cos 45^\circ & -[-3+(1-\alpha)\cos 45^\circ][6\sqrt{2}+(1-\alpha)\sin 45^\circ] & 1 \\ \left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & \left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & -\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & -\left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]\left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right] & 1 \\ \left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]^2 & \left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right]^2 & \dfrac{5}{2}+(1-\alpha)\cos 45^\circ & -\left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]\left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right] & 1 \end{vmatrix}$$

$$= -267.139\alpha^3 + 1601.891\alpha^2 - 13501.485\alpha - 14836.237. \tag{32}$$

For the value of C,

$$\begin{vmatrix} [1+(1-\alpha)\cos 45^\circ]^2 & [(1-\alpha)\sin 45^\circ]^2 & 1+(1-\alpha)\cos 45^\circ & (1-\alpha)\sin 45^\circ & -[1+(1-\alpha)\cos 45^\circ][(1-\alpha)\sin 45^\circ] \\ \left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & \left[3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]^2 & 2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & 3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}} & -\left[2+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right]\left[3\sqrt{3}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+3\sin^2 45^\circ}}\right] \\ [-3+(1-\alpha)\cos 45^\circ]^2 & \left[6\sqrt{2}+(1-\alpha)\sin 45^\circ\right]^2 & -3+(1-\alpha)\cos 45^\circ & 6\sqrt{2}+(1-\alpha)\sin 45^\circ & -[-3+(1-\alpha)\cos 45^\circ][6\sqrt{2}+(1-\alpha)\sin 45^\circ] \\ \left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & \left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]^2 & -\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & -\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}} & -\left[-\dfrac{4}{3}+\dfrac{(1-\alpha)\cos 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right]\left[-\sqrt{7}+\dfrac{(1-\alpha)\sin 45^\circ}{\sqrt{1+8\cos^2 45^\circ}}\right] \\ \left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]^2 & \left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right]^2 & \dfrac{5}{2}+(1-\alpha)\cos 45^\circ & -\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ & -\left[\dfrac{5}{2}+(1-\alpha)\cos 45^\circ\right]\left[-\dfrac{3\sqrt{21}}{2}+(1-\alpha)\sin 45^\circ\right] \end{vmatrix}$$
$$= 210.014\alpha^4 - 551.863\alpha^3 - 19423.306\alpha^2 + 25857.334\alpha + 66389.578. \tag{33}$$

Let substitute these values in equation (23), and we obtain the coefficients $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$, and c_θ^α.

If we substitute this coefficient in the conic equation (27) and simplify the equation with $h_\theta^\alpha (h_\theta^\alpha \neq 0)$, we obtain the following equation:

$$420.029\alpha^4 + 1162.068\alpha^3 - 42262.096\alpha^2 - 147723.644\alpha + 127196.826. \tag{34}$$

By solving equation (34), the values of α are $0.717, 10.197, -4.516$, and -9.164.

As alpha represents the membership degree, it must be $[0, 1]$. So, the appropriate alpha real number is $\theta = 0.717$ as it is supremum.

Thus, for $\theta_0 = 45^\circ$, the set S_{θ_0} of all possible value of α is the set $\{0.717\}$.

We change θ_0 through $[0, 2\pi]$ and obtain the value of s_{θ_0}. Then, we get supremum of all s_{θ_0}'s.

It can be easily verified that the supremum value is 0.717 for the point $(1.2, 0.4)$, and it is obtained for $\theta_0 = 45^\circ$. In conclusion, we obtain the hyperbola conic passing through to the same points for $\theta_0 = 45^\circ$ and $\alpha = 0.717$. As a result, we find that the conic which passing through to the same points are $(1.205, 0.205) \in \overline{H}_1$, $(2.129, 5.325) \in \overline{H}_2$, $(-2.784, 8.69) \in \overline{H}_3$, $(-1.203, -2.516) \in \overline{H}_4$, and $(2.705, -6.668) \in \overline{H}_5$.

Then, we obtain the conic equation (35) which passing through to the same points:

$$-72372.62x^2 - 159.62xy + 7917.21y^2 + 25766.96x - 4541.24y + 74675.32 = 0. \tag{35}$$

This hyperbola equation contains the point $(1.2, 0.4)$. And, we have

$$\mu((1.2, 0.4)|\overline{FH}) = 0.717, \tag{36}$$

alpha cut of fuzzy hyperbola.

Let also give an example of a fuzzy hyperbola whose core hyperbola is noncentral.

Example 2. Let $\overline{H}_1(6, 2\sqrt{17}), \overline{H}_2(2, 2), \overline{H}_3(0, 2\sqrt{5}), \overline{H}_4(-5, -10\sqrt{2})$, and $\overline{H}_5(4, -2\sqrt{5})$ be five fuzzy points. Let's get the fuzzy hyperbola that passes through these points.

In this example, we take core of that the points are in different regions on the curve. The membership functions of these selected five fuzzy points are cones with circular and elliptical basessimilar to the display of the points in Figure 1;

$$\begin{aligned} &\left\{(x, y)\colon (x-6)^2 + (y - 2\sqrt{17})^2 \leq 1\right\} \text{(circular)}, \\ &\left\{(x, y)\colon (x-2)^2 + 9(y-2)^2 \leq 1\right\} \text{(elliptical)}, \\ &\left\{(x, y)\colon x^2 + (y - 2\sqrt{5})^2 \leq 1\right\} \text{(circular)}, \\ &\left\{(x, y)\colon 9(x+5)^2 + (y + 10\sqrt{2})^2 \leq 1\right\} \text{(elliptical)}, \\ &\left\{(x, y)\colon (x-4)^2 + (y + 2\sqrt{5})^2 \leq 1\right\} \text{(circular)}. \end{aligned} \tag{37}$$

The vertices of the membership functions are $\overline{H}_1(6, 2\sqrt{17}), \overline{H}_2(2, 2), \overline{H}_3(0, 2\sqrt{5}), \overline{H}_4(-5, -10\sqrt{2})$, and $\overline{H}_5(4, -2\sqrt{5})$, respectively.

Now, for $\alpha \in [0,1]$, we may find the same points with membership degree α on $\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4$, and $\overline{H}_5$ as follows:

$$\begin{aligned}
Q_{1\theta}^{\alpha} &= \left(x_{1\theta}^{\alpha}, y_{1\theta}^{\alpha}\right) = \left(6 + (1-\alpha)\cos\,\theta, 2\sqrt{17} + (1-\alpha)\sin\,\theta\right),\\
Q_{2\theta}^{\alpha} &= \left(x_{2\theta}^{\alpha}, y_{2\theta}^{\alpha}\right) = \left(2 + (1-\alpha)\cdot\frac{\cos\,\theta}{\sqrt{1+8\,\sin^2\,\theta}}, 2\sqrt{5} + (1-\alpha)\cdot\frac{\sin\,\theta}{\sqrt{1+8\,\sin^2\,\theta}}\right),\\
Q_{3\theta}^{\alpha} &= \left(x_{3\theta}^{\alpha}, y_{3\theta}^{\alpha}\right) = \left((1-\alpha)\cos\,\theta, 2\sqrt{5} + (1-\alpha)\sin\,\theta\right),\\
Q_{4\theta}^{\alpha} &= \left(x_{4\theta}^{\alpha}, y_{4\theta}^{\alpha}\right) = \left(-5 + (1-\alpha)\cdot\frac{\cos\,\theta}{\sqrt{1+8\,\cos^2\,\theta}}, -2\sqrt{5} + (1-\alpha)\cdot\frac{\sin\,\theta}{\sqrt{1+8\,\cos^2\,\theta}}\right),\\
Q_{5\theta}^{\alpha} &= \left(x_{5\theta}^{\alpha}, y_{5\theta}^{\alpha}\right) = \left(4 + (1-\alpha)\cos\,\theta, -2\sqrt{5} + (1-\alpha)\sin\,\theta\right).
\end{aligned} \tag{38}$$

The equation of the core hyperbola through the points is

$$\frac{y^2}{4} - (x-2)^2 = 1. \tag{39}$$

The hyperbola H_θ^α that passes through $Q_{1\theta}^{\alpha}, Q_{2\theta}^{\alpha}, Q_{3\theta}^{\alpha}, Q_{4\theta}^{\alpha}$, and $Q_{5\theta}^{\alpha}$ can be determinant by the following equation:

$$a_\theta^\alpha x^2 + 2h_\theta^\alpha xy + b_\theta^\alpha y^2 + 2g_\theta^\alpha x + 2f_\theta^\alpha y + c_\theta^\alpha = 0, \tag{40}$$

with $h_\theta^{\alpha^2} > a_\theta^\alpha \cdot b_\theta^\alpha$. In this equation, the determinants used to find $a_\theta^\alpha, h_\theta^\alpha, b_\theta^\alpha, g_\theta^\alpha, f_\theta^\alpha$, and c_θ^α will be taken as in Example 1, and the membership of a point that we will choose is calculated.

Let $A, B, C, F, G, an\ d\, K$ be the determinant values used to find the values of $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$, and k_θ^α, respectively. So,

$$\begin{aligned}
a_\theta^\alpha &= \frac{2h_\theta^\alpha}{k_\theta^\alpha}\cdot A,\\
b_\theta^\alpha &= \frac{2h_\theta^\alpha}{k_\theta^\alpha}\cdot B,\\
c_\theta^\alpha &= \frac{2h_\theta^\alpha}{k_\theta^\alpha}\cdot C,\\
f_\theta^\alpha &= \frac{h_\theta^\alpha}{k_\theta^\alpha}\cdot F,\\
g_\theta^\alpha &= \frac{h_\theta^\alpha}{k_\theta^\alpha}\cdot G,\\
k_\theta^\alpha &= K.
\end{aligned} \tag{41}$$

The fuzzy hyperbola $\overline{FH}$ that passes through $\overline{H}_i$'s, ($i = 1, 2, 3, 4, 5$) is the union of all possible hyperbola H_θ^α's that lies between $Q_{1\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$'s.

That is,

$$\overline{FH}_{1,\ldots,5} = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \left\{(x,y) = a_\theta^\alpha x^2 + 2h_\theta^\alpha xy + b_\theta^\alpha y^2 + 2g_\theta^\alpha x + 2f_\theta^\alpha y + c_\theta^\alpha = 0\right\}. \tag{42}$$

The infinite tails $\overline{FH}_{1\infty}, \overline{FH}_{3\infty}, \overline{FH}_{4\infty}$, and $\overline{FH}_{5\infty}$ are determined by the following membership functions:

$$\begin{aligned}
\mu\left((x,y)|\overline{FH}_{1\infty}\right) &= \begin{cases} \alpha, & y^2 - y_{1\theta}^{\alpha^2} = 4\left((x-2)^2 + x_{1\theta}^{\alpha^2}\right), \theta = 27.2^{\circ}, \\ 0, & \text{elsewhere,} \end{cases} \\
\mu\left((x,y)|\overline{FH}_{3\infty}\right) &= \begin{cases} \alpha, & y^2 - y_{3\theta}^{\alpha^2} = 4\left((x-2)^2 + x_{3\theta}^{\alpha^2}\right), \theta = 29.2^{\circ}, \\ 0, & \text{elsewhere,} \end{cases} \\
\mu\left((x,y)|\overline{FH}_{4\infty}\right) &= \begin{cases} \alpha, & y^2 - y_{4\theta}^{\alpha^2} = 4\left((x-2)^2 + x_{4\theta}^{\alpha^2}\right), \theta = 26.7^{\circ}, \\ 0, & \text{elsewhere,} \end{cases} \\
\mu\left((x,y)|\overline{FH}_{5\infty}\right) &= \begin{cases} \alpha, & y^2 - y_{5\theta}^{\alpha^2} = 4\left((x-2)^2 + x_{5\theta}^{\alpha^2}\right), \theta = 29.2^{\circ}, \\ 0, & \text{elsewhere.} \end{cases}
\end{aligned} \tag{43}$$

The angles $\theta = 27.2^{\circ}, 29.2^{\circ}, 26.7^{\circ}$, and 29.2° are, respectively, the angles of the normals to the core hyperbola at $(6, 2\sqrt{17})$, $(0, 2\sqrt{15})$, $(-5, -10\sqrt{2})$, and $(4, -2\sqrt{5})$ with the positive x-axis.

$(2.05, 2.05)$ is a point on the fuzzy hyperbola $\overline{FH}$. First, we adjust the set of hyperbolas H_{θ}^{α}'s which the point $(2.05, 2.05)$ lies. We will calculate the membership degree of this point.

Let replace point $(2.05, 2.05)$ in (40). We need to identify the possible values of α.

Then, we get the equation as follows:

$$\begin{aligned}
a_{\theta}^{\alpha}(2.05)^2 + 2h_{\theta}^{\alpha}(2)(2.05)(2.05) + b_{\theta}^{\alpha}(2.05)^2 + 2g_{\theta}^{\alpha}(2.05) + 2f_{\theta}^{\alpha}(2.05) + c_{\theta}^{\alpha} &= 0, \\
4.2025a_{\theta}^{\alpha} + 8.405h_{\theta}^{\alpha} + 4.2025b_{\theta}^{\alpha} + 4.1g_{\theta}^{\alpha} + 4.1f_{\theta}^{\alpha} + c_{\theta}^{\alpha} &= 0.
\end{aligned} \tag{44}$$

First, we admit that $\theta_0 = 60^{\circ}$. We will calculate determinant values for this angle. These determinant values are calculated as in Example 1. We substitute $\theta_0 = 60^{\circ}$ in $(x_{1\theta}^{\alpha}, y_{1\theta}^{\alpha})$, $(x_{2\theta}^{\alpha}, y_{2\theta}^{\alpha})$, $(x_{3\theta}^{\alpha}, y_{3\theta}^{\alpha})$, $(x_{4\theta}^{\alpha}, y_{4\theta}^{\alpha})$, and $(x_{5\theta}^{\alpha}, y_{5\theta}^{\alpha})$ for determinants and calculate the determinant values approximately. We find them and replace in (44); then, we obtain the following nonlinear equation which determinants are found:

$$-3.184\alpha^4 - 17.469\alpha^3 + 14514.314\alpha^2 - 204227.678\alpha + 204987.984 = 0. \tag{45}$$

By solving this equation, real alpha values are found.

$$\alpha = 0.842, 42.001. \tag{46}$$

But, we get 0.842 from 0 to 1 from these real two values.

Thus, for $\theta_0 = 60^{\circ}$, the set S_{θ_0} of all possible values of α is the set $\{0.842\}$.

We change θ_0 through $[0, 2\pi]$ and obtain the value of s_{θ_0}. Finally, we get supremum of all s_{θ_0}'s. It can be easily verified that the supremum value is $0,842$ for the point $(2.05, 2.05)$, and it is obtained for $\theta_0 = 60^{\circ}$. In conclusion, we obtain the hyperbola conic passing through to the same points, for $\theta_0 = 60^{\circ}$ and $\alpha = 0.842$.

Substituting α and θ in Equation (38), we get the same points as $(6.079, 8.383) \in \widetilde{H}_1$, $(2.029, 2.051) \in \widetilde{H}_2$, $(0.079, 4.608) \in \widetilde{H}_3$, $(-4.954, -14.063) \in \widetilde{H}_4$ and $(4.079, -4335) \in \widetilde{H}_5$.

Then, we obtain the following conic (47) which passes through to the same points

$$348199.353x^2 - 1624.867xy - 86002.529y^2 - 1445151.495x + 27947.976y + 1809949.889 = 0. \tag{47}$$

The conic equation contains the point $(2.05, 2.05)$. And, we have

$$\mu((2.05, 2.05)|\overline{FE}) = 0.842. \tag{48}$$

4. Conclusion

The concept of fuzzy hyperbola has been examined, and the basic properties of fuzzy hyperbola have been explained in this study in detail. The membership degrees of fuzzy points have a specific role in the graph of hyperbola. The needed explanations based on these roles were made on the drawn graphics. Equations of conics such as hyperbola and ellipse can be obtained by determining five points. Starting from these, we developed a method for obtaining the fuzzy hyperbola in the study. But we cannot do this with five random points. We used the points which are called as same points. In the study, we have presented a method by determining the necessary properties for selecting points. As seen in the figures, the fuzzy hyperbolas can be depicted with different curves. The interpretations of this method and graphics presented in Section 3 are described in detail. The method used is proven by theorem and prepositions. Fuzzy space has an important place in the coding and programing of technological devices. On the other hand, hyperbolic curves are curvilinear structures that are frequently encountered in the photosynthesis, CO2, and heat distributions of plants in biology. In addition, the distributions of radar waves appear especially in broadcasts based on spectral radar and even neural network. It is used when scanning documents. When these areas are combined with each other, there is a need to create fuzzy hyperbola as a result of the mentioned technological structures taking up in fuzzy space. Mathematically, we aim to define and analyze the fuzzy hyperbola, thinking that it will be useful in the future. In this study, we have presented the fuzzy hyperbola in detail in a way that can guide the studies to be done in the above-mentioned areas.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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