International Electronic Journal of Geometry
Volume 15 No. 2 Page 214–224 (2022)
DOI: https://doi.org/10.36890/iejg.1110327

A New Generalization of Some Curve Pairs

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(Communicated by Murat Tosun)

Abstract

In this study, we give a new curve pair that generalizes some of the famous pairs of curves as Bertrand and constant torsion curves. This curve pair is defined with the help of a vector obtained by the intersection of the osculating planes such that this vector makes the same angle γ with the tangents of the curves. We examine the relations between torsions and curvatures of these curve mates. Also, We have seen that the unit quaternion corresponding to the rotation matrix between the Frenet vectors of the curves is $q = \cos(\theta/2) - \mathbf{i}\sin(\theta/2)\cos\gamma - \mathbf{j}\sin(\theta/2)\sin\gamma$, where θ is the angle between the reciprocal binormals of the curves. Finally, we show in which specific case which well-known pairs of curves will be obtained.

Keywords: *Bertrand mate, Backlund transformation, constant torsion curves, curve mates.*

AMS Subject Classification (2020): *Primary: 53A04.*

1. Introduction

In classical differential geometry, establishing a relationship between the corresponding points of two curves and obtaining pair of curves is a topic that attracts many researchers. These curve pairs were applied in the areas such as computer-aided geometric design, robotics, and path planning. Well-known pairs of curves are Involute-Evolute, Parallel, Bertrand, Mannheim, and Natural mates. Characterizations of these curves mates with some properties can be found in various papers : [2, 6, 12, 13, 14, 15, 20, 23, 25, 27, 31, 32, 39, 42]. Some of these curves mates have been generalized to larger dimensions and have been studied by many authors [11, 17, 19, 28, 35]. Also, these pairs of curves have been studied by many authors in the Lorentzian space [3, 5, 16, 18, 21, 24, 34, 40, 41, 43].

In addition to these curves mates, we will encounter the Backlund curve pairs obtained by the Backlund transformation and also called the constant torsion curve pair. Detailed information on these curve pair can be found in articles [7, 29, 30].

In this study, the aim of the authors is to give a new generalization of these curves mates in three-dimensional Euclidean space and to investigate whether special cases in the general definition give one of them. For this, a new definition of curve pair, called *osculating mate*, has been given. This pair of curves is called the osculating mate since the curves are defined by a vector obtained by the intersection of the osculating planes such that this vector makes the same angle with the tangents of the curves. Similarly, definitions of the rectifying and normal curve mates can be given. However, in this study, we will examine only the osculating mate. Besides, we will give the relations between the Frenet vectors of these curve pairs, the relations between their curvatures, and torsions, and finally, which famous curve mates they correspond to in special cases.

Received : *09-06-2022,* ***Accepted :*** *24-07-2022*
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2. Preliminaries

In this section, let's briefly recall the Frenet elements and some famous curve mates. Let $\alpha : I \to \mathbb{E}^3$ be a regular curve with $v = \|\alpha'\| \neq 0$. Tangent, normal and binormal vector fields of α are defined as

$$\mathbf{T} = \frac{\alpha'}{v}, \quad \mathbf{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \quad \text{and} \quad \mathbf{N} = \mathbf{B} \times \mathbf{T},$$

respectively. Also, curvature and torsion of the curve α are

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \qquad \text{and} \qquad \tau = \frac{\det(\alpha', \alpha'', \alpha'')}{\|\alpha' \times \alpha''\|^2}.$$

Frenet–Serret formula for the curve α is

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & v\kappa & 0 \\ -v\kappa & 0 & v\tau \\ 0 & -v\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

At each point of a curve, the planes spanned by $\{\mathbf{T}, \mathbf{N}\}$, $\{\mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}, \mathbf{B}\}$ are called the osculating, the normal, and the rectifying plane, respectively. In this paper, the curve pairs whose lines joining corresponding points lie in the osculating planes will be examined. The most well-known pairs of curves are as follows.

- *Involute-Evolute :* Evolute of a curve is defined as the locus of the centers of curvatures of the curve. Involute of a given regular curve α is a curve $\mathcal{I}$ to which all tangent vectors of α are normal vector. The evolute of an involute is the original curve. The concept of Involute and Evolute was introduced by Dutch mathematician Christian Huygens [38].
- *Bertrand Curves :* A Bertrand curve is a space curve $\mathcal{B}$ whose normal vector is the same as the normal vector of another curve $\mathcal{B}^*$, which is called Bertrand mate of $\mathcal{B}$. We say that $\mathcal{B}^*$ and $\mathcal{B}$ are Bertrand mates if there exists a nonzero constant λ such that

$$\mathcal{B}^*(t) = \mathcal{B}(t) + \lambda \mathbf{N}(t) \qquad \text{and} \qquad \mathbf{N}^*(t) = \pm \mathbf{N}(t)$$

  for all $t \in I$. It was used by Bertrand in 1850 ([2, 6, 20, 23, 27]).
- *Mannheim Curves :* A Mannheim curve is a space curve $\mathcal{M}$ whose normal vector is the same as the binormal vector of another curve $\mathcal{M}^*$. We say that $\mathcal{M}^*$ and $\mathcal{M}$ are Mannheim mates if there exists a nonzero constant λ such that

$$\mathcal{M}^*(t) = \mathcal{M}(t) + \lambda \mathbf{N}(t) \qquad \text{and} \qquad \mathbf{B}^*(t) = \pm \mathbf{N}(t)$$

  for all $t \in I$. Mannheim curves were expressed in 1878 by A. Mannheim [20, 25, 31].
- *Natural Mate of a curve :* For a given Frenet curve α, there exists a unique unit speed curve β tangent to the principal normal vector field of α. The curve β is called the natural mate of curve α [9, 14].
- *Parallel Mate of a curve :* We say that α and α^* are parallel curves, if tangent vectors at the corresponding points of α and α^* are parallel and the join of corresponding points is perpendicular to tangents.
- *Backlund Curves (Constant torsion curves) :* A pair of curves that occur as a result of the Backlund transformation, known as pairs of constant torsion curves, but in this paper, we will call them Backlund curves. In classical differential geometry, a Backlund map transform a new surface with constant negative Gauss curvature surface into a new surface with constant negative Gauss curvature. Backlund transformation can be restricted to get a transformation that carries constant torsion curves to constant torsion curves [7]. It can be given the relation

$$\beta^*(s) = \beta(s) + \frac{2C}{C^2 + \tau^2}\left((\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N}\right)$$

  where, $\gamma' = C \sin\gamma - \kappa$. The torsions at corresponding points of the Backlund curves are constant and determined by

$$\tau^* = \tau = \frac{\sin\theta}{\lambda}$$

  where θ is the angle between binormals of curves at the corresponding points and λ is the distance between these points . Detailed information on Backlund mates can be found in the articles [7, 29, 30]. Also, non-lightlike Backlund mates in the Lorentzian space can be found in the paper [33].

- *Combescure Curves :* Curves obtained from a curve with the help of Combescure transformation. Using this transformation, some special curves as Bertrand, Mannheim, Salkowski curves can be obtained [10]. These curves will not be dealt with in this study.

In this study, pairs of curves whose vectors connecting corresponding points are always lie in the osculating plane will be examined. We will call these curves osculating mates so that they do not confuse these curves with the concept of the osculating curve given in the literature. There are two important definitions encountered in the literature : **i.** An osculating curve is a plane curve that has the highest order of contact with another curve. [37] **ii.** An osculating curve is a space curve in which the position vector always lie in their osculating plane. [22].

Notice that, the curve $\mathcal{G}$ in the osculating mate $\{\mathcal{G}, \mathcal{G}^*\}$, called the osculating curve, does not mean a curve given in these two definitions. First of all, two curves are discussed in this study, not a single curve and the vector connecting their corresponding points is in the osculating plane.

3. Osculating Mates in the Euclidean 3-space

Now let's define a new pair of curves that can generalize to the curve mates mentioned above.

Definition 3.1. Let $\mathcal{G}$ and $\mathcal{G}^*$ be regular space curves defined on an open interval $I \subset \mathbb{R}$ for each $s \in I$. Let's consider the map

$$\mathcal{G}^*(s) = \mathcal{G}(s) + \lambda(s)\,\mathbf{u}(s)$$

where λ is the distance function and $\mathbf{u}$ is a unit vector for each $s \in I$. If the vector $\mathbf{u}$ is located on the intersection line of the osculating planes of the curves $\mathcal{G}$ and $\mathcal{G}^*$ and makes an angle γ with the tangent vector fields at the corresponding points of both curves for each $s \in I$, then the pair of curves $\{\mathcal{G}, \mathcal{G}^*\}$ is called the osculating curve pairs or osculating mates. Here, $\mathcal{G}$ and $\mathcal{G}^*$ are called osculating curve and equiosculating curve, respectively. Also, $\mathcal{G}^*(s)$ is called the osculating mate of $\mathcal{G}$.

That is, if the pair of curves $\{\mathcal{G}, \mathcal{G}^*\}$ is an osculating mate, then the followings hold

$$\begin{aligned} &\textbf{i. } \mathcal{G}^*(s) - \mathcal{G}(s) \in \mathbb{OP}_{\mathcal{G}(s)} \cap \mathbb{OP}_{\mathcal{G}^*(s)} \\ &\textbf{ii. } \lambda(s) = |\mathcal{G}(s)\,\mathcal{G}^*(s)| \\ &\textbf{iii. } \langle \mathbf{T}(s), \mathbf{u}(s)\rangle = \langle \mathbf{T}^*(s), \mathbf{u}(s)\rangle = \cos\gamma(s) \end{aligned}$$

where $\mathbb{OP}_{\mathcal{G}(s)}$ and $\mathbb{OP}_{\mathcal{G}^*(s)}$ are the osculating planes of $\mathcal{G}$ and $\mathcal{G}^*$ at the points $\mathcal{G}(s)$ and $\mathcal{G}^*(s)$, respectively.

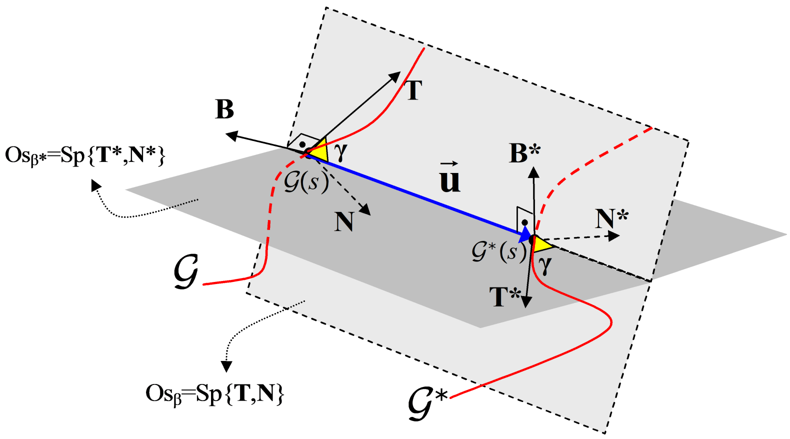

Figure 1. Osculating curve mate

Theorem 3.1. *Let $\mathcal{G}^*$ be the osculating mate of the curve $\mathcal{G}$. If the tangent and normal vector fields of the curve $\mathcal{G}(s)$ are $\mathbf{T}(s)$ and $\mathbf{N}(s)$, it can be written as*

$$\mathcal{G}^*(s) = \mathcal{G}(s) + \lambda(s)\left(\mathbf{T}(s)\cos\gamma(s) + \mathbf{N}(s)\sin\gamma(s)\right) \tag{3.1}$$

where $\lambda(s) \neq 0$ is the distance function between these two curves. Also, there is the equality

$$(\cos\gamma(s))(v^*(s) - v(s)) = \lambda' \tag{3.2}$$

between the corresponding points of the curves $\mathcal{G}$ and $\mathcal{G}^*$ where $v = \|\mathcal{G}'\|$ and $v^* = \|\mathcal{G}^{*\prime}\|$.

Proof. If $\mathcal{G}^*$ is the osculating mate of $\mathcal{G}$, then the vector $\lambda u(s) = \mathcal{G}^*(s) - \mathcal{G}(s)$ will make an angle $\gamma(s)$ with tangent vector fields $\mathbf{T}(s)$ and $\mathbf{T}^*(s)$ along the curves $\mathcal{G}$ and $\mathcal{G}^*$. On the other hand, considering that the unit vector $u(s)$ lie in the osculating plane, we can write

$$\mathbf{u} = (\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N}$$

and

$$\mathcal{G}^* = \mathcal{G} + \lambda\left((\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N}\right).$$

Let's take the derivative of this equation with respect to s. Using the Frenet formulas, we obtain

$$\begin{array}{l} v^*\mathbf{T}^* = (v + \lambda'\cos\gamma - \gamma'\lambda\sin\gamma - v\kappa\lambda\sin\gamma)\,\mathbf{T} \\ \quad + (\lambda'\sin\gamma + \lambda v\kappa\cos\gamma + \gamma'\lambda\cos\gamma)\,\mathbf{N} + (\lambda v\tau\sin\gamma)\,\mathbf{B} \end{array}. \tag{3.3}$$

Taking the dot product of both sides by the vector u, we have

$$\begin{aligned} v^*\cos\gamma &= (v + \lambda'\cos\gamma - \gamma'\lambda\sin\gamma - v\kappa\lambda\sin\gamma)\cos\gamma \\ &\quad + (\lambda'\sin\gamma + \lambda v\kappa\cos\gamma + \gamma'\lambda\cos\gamma)\sin\gamma \\ &= \lambda' + v\cos\gamma. \end{aligned}$$

Therefore, we obtain the equality

$$(\cos\gamma)(v^* - v) = \lambda'.$$

□

Corollary 3.1. *$\{\mathcal{G}, \mathcal{G}^*\}$ is an osculating mate if and only if $(\cos\gamma(s))(v^*(s) - v(s)) = \lambda'$ except for $\gamma \neq \pi/2$.*

Corollary 3.2. *If $\gamma \neq \pi/2$, the distance between the corresponding points of the curves $\mathcal{G}$ and $\mathcal{G}^*$ is constant if and only if $v^* = v$.*

Corollary 3.3. *If the angle γ is $\pi/2$ along the curves, since $\lambda' = 0$, the distance between the corresponding points of the curves $\mathcal{G}$ and $\mathcal{G}^*$ is*

$$d = |\mathcal{G}^*(s) - \mathcal{G}(s)| = |\lambda|$$

will be constant along curves. This does not mean that $v^* = v$.

4. The Relation Between Frenet Apparatus of Pair of Osculating Curves

In this section, first of all, the relationship between Frenet frames of osculating curve and its mate will be examined. Also the curvature and the torsion of these curves will be given. Besides, it will be examined whether the distance function between the corresponding points of the curves $\mathcal{G}$ and is constant or not. Besides, whether the angle γ is constant or not will be discussed together with special cases as in the cases γ constant, $\gamma = \pi/2$, or $\gamma = 0$. It will be investigated in which specific cases to obtain a pair of curves in Mannheim or a Bertrand, or a Backlund.

Theorem 4.1. *Let $\mathcal{G}$ be a regular curve and $\mathcal{G}^*$ be the osculating mate of $\mathcal{G}$ given by the equation*

$$\mathcal{G}^* = \mathcal{G} + \lambda\left((\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N}\right).$$

Let the Frenet vector fields of the curves $\mathcal{G}$ and $\mathcal{G}^*$ be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*\}$, respectively. If θ is the angle between the reciprocal binormals of the curves and $\mu = 1 - \cos\theta$, we have the matrix equality

$$\begin{bmatrix} \mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^* \end{bmatrix} = \begin{bmatrix} 1 - \mu\sin^2\gamma & \mu\sin\gamma\cos\gamma & -\sin\gamma\sin\theta \\ \mu(\sin\gamma\cos\gamma) & 1 - \mu\cos^2\gamma & \cos\gamma\sin\theta \\ \sin\theta\sin\gamma & -\sin\theta\cos\gamma & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{4.1}$$

Proof. Since the vector field

$$\mathbf{u} = (\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N}$$

lies in the osculating plane, it is perpendicular to the binormal vectors $\mathbf{B}$ and $\mathbf{B}^*$. Therefore, the vector fields $\mathbf{v} = \mathbf{B} \times \mathbf{u}$ and $\mathbf{v}^* = \mathbf{B}^* \times \mathbf{u}$ lie on the osculating planes of the curves. Also, each of the sets $\{\mathbf{u}, \mathbf{v}^*, \mathbf{B}^*\}$ and $\{\mathbf{u}, \mathbf{v}, \mathbf{B}\}$ are orthonormal frames for the curves $\mathcal{G}$ and $\mathcal{G}^*$. Accordingly, we can write the matrix equations

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \end{bmatrix} \tag{4.2}$$

and

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v}^* \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \mathbf{T}^* \\ \mathbf{N}^* \end{bmatrix}. \tag{4.3}$$

Let θ be the angle between $\mathbf{B}^*$ and $\mathbf{B}$. Since all of the vectors $\mathbf{B}^*$, $\mathbf{v}^*$, $\mathbf{B}$, $\mathbf{v}$ are perpendicular to the vector $\mathbf{u}$, if the equality (4.2) is used, we obtain

$$\begin{aligned} \mathbf{B}^* &= -(\sin\theta)\,\mathbf{v} + (\cos\theta)\,\mathbf{B} \\ &= -(\sin\theta)\,(-\sin\gamma\mathbf{T} + \cos\gamma\mathbf{N}) + (\cos\theta)\,\mathbf{B} \\ &= (\sin\theta\sin\gamma)\,\mathbf{T} + (-\sin\theta\cos\gamma)\,\mathbf{N} + (\cos\theta)\,\mathbf{B} \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}^* &= (\cos\theta)\,\mathbf{v} + (\sin\theta)\,\mathbf{B} \\ &= (\cos\theta)\,(-\sin\gamma\mathbf{T} + \cos\gamma\mathbf{N}) + (\sin\theta)\,\mathbf{B} \\ &= (-\cos\theta\sin\gamma)\,\mathbf{T} + (\cos\theta\cos\gamma)\,\mathbf{N} + (\sin\theta)\,\mathbf{B}. \end{aligned}$$

Therefore, using the equalities (4.3) and (4.2), we have

$$\begin{aligned} \begin{bmatrix} \mathbf{T}^* \\ \mathbf{N}^* \end{bmatrix} &= \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v}^* \end{bmatrix} \\ &= \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} (\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N} \\ -(\cos\theta\sin\gamma)\,\mathbf{T} + (\cos\theta\cos\gamma)\,\mathbf{N} + (\sin\theta)\,\mathbf{B} \end{bmatrix} \\ &= \begin{bmatrix} \left(1 + \sin^2\gamma\,(\cos\theta - 1)\right)\mathbf{T} + \sin\gamma\cos\gamma\,(1 - \cos\theta)\,\mathbf{N} - (\sin\gamma\sin\theta)\,\mathbf{B} \\ (\sin\gamma\cos\gamma)\,(1 - \cos\theta)\,\mathbf{T} + \left(1 + \cos^2\gamma\,(\cos\theta - 1)\right)\mathbf{N} + (\cos\gamma\sin\theta)\,\mathbf{B} \end{bmatrix}. \end{aligned}$$

As a conclusion, the vector fields $\mathbf{T}^*, \mathbf{N}^*$ and $\mathbf{B}^*$ can be written

$$\begin{bmatrix} \mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^* \end{bmatrix} = \begin{bmatrix} 1 - \mu\sin^2\gamma & \mu\sin\gamma\cos\gamma & -\sin\gamma\sin\theta \\ \mu\,(\sin\gamma\cos\gamma) & 1 - \mu\cos^2\gamma & \cos\gamma\sin\theta \\ \sin\theta\sin\gamma & -\sin\theta\cos\gamma & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

where $1 - \mu = \cos\theta$. □

4.1. Curvature and Torsion of The Osculating Curve $\mathcal{G}$

Theorem 4.2. *Let $\{\mathcal{G}, \mathcal{G}^*\}$ be an Osculating curve mate with the relation*

$$\mathcal{G}^* = \mathcal{G} + \lambda\left((\cos\gamma)\,\mathbf{T} + (\sin\gamma)\,\mathbf{N}\right)$$

where $\gamma \neq 0$, $\gamma \neq \pi/2$. If the angle between binormal of the corresponding points of $\mathcal{G}$ and $\mathcal{G}^*$ is $\theta \neq 0$, then the torsion and the curvature of the curve $\mathcal{G}\,(s)$ is

$$\tau = \frac{-v^*\sin\theta}{v\lambda} \qquad \textit{and} \qquad \kappa = \frac{(v - v^*\cos\theta)\sin\gamma - \gamma'\lambda}{v\lambda}, \tag{4.4}$$

respectively, where $\mu = 1 - \cos\theta$.

Proof. If we compare the equality (3.3) with

$$\mathbf{T}^{*}=\left(1-\mu\sin^{2}\gamma\right)\mathbf{T}+\mu\sin\gamma\cos\gamma\mathbf{N}-\left(\sin\gamma\sin\theta\right)\mathbf{B},$$

we obtain

$$\begin{cases} v^{*}\left(1-\mu\sin^{2}\gamma\right)=v+\lambda'\cos\gamma-\gamma'\lambda\sin\gamma-v\kappa\lambda\sin\gamma \\ v^{*}\mu\sin\gamma\cos\gamma=\lambda'\sin\gamma+\lambda v\kappa\cos\gamma+\gamma'\lambda\cos\gamma \\ -v^{*}\sin\gamma\sin\theta=\lambda v\tau\sin\gamma \end{cases}. \tag{4.5}$$

In the case $\gamma\neq 0,\pi/2$,considering $(v^{*}-v)\cos\gamma=\lambda'$ and $\mu=1-\cos\theta$, we find

$$\tau=\frac{-v^{*}\sin\theta}{v\lambda}$$

and

$$\kappa=\frac{(v-v^{*}\cos\theta)\sin\gamma-\gamma'\lambda}{v\lambda}.$$

□

Proposition 4.1. *Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T}^{*},\mathbf{N}^{*},\mathbf{B}^{*}\}$ be the Frenet frames of the Osculating curve $\mathcal{G}$ and its mate $\mathcal{G}^{*}$ at the corresponding points. Then we have the matrix equalities*

$$\begin{bmatrix}\mathbf{T}^{*}\\ \mathbf{N}^{*}\\ \mathbf{B}^{*}\end{bmatrix}=\begin{bmatrix}1-\mu\sin^{2}\gamma & \mu\sin\gamma\cos\gamma & -\sin\gamma\sin\theta\\ \mu\left(\sin\gamma\cos\gamma\right) & 1-\mu\cos^{2}\gamma & \cos\gamma\sin\theta\\ \sin\theta\sin\gamma & -\cos\gamma\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix},$$

and

$$\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\cos\theta+\mu\cos^{2}\gamma & \mu\cos\gamma\sin\gamma & \sin\gamma\sin\theta\\ \mu\cos\gamma\sin\gamma & 1-\mu\cos^{2}\gamma & -\cos\gamma\sin\theta\\ -\sin\theta\sin\gamma & \cos\gamma\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{T}^{*}\\ \mathbf{N}^{*}\\ \mathbf{B}^{*}\end{bmatrix}.$$

Here, the matrix

$$R_{\theta}=\begin{bmatrix}1-\mu\sin^{2}\gamma & \mu\cos\gamma\sin\gamma & -\sin\theta\sin\gamma\\ \mu\cos\gamma\sin\gamma & 1-\mu\cos^{2}\gamma & \cos\gamma\sin\theta\\ \sin\theta\sin\gamma & -\cos\gamma\sin\theta & \cos\theta\end{bmatrix}$$

is a rotation matrix where it rotates a vector through the angle θ about the axis $\overrightarrow{\mathbf{w}}=(\cos\gamma,\sin\gamma,0)$. The unit quaternion q corresponds to R_{θ} is

$$q=\cos\left(\frac{\theta}{2}\right)-\mathbf{i}\sin\left(\frac{\theta}{2}\right)\cos\gamma-\mathbf{j}\sin\left(\frac{\theta}{2}\right)\sin\gamma.$$

Proof. It is well known that a unit quaternion $q=q_1+q_2\mathbf{i}+q_3\mathbf{j}+q_4\mathbf{k}$ corresponds to the rotation matrix,

$$R=\begin{bmatrix}q_1^2+q_2^2-q_3^2-q_4^2 & -2q_1q_4+2q_2q_3 & 2q_1q_3+2q_2q_4\\ 2q_1q_4+2q_2q_3 & q_1^2-q_2^2+q_3^2-q_4^2 & -2q_1q_2+2q_3q_4\\ -2q_1q_3+2q_2q_4 & 2q_1q_2+2q_3q_4 & q_1^2-q_2^2-q_3^2+q_4^2\end{bmatrix} [1].$$

In the case, $q_1\neq 0$, we have the equalities,

$$\begin{aligned} q_1^2 &= \frac{1}{4}\left(1+R_{11}+R_{22}+R_{33}\right), \quad q_2=\frac{1}{4q_1}\left(\widehat{R}_{32}-\widehat{R}_{23}\right),\\ q_3 &= -\frac{1}{4q_1}\left(\widehat{R}_{13}-\widehat{R}_{31}\right) \qquad q_4=\frac{1}{4q_1}\left(\widehat{R}_{21}-\widehat{R}_{12}\right). \end{aligned}$$

Therefore, using these equalities, we obtain $q_1=\pm\cos(\frac{\theta}{2})$, $q_2=-\sin(\frac{\theta}{2})\cos\gamma$, $q_3=-\sin(\frac{\theta}{2})\sin\gamma$ and $q_4=0$. □

Corollary 4.1. *For a given rotation matrix R in 3-dimensional Euclidean space, the rotation angle t is determined by the equation*

$$\cos t=\frac{\mathrm{Tr}R-1}{2}.$$

Accordingly, we have $t=\theta$, since

$$\cos t=\frac{1-\mu\sin^{2}\gamma+1-\mu\cos^{2}\gamma+\cos\theta-1}{2}=\frac{1-\mu+\cos\theta}{2}=\cos\theta.$$

Theorem 4.3. *Let $\{\mathcal{G}, \mathcal{G}^*\}$ be an Osculating curve mate with the relation*

$$\mathcal{G}^* = \mathcal{G} + \lambda \left((\cos\gamma)\, \mathbf{T} + (\sin\gamma)\, \mathbf{N} \right)$$

for $\gamma \neq 0, \pi/2$. If the angle between binormal of the corresponding points of $\mathcal{G}$ and $\mathcal{G}^*$ is $\theta \neq 0$, then the torsion and the curvature of the curve $\mathcal{G}^*(s)$ is

$$\begin{aligned} \tau^* &= \frac{\theta'}{\upsilon^* \cos\gamma} - \frac{\sin\theta}{\lambda} \\ \kappa^* &= -\frac{\gamma'}{\upsilon^*} - \frac{\theta'}{\upsilon^*} \tan\gamma \cot\theta - \frac{1-\cos\theta}{\lambda} \sin\gamma. \end{aligned} \tag{4.6}$$

Moreover, there is the relation

$$\lambda = \frac{1-\cos\theta}{\sin\theta} + c$$

between the angle θ and the distance function λ, where $c \in \mathbb{R}$.

Proof. Let's take derivative of $\mathbf{T}^*$ given in (4.1). Using the Frenet formulas we obtain

$$\begin{aligned} \upsilon^* \kappa^* \mathbf{N}^* = \; & -\left(\left(\mu' \sin^2\gamma \right) + \left(\gamma' \mu \sin 2\gamma \right) + \mu\upsilon\kappa \sin\gamma \cos\gamma \right) \mathbf{T} \\ & + \left(\frac{\mu'}{2} \sin 2\gamma + \mu\gamma' \cos 2\gamma + \left(1 - \mu \sin^2\gamma \right) \upsilon\kappa + (\sin\gamma \sin\theta)\, \upsilon\tau \right) \mathbf{N} \\ & + \left(-(\gamma' \cos\gamma \sin\theta) - (\theta' \sin\gamma \cos\theta) + \mu\upsilon\tau \sin\gamma \cos\gamma \right) \mathbf{B}. \end{aligned} \tag{4.7}$$

If we take the inner product of each side by $\mathbf{T}$, we obtain

$$\upsilon^* \kappa^* \mu \sin\gamma \cos\gamma = -\mu' \sin^2\gamma - \gamma' \mu \sin 2\gamma - \mu\upsilon\kappa \sin\gamma \cos\gamma \tag{4.8}$$

since $\langle \mathbf{N}^*, \mathbf{T} \rangle = \mu \sin\gamma \cos\gamma$. Therefore, we get

$$\upsilon^* \kappa^* = -\frac{\mu'}{\mu} \tan\gamma - 2\gamma' - \upsilon\kappa$$

for $\gamma \neq 0$ and $\gamma \neq \pi/2$. Moreover, in the case $\theta \neq 0$, we write $\mu = 1 - \cos\theta$, $\mu' = \theta' \sin\theta$ and

$$\frac{\mu'}{\mu} = \frac{\theta' \sin\theta}{1 - \cos\theta} = \frac{\theta' (1 + \cos\theta)}{\sin\theta}.$$

Using these equalities, we obtain

$$\upsilon^* \kappa^* = \frac{-\theta' (1 + \cos\theta) \tan\gamma}{\sin\theta} - 2\gamma' - \upsilon\kappa. \tag{4.9}$$

If the inner product of both sides of the equation (4.7) is taken by the vector field $\mathbf{B}$, we get

$$\upsilon^* \kappa^* = -\gamma' - \theta' \tan\gamma \cot\theta + \mu\upsilon\tau \frac{\sin\gamma}{\sin\theta}$$

since $\langle \mathbf{N}^*, \mathbf{B} \rangle = \cos\gamma \sin\theta$. Let's write $\tau = \dfrac{-\upsilon^* \sin\theta}{\upsilon\lambda}$ in this equation. Therefore, we have

$$\upsilon^* \kappa^* = -\gamma' - \theta' \tan\gamma \cot\theta - \frac{\mu\upsilon^* \sin\gamma}{\lambda}. \tag{4.10}$$

Also, using the equalities (4.9) and (4.10), we find the equality

$$\frac{-\theta' \tan\gamma}{\sin\theta} - \gamma' - \upsilon\kappa = -\frac{(1 - \cos\theta)\, \upsilon^* \sin\gamma}{\lambda}.$$

If we write the curvature κ found in the Theorem 4.2, we get

$$\frac{\lambda\theta'}{\sin\theta} = (\upsilon^* - \upsilon) \cos\gamma.$$

Thus, considering the equality $(\cos\gamma)(v^* - v) = \lambda'$, we obtain the separable differential equation

$$\frac{\theta'}{\sin\theta} = \frac{\lambda'}{\lambda},$$

and we find

$$\lambda = \frac{1-\cos\theta}{\sin\theta} + c \tag{4.11}$$

with $c \in \mathbb{R}$. The relation (4.11) indicates that when θ is constant, λ will also be constant. Now, let's take derivative of $\mathbf{B}^*$ given in given in (4.1). Then we obtain

$$\begin{aligned} -v^*\tau^*\mathbf{N}^* &= \theta'(\cos\theta\sin\gamma)\mathbf{T} + \gamma'(\sin\theta\cos\gamma)\mathbf{T} + v\kappa\sin\theta\cos\gamma\mathbf{T} \\ &- \theta'(\cos\theta\cos\gamma)\mathbf{N} + \gamma'\sin\theta\sin\gamma\mathbf{N} + (\sin\theta\sin\gamma)(v\kappa\mathbf{N}) - (\cos\theta)(v\tau\mathbf{N}) \\ &- \theta'(\sin\theta)\mathbf{B} - v\tau\sin\theta\cos\gamma\mathbf{B}. \end{aligned}$$

Using the equality $\langle\mathbf{N}^*, \mathbf{B}\rangle = \cos\gamma\sin\theta$, we obtain

$$\tau^* = \frac{\theta' + v\tau\cos\gamma}{v^*\cos\gamma} \tag{4.12}$$

for $\theta \neq 0$. Moreover, if we use the equality $\tau = \dfrac{-v^*\sin\theta}{v\lambda}$, we have

$$\tau^* = \frac{\theta'}{v^*\cos\gamma} - \frac{\sin\theta}{\lambda}.$$

□

Remark 4.1. Finding the pair of curves satisfying the above conditions requires to solve of difficult and nonlinear differential equations. This is another research topic. However, in some special cases of the above conditions, normal, Backlund and Bertrand curve pairs are obtained. For example, Backlund curves, which are constant torsion curves and known in the literature as Backlund transformation for curves, and which is an important tool for in soliton theory and integrable systems, can be found in many sources. A few of them are given in the references [36, 29, 30, 7].

Let us now see which pairs of known curves are obtained in which particular cases of the pair of osculating curves. For this, we will examine two main cases according to whether the angle θ is zero or not.

4.2. *Special Cases for $\theta \neq 0$.*

Let $\{\mathcal{G}, \mathcal{G}^*\}$ be an Osculating curve mate. If the angle between binormal of the corresponding points of $\mathcal{G}$ and $\mathcal{G}^*$ is $\theta \neq 0$, we have the relation

$$\mathcal{G}^* = \mathcal{G} + \left(\tfrac{1-\cos\theta}{\sin\theta} + c\right)((\cos\gamma)\mathbf{T} + (\sin\gamma)\mathbf{N})$$

for $\gamma \neq 0, \pi/2$. Torsions and curvatures of $\mathcal{G}$ and $\mathcal{G}^*$are

$$\tau = \frac{-v^*\sin\theta}{v\lambda}, \qquad \tau^* = \frac{\theta'}{v^*\cos\gamma} - \frac{\sin\theta}{\lambda}$$

$$\kappa = \frac{(v - v^*\cos\theta)\sin\gamma - \gamma'\lambda}{v\lambda}, \quad \kappa^* = -\frac{\gamma'}{v^*} - \frac{\theta'}{v^*}\tan\gamma\cot\theta - \frac{1-\cos\theta}{\lambda}\sin\gamma.$$

Now, let's examine spacial cases.

Case 1. **θ is constant and γ is non-constant with** $\gamma \neq 0, \gamma \neq \pi/2$.

We show that when θ is constant, λ will also be constant and, $\lambda = \dfrac{1-\cos\theta}{\sin\theta} + c$. Also, if $\gamma \neq \pi/2$, the equality $(\cos\gamma)(v^* - v) = \lambda'$ implies $v^* = v$. So, in the case θ is constant, then the torsions and the curvatures of and $\mathcal{G}$ and $\mathcal{G}^*$ will be

$$\tau^* = \tau = -\frac{\sin\theta}{\lambda}, \ \kappa = -\frac{\gamma'}{v} + \frac{\mu\sin\gamma}{\lambda} \text{ and}, \ \kappa^* = -\frac{\gamma'}{v} - \frac{\mu\sin\gamma}{\lambda}.$$

That is, we have the relation

$$\kappa = \kappa^* + (1 - c\sin\theta)\sin\gamma.$$

Thus, $\mathcal{G}$ and $\mathcal{G}^*$ are a Backlund mate. The transformation between $\mathcal{G}$ and $\mathcal{G}^*$ is a Backlund transformation.

Case 2. θ **and** $\gamma \neq \pi/2$ **are constant**
If both θ and γ are constant, we have

$$\tau = \tau^* = -\frac{\sin\theta}{\lambda} \quad \text{and} \quad \kappa = -\kappa^* = \frac{\mu\sin\gamma}{\lambda}.$$

But, for a space curve $\kappa = -\kappa^*$ if and only if $\kappa = \kappa^* = 0$. It means that $\gamma = 0$. Therefore, $\mathcal{G}, \mathcal{G}^*$ are lines, and the Frenet frame of $\mathcal{G}^*$ can be obtained by rotating the Frenet frame of $\mathcal{G}$ about the tangent vector by the angle θ and it can written

$$\begin{bmatrix} \mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

Case 3. $\theta \neq 0$ **and** $\gamma = \pi/2$. **(In the case** $\{\mathcal{G}, \mathcal{G}^*\}$ **is a Bertrand mate)**
If the angle γ is $\pi/2$, then we have $\lambda' = (\cos\gamma)(v^* - v) = 0$. That is, λ is constant and we write

$$\mathcal{G}^* = \mathcal{G} + \lambda\mathbf{N}.$$

On the other hand, according to the equations (3.3) and (4.1), we find

$$v^*\mathbf{T}^* = (v - v\kappa\lambda)\,\mathbf{T} + v\lambda\tau\mathbf{B}$$

and

$$\begin{bmatrix} \mathbf{T}^* \\ \mathbf{N}^* \\ \mathbf{B}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

Therefore, since $\mathcal{G}^* = \mathcal{G} + \lambda\mathbf{N}$, $\lambda \in \mathbb{R}$ and $\mathbf{N}^* = \mathbf{N}$, $\{\mathcal{G}, \mathcal{G}^*\}$ is a Bertrand mate. Also, using the Theorems given above, we get

$$\kappa = \frac{v - v^*\cos\theta}{v\lambda} \quad \text{and} \quad \tau = \frac{-v^*\sin\theta}{v\lambda}. \tag{4.13}$$

for Bertrand curve $\mathcal{G}$. With the similar calculations for $\mathcal{G} = \mathcal{G}^* - \lambda\mathbf{N}^*$, we can obtain that

$$\kappa^* = \frac{v\cos\theta - v^*}{v^*\lambda} \quad \text{and} \quad \tau^* = \frac{-v\sin\theta}{v^*\lambda}.$$

Thus, we get the following well known propositions for Bertrand mates.
i. The angle between tangent vectors of Bertrand curves at the corresponding point is constant θ, since the coefficients of B and T must be zero in the derivative of $\mathbf{B}^*$.
ii. Product of the torsions of Bertrand curves at the corresponding points is constant, since

$$\tau\tau^* = \frac{\sin^2\theta}{\lambda^2}.$$

iii. A curve for which $a\kappa + b\tau = 1$, where a and b are constants different form zero, is Bertrand curve. Notice that, it can be obtained

$$\cot\theta = \frac{\kappa\lambda - 1}{\tau\lambda} \Leftrightarrow \kappa\lambda - (\cot\theta)\,\lambda\tau = 1$$

from (4.13). It means that, $a = \lambda$ and $b = -\lambda\cot\theta$.

4.3. Special Cases for $\theta = 0$.

Case 4 (1). $\theta = 0$ **and** γ **is non-constant or** $\theta = 0$ **and** γ **is constant**
If $\theta = 0$, the Frenet frames at corresponding points of the curves are the same. On the other hand, the torsion of the curves are $\tau = \tau^* = 0$. So, the curves $\mathcal{G}$ and $\mathcal{G}^*$ are planar and lie in the osculating plane. Also, if we write $\mu = 0$, $\lambda' = 0$, $v^* = v$ in the equalities (4.5), we find

$$\begin{cases} 0 = -\lambda\,(\gamma' + v\kappa)\sin\gamma \\ 0 = \lambda\,(v\kappa + \gamma')\cos\gamma \end{cases}.$$

The necessary and sufficient condition to satisfy these two equations is that $\lambda = 0$ or $\gamma' = -v\kappa$. $\lambda = 0$ means that $\mathcal{G} = \mathcal{G}^*$. So, for an osculating mate $\{\mathcal{G}, \mathcal{G}^*\}$ such that $\theta = 0$ and $\lambda \neq 0$, curvatures are equal and satisfy

$$\kappa = \kappa^* = -\frac{\gamma'}{v}.$$

If the angle γ is also chosen constant, there will be a straight lines for Osculating mate.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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