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Almost absolute weighted summability with index k and matrix transformations

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Abstract

In this paper we generalize the space $\hat{\ell}_k$ of absolutely almost convergent series (J. Math. Anal. Appl. 161:50–56, 1991) via weighted mean transformations. We study some inclusion relations and their topological properties. Further we characterize certain matrix transformations.

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1 Introduction

Let w be the set of all sequences of complex numbers. A Banach sequence space X is *BK* if the map $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \geq 0$. A *BK*-space X is said to have the *AK* property if $\phi \subset X$ and $(e^{(v)})$ is a basis for X, where $e^{(v)} = \{0, 0, 0, \ldots, 1^{v^{th} \text{ term}}, 0, 0, \ldots\}$ and $\phi = \text{span}\{e^{(v)}\}$. If ϕ is dense in X, then it is called an *AD*-space, so *AK* implies *AD*.

Let ℓ_∞ be the space of all bounded sequences. A sequence $(x_n) \in \ell_\infty$ is said to be almost convergent to γ if all of its Banach limits [1] coincide to γ. Lorentz [8] (see [10] for double sequences) characterized almost convergence by saying that a sequence (x_n) is almost convergent to γ if and only if

$$\frac{1}{r+1} \sum_{v=0}^{r} x_{n+v} \to \gamma \quad \text{as } r \to \infty \text{ (uniformly in } n). \tag{1.1}$$

This notion plays an important role in summability theory and was investigated by several authors. For example, it was later used to define and study some concepts such as conservative and regular matrices, some sequence spaces, and matrix transformations (see [2, 6, 7, 9, 11, 12, 15]).

Absolute almost convergence emerges naturally as an absolute analogue of almost convergence. To introduce this concept, let $s_n = \Sigma_{v=1}^{n} a_v$ be a partial sum of Σa_v. The series

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Σa_v is said to be absolutely almost convergent series if (see [3])

$$\sum_{m=0}^{\infty} |\psi_{m,n}|^k < \infty, \quad k > 0,$$

uniformly in n, where

$$\psi_{m,n} = \begin{cases} a_n, & m = 0, \\ \frac{1}{m(m+1)} \sum_{v=1}^{m} v a_{n+v}, & m \geq 1. \end{cases}$$

The space of all absolutely almost convergent series

$$\widehat{\ell}_k = \left\{ a = (a_n)_{n \in \mathbb{N}} : \sum_{m=0}^{\infty} |\psi_{m,n}|^k < \infty, \text{ uniformly in } n, k > 0 \right\}$$

was first defined and studied in [4]. We note an important relation between $\widehat{\ell}_k$ and absolute Cesaro summability $|C, 1|$ in Flett's notation [5], $\widehat{\ell}_k \subset |C, 1|$ (see [4]).

The purpose of the present paper is to define an absolute almost weighted summability using some factors and weighted means and to study its topological structures. This new method of summability extends the well-known concept of absolute almost convergence of Das et al. [4], and space $\widehat{\ell}_k$ of Das et al. [4] becomes a special case of our space $|f(\overline{N}_p^u)|_k$. We investigate relations between classical sequence spaces and show that the space $|f(\overline{N}_p^u)|_k$ is not separable for $k > 1$. Also, we characterize the matrix classes $(c, |f(\overline{N}_p)|_k)$ and $(c, |f(\overline{N}_p)|_k), 1 \leq k < \infty$.

2 Main results

For any sequence (s_n), we define $T_{m,n}$ by

$$T_{-1,n}(s) = s_{n-1}, \qquad T_{m,n}(s) = \frac{1}{P_m} \sum_{v=0}^{m} p_v s_{n+v}, \quad m \geq 0,$$

where (p_n) is a sequence of positive real numbers with

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty \quad \text{as } n \to \infty, P_{-1} = p_{-1} = 0.$$

A straightforward calculation then shows that

$$F_{m,n}(a) = T_{m,n}(s) - T_{m-1,n}(s) = \begin{cases} a_n, & m = 0, \\ \frac{p_m}{P_m P_{m-1}} \sum_{v=1}^{m} P_{v-1} a_{n+v}, & m \geq 1. \end{cases}$$

So, we can give the following definition.

Definition 2.1 Let Σa_v be an infinite series with partial summations (s_n). Let (p_n) and (u_n) be sequences of positive real numbers. The series Σa_v is said to be absolute almost

weighted summable $|f(\overline{N}_p), u_m|_k, k \geq 1$, if

$$\sum_{m=0}^{\infty} u_m^{k-1} \big|F_{m,n}(a)\big|^k < \infty$$

uniformly in n.

For $|f(\overline{N}_p^u)|_k, k \geq 1$, we write the set of all series summable by the method $|f(\overline{N}_p), u_m|_k$. Then Σa_v is summable $|f(\overline{N}_p), u_m|_k$ iff the series $\Sigma a_v \in |f(\overline{N}_p^u)|_k$. Note that, in the case $u_m = p_m = 1$ for $m \geq 0$, it reduces to the set of absolutely almost convergent series $\widehat{\ell}_k$ given by Das, Kuttner, and Nanda [3]. Further, it is clear that the space $|\overline{N}_p^u|_k$ is derived from $|f(\overline{N}_p^u)|_k$ by putting $n = 0$ [9, 13, 14], and also $|f(\overline{N}_p^u)|_k \subset |\overline{N}_p^u|_k$, but the converse is not true.

First we give some relations between the new method and classical sequence spaces such as *bs* and ℓ_∞, which are the sets of all bounded series and bounded sequences, respectively.

Theorem 2.2 *Let (p_m) and (u_m) be sequences of positive numbers.*
  (i) *If*

$$\left(\frac{1}{u_m}\right) \in \ell_\infty, \tag{2.1}$$

  then $|f(\overline{N}_p^u)|_k \subset \ell_\infty, k \geq 1$.
 (ii) *If*

$$\sum_{m=0}^{\infty} u_m^{k-1} \left(\frac{p_m}{P_m}\right)^k < \infty, \tag{2.2}$$

  then $bs \subset |f(\overline{N}_p^u)|_k, k > 1$.

Proof (i) Let $a = (a_v) \in |f(\overline{N}_p^u)|_k$. Then, by the definition, there exists an integer M such that

$$\sum_{m=M}^{\infty} u_m^{k-1} \big|F_{m,n}(a)\big|^k \leq 1 \tag{2.3}$$

holds for all n. So, it is sufficient to show that the sequence $(|F_{m,n}(a)|)$ is bounded for a fixed number m. By (2.1) and (2.3), we have $|F_{m,n}(a)| \leq u_m^{-1+1/k}$ for $m \geq M$ and all n. On the other hand, for $m \geq 1$,

$$a_{m+n} = \frac{P_m}{p_m}F_{m,n}(a) - \frac{P_{m-2}}{p_{m-1}}F_{m-1,n}(a). \tag{2.4}$$

It follows by applying (2.4) for any $m \geq M + 1$ that $a = (a_v) \in \ell_\infty$, which completes the proof.

(ii) Let $a = (a_v) \in bs$. Denote $M = \sup_v |\sum_{j=0}^v a_j|$. Then we have

$$\left|\sum_{v=1}^{m} P_{v-1} a_{n+v}\right| = \left|\sum_{v=1}^{m-1}(-p_v)\sum_{j=1}^{v} a_{n+j} + P_{m-1}\sum_{j=1}^{m} a_{n+j}\right|$$

$$\leq \sum_{\nu=1}^{m-1} p_\nu \left| \sum_{j=n+1}^{n+\nu} a_j \right| + P_{m-1} \left| \sum_{j=n+1}^{n+m} a_{n+j} \right|$$

$$\leq \sum_{\nu=1}^{m-1} p_\nu \left| \sum_{j=0}^{n+\nu} a_j - \sum_{j=0}^{n} a_j \right| + P_{m-1} \left| \sum_{j=0}^{n+m} a_{n+j} - \sum_{j=0}^{n} a_j \right|$$

$$\leq 4MP_{m-1},$$

which implies that

$$\sum_{m=0}^{\infty} u_m^{k-1} \left| F_{m,n}(a) \right|^k = u_0^{k-1} |a_n|^k + \sum_{m=1}^{\infty} u_m^{k-1} \left| \frac{p_m}{P_m P_{m-1}} \sum_{\nu=1}^{m} P_{\nu-1} a_{n+\nu} \right|^k$$

$$\leq (4M)^k \sum_{m=0}^{\infty} u_m^{k-1} \left(\frac{p_m}{P_m} \right)^k < \infty.$$

This completes the proof.   □

For the special case $u_m = p_m = 1$ for all $m \geq 0$, $|f(\overline{N}_p^u)|_k = \widehat{\ell}_k$ and (2.2) reduces to $\sum_{m=0}^{\infty} (m+1)^{-k} < \infty$. So we have the following result in [3].

Corollary 2.3 *For $k > 1$, $bs \subset \widehat{\ell}_k$.*

Theorem 2.4 *Let (u_m) be a sequence of positive numbers such that* (2.4) *holds. Then $|f(\overline{N}_p^u)|_k$, $k \geq 1$, is a BK space with respect to the norm*

$$\|a\|_{|f(\overline{N}_p^u)|_k} := \sup \left\{ \sum_{m=0}^{\infty} u_m^{k-1} \left| F_{m,n}(a) \right|^k : n \in \mathbb{N} \right\}^{1/k}. \tag{2.5}$$

Proof It is routine to prove that the norm conditions are satisfied by (2.5). We only note that (2.5) is well defined. In fact, if $a \in |f(\overline{N}_p^u)|_k$, then, as in the proof of part (ii) of Theorem 2.2, there exists an integer M such that, for all n,

$$\sum_{m=M}^{\infty} u_m^{k-1} \left| F_{m,n}(a) \right|^k \leq 1,$$

and $(u_m^{1-1/k} |F_{m,n}(a)|)$ is bounded for all $m, n \geq 0$. This gives

$$\sup \left\{ \sum_{m=0}^{\infty} u_m^{k-1} \left| F_{m,n}(a) \right|^k : n \in \mathbb{N} \right\} \leq 1 + \sup \left\{ \sum_{m=0}^{M} u_m^{k-1} \left| F_{m,n}(a) \right|^k : n, m \in \mathbb{N} \right\} < \infty.$$

To show that it is a Banach space, let us take an arbitrary Cauchy sequence $(a^m) = (a_\nu^m) = (a_\nu^1, a_\nu^2, \ldots) = ((a_1^1, a_2^1, \ldots), \ldots, (a_1^m, a_2^m, a_3^m, \ldots), \ldots)$, where $a^m = (a_\nu^m) \in |f(\overline{N}_p^u)|_k$ for $m \geq 0$. Given $\varepsilon > 0$. Then there exists an integer m_0 such that $\|a^{m_1} - a^{m_2}\|_{|f(\overline{N}_p^u)|_k} < \varepsilon$ for $m_1, m_2 > m_0$ and all ν, or, equivalently,

$$\left\{ u_0^{k-1} \left| a_n^{m_1} - a_n^{m_2} \right|^k + \sum_{m=1}^{\infty} u_m^{k-1} \left| \frac{p_m}{P_m P_{m-1}} \sum_{\nu=1}^{m} P_{\nu-1} \left(a_{n+\nu}^{m_1} - a_{n+\nu}^{m_2} \right) \right|^k \right\}^{1/k} < \varepsilon. \tag{2.6}$$

This also gives that $|a_\nu^{m_1} - a_\nu^{m_2}|^k < \varepsilon/u_0^{1-1/k}$ holds for all $m_1, m_2 > m_0$, i.e., the sequence (a_ν^m) is a Cauchy sequence in the set of complex numbers $\mathbb{C}$. So, it converges to a number a_ν ($\nu = 0, 1, \ldots$), i.e., $\lim_{m \to \infty} a_\nu^m = a_\nu$. Now, letting $m_2 \to \infty$, by (2.6) we have for $\|a^{m_1} - a\|_{|f(\overline{N}_p^u)|_k} < \varepsilon$ for $m_1 > m_0$. This means $\lim_{m \to \infty} a^m = a$. Further, since

$$\|a\|_{|f(\overline{N}_p^u)|_k} \leq \|a^{m_1} - a\|_{|f(\overline{N}_p^u)|_k} + \|a^{m_1}\|_{|f(\overline{N}_p^u)|_k} < \infty,$$

then $a \in |f(\overline{N}_p^u)|_k$. So, $|f(\overline{N}_p^u)|_k$ is a Banach space. This completes the proof. □

We note that if E is a BK-space such that $bs \subset E \subset \ell_\infty$, then E is not separable (and hence not reflexive) (see [4]). Hence the following result at once follows from Theorem 2.2.

Corollary 2.5 *If (p_m) and (u_m) are sequences of positive numbers satisfying (2.1) and (2.2), then $|f(\overline{N}_p^u)|_k$ is not separable for $k > 1$.*

3 Matrix transformations on space $|f(\overline{N}_p)|_k$

In this section we characterize certain matrix transformations on the space $|f(\overline{N}_p)|_k$. First we recall some notations. Let X, Y be any subsets of ω and $A = (a_{n\nu})$ be an infinite matrix of complex numbers. By $A(x) := (A_n(x))$, we indicate the A-transform of a sequence $x = (x_\nu)$ if the series

$$A_n(x) := \sum_{\nu=0}^{\infty} a_{n\nu} x_\nu$$

are convergent for $n \geq 0$. If $Ax \in Y$, whenever $x \in X$, then we say that A defines a matrix mapping from X into Y and denotes the class of all infinite matrices A such that $A : X \to Y$ by (X, Y). Also, we denote the set of all k-absolutely convergent series by $\ell_k, 1 \leq k < \infty$, i.e.,

$$\ell_k = \left\{ x = (x_\nu) \in w : \sum_{\nu=0}^{\infty} |x_\nu|^k < \infty \right\},$$

which is a BK-space with respect to the norm

$$\|x\|_{\ell_k} = \left(\sum_{\nu=0}^{\infty} |x_\nu|^k \right)^{1/k}.$$

Also we make use of the following lemma in [15].

Lemma 3.1 *Suppose that $A = (a_{n\nu})$ is an infinite matrix with complex numbers and $p = (p_\nu)$ is a bounded sequence of positive numbers such that $H = \sup_\nu p_\nu$ and $C = \max\{1, 2^{H-1}\}$. Then*

$$\left(4C^2\right)^{-1} U_p(A) \leq L_p(A) \leq U_p(A)$$

provided that

$$U_p(A) = \sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{n\nu}| \right)^{p_\nu} < \infty$$

and

$$L_p(A) = \sup\left\{ \sum_{v=0}^{\infty} \left| \sum_{n \in N} a_{nv} \right|^{p_v} : N \subset N_0 \text{ is finite} \right\} < \infty.$$

Now we begin with the first theorem given the characterization of the class $(\ell_1, |f(\overline{N}_p)|_k)$.

Theorem 3.2 *Let $u = (u_n)$ be a sequence of positive numbers, and let $A = (a_{vj})$ be an infinite matrix. Then $A \in (\ell_1, |f(\overline{N}_p)|_k), 1 \le k < \infty$, if and only if*

$$(a_{vj}) \in \left| f(\overline{N}_p) \right|_k \quad \text{for each } j \tag{3.1}$$

and

$$L := \sup_{n,j} \sum_{m=0}^{\infty} u_m^{k-1} \left| b(m, n, j) \right|^k < \infty, \tag{3.2}$$

where

$$b(m, n, j) = \begin{cases} a_{nj}, & m = 0, \\ \frac{p_m}{P_m P_{m-1}} \sum_{v=1}^{m} P_{v-1} a_{n+v,,j}, & m \ge 1. \end{cases}$$

Proof Necessity. Suppose $A \in (\ell_1, |f(\overline{N}_p)|_k)$. Then $A(x) \in |f(\overline{N}_p)|_k$ for all $x \in \ell_1$, i.e.,

$$\sum_{m=0}^{\infty} u_m^{k-1} \left| F_{m,n}(A(x)) \right|^k$$

$$= u_0^{k-1} \left| A_n(x) \right|^k + \sum_{m=1}^{\infty} u_m^{k-1} \left| \sum_{v=1}^{m} \frac{p_m P_{v-1}}{P_m P_{m-1}} A_{n+v}(x) \right|^k$$

$$= u_0^{k-1} \left| \sum_{j=0}^{\infty} a_{n,j} x_j \right|^k + \sum_{m=1}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} \left(\sum_{v=1}^{m} \frac{p_m P_{v-1}}{P_m P_{m-1}} a_{n+v,j} \right) x_j \right|^k$$

$$= \sum_{m=0}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} b(m, n, j) x_j \right|^k$$

converges uniformly in n.

   If we put $x = (e^j) = (e_r^j) \in \ell_1$ such that $e_r^j = 1$ for $r = j$ and zero otherwise, $A(e^j) \in |f(\overline{N}_p)|_k$, which gives that (3.1) holds. Further, since ℓ_1 is a Banach space, by the Banach–Steinhaus theorem, $A : \ell_1 \to \mathbb{C}$ is a continuous linear map. So, for fixed n and s,

$$q_{sn}(x) = \left(\sum_{m=0}^{s} u_m^{k-1} \left| F_{m,n}(A(x)) \right|^k \right)^{1/k}$$

is a continuous seminorm on ℓ_1, which implies that $\lim_{s \to \infty} q_{sn}(x) = q_n(x)$ is a continuous seminorm, or, equivalently, there exists a constant K such that

$$q_n(x) = \left(\sum_{m=0}^{\infty} u_m^{k-1} \left| F_{m,n}(A(x)) \right|^k \right)^{1/k} \le K \|x\|_{\ell_1} \tag{3.3}$$

for every $x \in \ell_1$. Applying (3.3) with $x = (e_v^j) \in \ell_1$ we have, for all $j, n \geq 0$,

$$\left(\sum_{m=0}^{\infty} u_m^{k-1} |b(m,n,j)|^k \right)^{1/k} \leq K,$$

which gives (3.2).

Sufficiency. Suppose that (3.1) and (3.2) hold. Take $x \in \ell_1$. Then we should show $A(x) \in |f(\overline{N}_p)|_k$. For this, it is enough to prove that

$$\sum_{m=l}^{\infty} u_m^{k-1} |F_{m,n}(A(x))|^k \to 0 \quad \text{as } l \to \infty,$$

uniformly in n. By applying generalized Minkowski's inequality, we get

$$\left(\sum_{m=l}^{\infty} u_m^{k-1} |F_{m,n}(A(x))|^k \right)^{1/k} = \left(\sum_{m=l}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} b(m,n,j)x_j \right|^k \right)^{1/k}$$

$$\leq \sum_{j=0}^{\infty} |x_j| \left(\sum_{m=l}^{\infty} u_m^{k-1} |b(m,n,j)|^k \right)^{1/k} \qquad (3.4)$$

$$= \sum_{j=0}^{\infty} |x_j| R(l,n,j),$$

where

$$R(l,n,j) = \left(\sum_{m=l}^{\infty} u_m^{k-1} |b(m,n,j)|^k \right)^{1/k}.$$

On the other hand, it follows from (3.2) that, for all $l, n, j \geq 0$,

$$R(l,n,j)|x_j| \leq L^{1/k} |x_j|,$$

which gives

$$\sum_{j=0}^{\infty} |x_j| R(l,n,j) \leq L^{1/k} \sum_{j=0}^{\infty} |x_j| < \infty.$$

Now, let $\varepsilon > 0$. Then there exists an integer j_0 such that, for all l and n,

$$\sum_{j=j_0}^{\infty} |x_j| R(l,n,j) < \frac{\varepsilon}{2}.$$

Also, by (3.1), for each j,

$$R(l,n,j) \to 0 \quad \text{as } l \to \infty \text{ uniformly in } n,$$

there exists an integer l_0 so that, for $l \geq l_0$ and all n,

$$\sum_{j=0}^{j_0-1} |x_j| R(l, n, j) < \frac{\varepsilon}{2}.$$

So, we have, for $l \geq l_0$ and all n,

$$\sum_{j=0}^{\infty} |x_j| R(l, n, j) < \varepsilon,$$

which implies, by (3.4),

$$\left(\sum_{m=l}^{\infty} u_m^{k-1} \left| F_{m,n}(A(x)) \right|^k \right)^{1/k} < \varepsilon.$$

This states that

$$\sum_{m=l}^{\infty} u_m^{k-1} \left| F_{m,n}(A(x)) \right|^k \to 0 \quad \text{as } l \to \infty \text{ uniformly in } n. \qquad \square$$

This completes the proof.

In the special case $p_m = u_m = 1$ for all $m \geq 0$, we have $|f(\overline{N}_p)| = \widehat{\ell}_k$, and so the following result follows from Theorem 3.2.

Corollary 3.3 $A \in (\ell_1, \widehat{\ell}_k), 1 \leq k < \infty$, *if and only if*

$$(a_{vj}) \in \widehat{\ell}_k \quad \text{for each } j$$

and

$$\sup_{n,j} \sum_{m=0}^{\infty} \left| b'(m, n, j) \right|^k < \infty,$$

where

$$b'(m, n, j) = \begin{cases} a_{nj}, & m = 0, \\ \frac{1}{m(m+1)} \sum_{v=1}^{m} v a_{n+v,j}, & m \geq 1. \end{cases} \tag{3.5}$$

Theorem 3.4 *Let $u = (u_n)$ be a sequence of positive numbers, and let $A = (a_{vj})$ be an infinite matrix. Then $A \in (c, |f(\overline{N}_p)|_k), 1 \leq k < \infty$, if and only if conditions (3.1),*

$$B = \sup_n \sum_{m=0}^{\infty} u_m^{k-1} \left(\sum_{j=0}^{\infty} \left| b(m, n, j) \right| \right)^k < \infty, \tag{3.6}$$

and

$$\sum_{m=0}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} b(m, n, j) \right|^k \quad \text{converges uniformly in } n \tag{3.7}$$

hold.

Proof Necessity. Let $A \in (c, |f(\overline{N}_p)|_k)$. Then $A(x) \in |f(\overline{N}_p)|_k$ for every $x \in c$. Now, take $x = e^{(j)}$ and $x = e = (1, 1, \ldots)$. Then (3.1) and (3.7) hold, respectively. Also, it follows as in the proof of Theorem 3.2 that

$$\left(\sum_{m=0}^{\infty} u_m^{k-1} \left| F_{m,n}(A(x)) \right|^k \right)^{1/k} \leq K \|x\|_\infty. \tag{3.8}$$

Let N be an arbitrary finite set of natural numbers, and define a sequence x by

$$x_j = \begin{cases} 1, & j \in N, \\ 0, & j \notin N, \end{cases} \tag{3.9}$$

then $x \in c$ and $\|x\| = 1$. Applying (3.8) with this sequence (3.9), we have

$$\left\{ \sum_{m=0}^{\infty} u_m^{k-1} \left(\left| \sum_{j \in N} b(m, n, j) \right| \right)^k \right\}^{1/k} \leq K. \tag{3.10}$$

Hence, it is seen from Lemma 3.1 together with $p_\nu = 1$ for all ν that (3.10) is equivalent to (3.6).

Sufficiency. Suppose that (3.1), (3.6), and (3.7) hold. Given $x \in c$ and say $\lim_j x_j = \beta$. Then, by (3.7), as in Theorem 3.2,

$$\sum_{m=0}^{\infty} u_m^{k-1} \left| F_{mn}(A(x)) \right|^k = \sum_{m=0}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} b(m, n, j) x_j \right|^k < \infty.$$

Now it is enough to show that the tail of this series tends to zero uniformly in n. To see that, we write

$$\sum_{m=M}^{\infty} u_m^{k-1} \left| F_{mn}(A(x)) \right|^k$$

$$= \sum_{m=M}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} \beta b(m, n, j) + \sum_{j=0}^{\infty} b(m, n, j)(x_j - \beta) \right|^k$$

$$\leq 2^k \left\{ \sum_{m=M}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} \beta b(m, n, j) \right|^k + \sum_{m=M}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} b(m, n, j)(x_j - \beta) \right|^k \right\}$$

$$= 2^k \left\{ F_{M,n}^1 + F_{M,n}^2(x') \right\}, \quad \text{say.}$$

It is clear from (3.7) that $F_{M,n}^1 \to 0$ as $M \to \infty$ uniformly in n. On the other hand, since $x_j \to \beta$, for any $\varepsilon > 0$, there exists an integer j_0 such that

$$|x_j - \beta| < \frac{1}{2} \left(\frac{\varepsilon}{B} \right)^{1/k} \quad \text{for } j \geq j_0,$$

which gives us, by (3.6), for all $n \geq 0$,

$$
F_{M,n}^2(x') = \sum_{m=M}^{\infty} u_m^{k-1} \left| \sum_{j=0}^{\infty} b(m,n,j)(x_j - \beta) \right|^k
$$

$$
\leq 2^k \sum_{m=M}^{\infty} u_m^{k-1} \left\{ \left| \sum_{j=0}^{j_0-1} b(m,n,j)(x_j - \beta) \right|^k + \left| \sum_{j=j_0}^{\infty} b(m,n,j)(x_j - \beta) \right|^k \right\}
$$

$$
= 2^k \sup_j |x_j - \beta| \sum_{m=M}^{\infty} u_m^{k-1} \left(\sum_{j=0}^{j_0-1} |b(m,n,j)| \right)^k + \frac{\varepsilon}{2}.
$$

By (3.1), the first term of the equality is smaller than $\varepsilon/2$ for sufficiently large M and all n. This means $F_{M,n}^2 \to 0$ as $M \to \infty$ uniformly in n. Hence, the theorem is established.

For $p_m = u_m = 1$, Theorem 3.4 also gives the characterization of the class $(c, \widehat{\ell}_k)$ as follows. □

Corollary 3.5 *Let $A = (a_{vj})$ be an infinite matrix and $(b'(m,n,j))$ be as in (3.5). Then $A \in (c, \widehat{\ell}_k)$, $1 \leq k < \infty$, if and only if conditions (3.1),*

$$
\sup_n \sum_{m=0}^{\infty} \left(\sum_{j=0}^{\infty} |b'(m,n,j)| \right)^k < \infty,
$$

and

$$
\sum_{m=0}^{\infty} \left| \sum_{j=0}^{\infty} b'(m,n,j) \right|^k \qquad \text{converges uniformly in } n
$$

hold.

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