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On Some Identities Involving Cauchy Products of Central Delannoy Numbers

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ABSTRACT: In this paper, we have examined Cauchy products of central Delannoy numbers. Moreover, using their recurrence relation we have derived some important identities such as the Cassini and Catalan's identities which contain these products.

Keywords: Recurrences, Delannoy numbers, Cauchy product, Identities

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INTRODUCTION

Delannoy Numbers, $d_{i,j}$, were defined in the 19th century by H. Delannoy using the following equation for $i, j \in \mathbb{Z}$

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1}\ .$$

$d_{0,0} = 1$ and for negative values i or j it is $d_{i,j} = 0$. The number $d_{i,j}$ gives geometrically the number of lattices that can be drawn from point (0,0) to point (i, j). For detailed information about these numbers, the studies of (Sulanke, 2003; Banderier et al, 2005; Wang et al, 2019; Deveci at al, 2021) can be viewed. Diagonal elements $d_{n,n}$ are called central Delannoy numbers while $n \geq 0$.

Asymmetric Delannoy numbers $\tilde{d}_{m,n}$ give starting from the origin point to $(m, n+1)$ the number paths. Some asymmetric Delannoy numbers can be listed as

$$\{\tilde{d}_{0,n}\}_{n\geq 0} = \{1, 2, 4, 8, 16, 32, \cdots\},$$
$$\{\tilde{d}_{1,n}\}_{n\geq 0} = \{1, 3, 8, 20, 48, 112, \cdots\},$$
$$\{\tilde{d}_{2,n}\}_{n\geq 0} = \{1, 4, 13, 38, 104,272, \cdots\}.$$

The numbers $\tilde{d}_{m,m} = D(m)$ are called central Delannoy numbers (Qi, 2019). The generating function of these numbers is given by

$$G(x) = \sum_{n=0}^{\infty} D(n)x^k = 1 + 3\mathrm{x} + 13x^2 + 63x^3 + \cdots = \frac{1}{\sqrt{1-6x+x^2}}\ .$$

For $k \in \mathbb{N}$, these numbers are calculated by the following determinant (Qi et al, 2016).

$$D(k) = (-1)^k \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ a_{k-2} & a_{k-3} & a_{k-4} & \cdots & a_1 & 1 & 0 \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_2 & a_1 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}.$$

Here the terms a_k are

$$a_k = \frac{(-1)^{k+1}}{6^k} \sum_{l=1}^{k} (-1)^l 6^{2l} \frac{(2l-3)!!}{(2l)!!} \binom{l}{k-l}\ .$$

When the above calculation is continued, the elements of the sequence $D(k)$ can be listed and some central Delannoy numbers, for $1 \leq k \leq 11$, as follows:

$$\{\ 1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, \ldots\}.$$

Since the Cauchy product of the two sequences such as (a_n), (b_n) is defined by a discrete convolution. For the central Delannoy numbers, this product can be also defined as

$$c_k = \sum_{l=0}^{k} D(l)D(k-l).$$

In (Qi et al, 2016)], the authors have given the following useful identities, including explicit formulas that give these numbers with the help of the generating function of central Delannoy numbers $D(k)$ and Cauchy products of these numbers.

i) For $k \geq 0$, the Cauchy product of the numbers $D(k)$ is calculated as

$$\sum_{l=0}^{k} D(l)D(k-l) = \frac{(-1)^k}{6^k}\sum_{l=0}^{k}(-1)^l 6^{2l}\binom{l}{k-l}.$$

ii) For $k \geq 2$, the Cauchy product of central Delannoy numbers provides the following property:

$$\sum_{l=0}^{k} D(l)D(k-l) - 6\sum_{l=0}^{k-1} D(l)D(k-l-1) + \sum_{l=0}^{k-2} D(l)D(k-l-2) = 0\,.$$

iii) For $k \geq 1$, the Cauchy product of central Delannoy numbers can be calculated by the following k order tridiagonal determinant:

$$(-1)^k\begin{vmatrix} -6 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -6 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -6 & 1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \ddots & . & . & . \\ . & . & . & . & \ddots & . & . & . \\ . & . & . & . & \ddots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & -6 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -6 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -6 \end{vmatrix}.$$

MATERIAL AND METHODS

Since each recurrence relation can be expressed as a difference equation, some authors use difference equations and recurrence relations interchangeably. The solution method of difference equations is used in the solution of recurrence relations. The coefficient of x^n in the power series of the function $g(x)$ obtained as the generating function for the given repeated relation can be obtained $g(x)$ can be solved algebraically, and $g(x)$ is expressed as a power series to obtain the term a_n. In other words, the recurrence relations can be solved with the help of the corresponding generating function. In [Qi(2019)], the matrix $M_k(c)$ related to central Delannoy numbers is defined and given generating function for the numbers $D_k(c)$. In this section, with the properties of the recurrence relations, we give some important identities. For $c \in \mathbb{C}$ and $k \in \mathbb{N}$, this matrix is

$$\mathrm{M_k(c)} = \begin{pmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \ddots & . & . & . \\ . & . & . & . & \ddots & . & . & . \\ . & . & . & . & \ddots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & c & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{pmatrix}_{k\times k}.$$

Then, using the determinant of the matrix $M_k(c)$ we get the following sequence

$$\{D_k(c)\}_{k\geq 1} = \{c, c^2-1, c^3-2c,\ c^4-3c^2+1, \cdots, D_n(c), \cdots\}.$$

Note that the terms of the above sequence satisfy in the below relation

$$D_k(c) = cD_{k-1}(c) - D_{k-2}(c),\ \ k \geq 2.$$

For the roots of the characteristic equation, $\alpha + \beta = c$, $\alpha\beta = 1$ can be written. For $k \geq 0$, using the generating function definition the following equations can be written.

$\sum_{k=0}^{\infty} D_k(c)x^k = D_0(c) + D_1(c)x + D_2(c)x^2 + \cdots,$
$x^2 \sum_{k=0}^{\infty} D_k(c)x^k = D_0(c)x^2 + D_1(c)x^3 + D_2(c)x^4 + \cdots,$
$-cx \sum_{k=0}^{\infty} D_k(c)x^k = -D_0(c)x - D_1(c)x^2 - D_2(c)x^3 - \cdots.$

If the recurrence relation is used and the necessary operations are performed, the generating function can be easily obtained:

$$\sum_{k=0}^{\infty} D_k(c)\, x^k = \frac{1}{x^2 - cx + 1}.$$

In (Qi, 2019), the authors also gave the following recursive formula for $c \in \mathbb{C}$ and $k \geq 0$ with the help of the generating function:

$$D_k(c) = \begin{cases} \dfrac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}, & c \neq \pm 2 \\ k + 1, & c = 2 \\ (-1)^k(k+1), & c = -2. \end{cases}$$

RESULTS AND DISCUSSION

Since recursive relations are very advantageous, we used them to find the Cauchy products and examine the properties of the sequence we just constructed.

As known that the Cassini's identity is a special case of Catalan's identity and gives information about the nth term of the sequence. In the following theorem, we give Cassini's identity involving the terms of the sequence $D_k(c)$.

Theorem 1. For $k \geq 1$, the following identity is true.

$$D_{k-1}(c)D_{k+1}(c) - D_k(c)^2 = -1.$$

Proof. While showing the accuracy of this equation, we use the recurrence relation. Since we need to examine the proof according to the cases c, let's first show the accuracy of this equation when $c \neq \pm 2$. Then,

$$D_{k-1}(c)D_{k+1}(c) - D_k(c)^2 = \left(\frac{2\alpha^{k+1}\beta^{k+1} - \alpha^k\beta^{k+2} - \beta^k\alpha^{k+2}}{(\alpha-\beta)^2}\right),$$
$$D_{k-1}(c)D_{k+1}(c) - D_k(c)^2 = \frac{-1}{(\alpha-\beta)^2}\alpha^k\beta^k(\alpha^2 - 2\alpha\beta + \beta^2),$$
$$D_{k-1}(c)D_{k+1}(c) - D_k(c)^2 = \frac{-1}{(\alpha-\beta)^2}(\alpha\beta)^k(\alpha-\beta)^2 = -1.$$

is obtained. Now, let us show the same equation when $c = 2$. The equations

$$D_{k-1}(2) = k,\ D_{k+1}(2) = k + 2, \quad D_k(2) = k + 1$$

if we write down equations, then

$$D_{k-1}(2)D_{k+1}(2) - D_k(2)^2 = k(k+2) - (k+1)^2 = -1$$

is obtained. For $c = -2$, using the following facts

$$D_{k-1}(-2) = (-1)^{k-1}k,\ D_{k+1}(-2) = (-1)^{k+1}k + 2,\ D_k(-2) = (-1)^k k + 1$$

we get

$$D_{k-1}(-2)D_{k+1}(-2) - D_k(-2)^2 = (-1)^{2k}(k^2 + 2k) - \left((-1)^{2k}(k^2 + 2k + 1)\right) = -1.$$

So, Cassini's identity ensures the following equality for each value c.

$$D_{k-1}(c)D_{k+1}(c) - D_k(c)^2 = -1.$$

For example, $k = 2, 3$ if we examine the last equation respectively, then

$$D_1(c)D_3(c) - D_2(c)^2 = \left(\frac{\alpha^6 - \alpha^2\beta^4 - \beta^2\alpha^4 + \beta^6 - \alpha^6 + 2\alpha^3\beta^3 - \beta^6}{(\alpha-\beta)^2}\right),$$
$$D_1(c)D_3(c) - D_2(c)^2 = \frac{-(\alpha\beta)^2(\alpha-\beta)^2}{(\alpha-\beta)^2} = -1$$

and

$$D_2(c)D_4(c)-D_3(c)^2=\left(\frac{\alpha^8-\alpha^3\beta^5-\beta^3\alpha^5+\beta^8-\alpha^8+2\alpha^4\beta^4-\beta^8}{(\alpha-\beta)^2}\right),$$
$$D_2(c)D_4(c)-D_3(c)^2=\frac{-\alpha^3\beta^3(\beta^2-2\alpha\beta+\alpha^2)}{(\alpha-\beta)^2}=-1$$

is obtained. We would like to specifically point out that the following results are obtained with the help of Cassini's identity. For sequences with positive terms, if the inequality

$${t_k}^2\geq t_{k-1}t_{k+1}$$

is satisfied while $k=1,2,3,...$ such type sequences are called logarithmic concave sequences. Thus, with the help of the Cassini identity, the sequence of Cauchy products of central Delannoy numbers becomes logarithmic concave:

i) ${D_k}^2(c)\geq\ D_{k-1}(c)D_{k+1}(c)$.

ii) Two consecutive $D_k(c)$ numbers are prime between them: $\left(D_k(c)\,,D_{k+1}(c)\right)=1$.

iii) ${D_k}^2(c)-{D_{k-1}}^2(c)-cD_k(c)D_{k-1}(c)=1$.

In the following corollary, we give the positive powers of the roots of the related equation of sequence $D_k(c)$.

Corollary 1. For the roots α and β the following equalities are satisfied.

i) $\alpha^n=\alpha D_{n-1}(c)-D_{n-2}(c)$

ii) $\beta^n=\beta D_{n-1}(c)-D_{n-2}(c)$

Proof. i) From the characteristic equation of the numbers $D_k(c)$, we write $\alpha^2=c\alpha-1$. If we multiply both sides of this equality by α and also

$$D_1(c)=c,D_2(c)=c^2-1,D_3(c)=c^3-2c$$

using the equations, respectively

$$\alpha^3=\alpha D_2(c)-D_1(c)\ \text{ ve }\alpha^4=\alpha D_3(c)-D_2(c)$$

obtained. For $n=k$, we assume the claim is true:

$$\alpha^k=\alpha D_{k-1}(c)-D_{k-2}(c).$$

We look for $n=k+1$, then we have

$$\alpha^{k+1}=\alpha\alpha^k=\alpha\left(\alpha D_{k-1}(c)-D_{k-2}(c)\right)=\alpha D_k(c)-D_{k-1}(c).$$

The other claim can be seen similarly. So, the proof ends.

In the following theorem, we give the Catalan's identity which is a generalization of Cassini's identity.

Theorem 2. For $n\geq k$, the following equality is true.

$$D_{n+k}(c)D_{n-k}(c)-D_n(c)^2=\begin{cases}-{D_{k-1}}^2(c), & c\neq\pm2\\ -k^2, & c=2\ ve\ c=-2\end{cases}$$

Proof. We first show the correctness for $c\neq\pm2$. For this, let's write the following equations using the recurrence relation.

$$D_{n+k}(c)D_{n-k}(c)-D_n(c)^2=\left(\frac{2\alpha^{n+1}\beta^{n+1}-\alpha^{n+k+1}\beta^{n-k+1}-\beta^{n+k+1}\alpha^{n-k+1}}{(\alpha-\beta)^2}\right).$$

We get

$$\frac{\alpha^{n+1}\beta^{n+1}(2-\alpha^k\beta^{-k}-\beta^k\alpha^{-k})}{(\alpha-\beta)^2}.$$

If the relationships between roots are used and the necessary simplification is done, then

$$\frac{1}{(\alpha-\beta)^2}\left(\frac{2\alpha^k\beta^k-\alpha^{2k}-\beta^{2k}}{(\alpha\beta)^k}\right)=-{D_{k-1}}^2(c).$$

For $c=2$, let's give the proof. Then, we get

$$D_{n+k}(2)D_{n-k}(2) - {D_n}^2(2) = (n+k+1)(n-k+1) - (n+1)^2 = -k^2.$$

Finally, for $c = -2$, the value $D_{n+k}(-2)D_{n-k}(-2) - {D_n}^2(-2)$ is equal to this:

$$(-1)^{2n}(n^2 - nk + n + kn - k^2 + k + n - k + 1) - ((-1)^{2n}(n^2 + 2n + 1).$$

Thus, we get

$$D_{n+k}(-2)D_{n-k}(-2) - {D_n}^2(-2) = -k^2.$$

We derive the Vajda's identity provided by the terms of the sequence we are working on in the following theorem.

Theorem 3. For positive numbers n, m, k the following equality is true.

$$D_{n+m}(c)D_{n+k}(c) - D_n(c)D_{n+m+k}(c) = \begin{cases} D_{m-1}(c)D_{k-1}(c), & c \neq \pm 2 \\ mk, & c = 2 \\ (-1)^{m+k}mk, & c = -2 \end{cases}$$

Proof. First, we show that equality is correct, $c \neq \pm 2$. If the recurrence relation is used for the first side of the equation given in the theorem, then the second side of the equation follows.

$$D_{n+m}(c)D_{n+k}(c) - D_n(c)D_{n+m+k}(c) = \frac{\alpha^{n+1}\beta^{n+1}(\beta^{m+k} + \alpha^{m+k} - \alpha^m\beta^k - \alpha^k\beta^m)}{(\alpha - \beta)^2},$$

$$D_{n+m}(c)D_{n+k}(c) - D_n(c)D_{n+m+k}(c) = \frac{\alpha^m(\alpha^k - \beta^k) - \beta^m(\alpha^k - \beta^k)}{(\alpha - \beta)^2} = \frac{(\alpha^k - \beta^k)(\alpha^m - \beta^m)}{(\alpha - \beta)^2},$$

$$D_{n+m}(c)D_{n+k}(c) - D_n(c)D_{n+m+k}(c) = D_{m-1}(c)D_{k-1}(c).$$

We show the correctness for $c = 2$:

$D_{n+m}(2)D_{n+k}(2) - D_n(2)D_{n+m+k}(2)$ is equal to this

$$(n+m+1)(n+k+1) - (n+1)(n+m+k+1) = mk.$$

And so, from this fact we get

$$D_{n+m}(2)D_{n+k}(2) - D_n(2)D_{n+m+k}(2) = mk.$$

Finally, $c = -2$, let's show that the claim is true.

$D_{n+m}(-2)D_{n+k}(-2) - D_n(-2)D_{n+m+k}(-2)$ is

$$(-1)^{n+m}(n+m+1)(-1)^{n+k}(n+k+1) - (-1)^n(n+1)(-1)^{n+m+k}(n+m+k)$$
$$= (-1)^{2n+m+k}(mk) = (-1)^{m+k}(mk) = (-1)^{m+k}mk.$$

Thus, the proof is completed.

Teorem 4 (D'Ocagne's identity). For $n \geq -1$ and $m \geq -1$, we have

$$D_m(c)D_{n+1}(c) - D_n(c)D_{m+1}(c) = \begin{cases} \dfrac{\alpha^m\beta^n - \alpha^n\beta^m}{\alpha - \beta}, & c \neq \pm 2 \\ m - n, & c = 2 \\ (-1)^{m+n+1}m - n, & c = -2 \end{cases}$$

Proof. First, let's show the truth of the equation for the case $c \neq \pm 2$.

$D_m(c)D_{n+1}(c) - D_n(c)D_{m+1}(c)$ equal to the following value**:**

$$D_m(c)D_{n+1}(c) - D_n(c)D_{m+1}(c) = \frac{\alpha^{n+1}\beta^{m+1}(\beta - \alpha) - \alpha^{m+1}\beta^{n+1}(\beta - \alpha)}{(\alpha - \beta)^2},$$

$$D_m(c)D_{n+1}(c) - D_n(c)D_{m+1}(c) = \frac{-(\alpha - \beta)(\alpha^n\alpha\beta^m\beta - \alpha^m\alpha\beta^n\beta)}{(\alpha - \beta)^2},$$

$$D_m(c)D_{n+1}(c) - D_n(c)D_{m+1}(c) = \frac{\alpha^m\beta^n - \alpha^n\beta^m}{\alpha - \beta}$$

which is the desired result. When $c = 2$ we have

$$\mathrm{D_m(2)D_{n+1}(2) - D_n(2)D_{m+1}(2) = (m+1)(n+2) - (n+1)(m+2)},$$

$$\mathrm{D_m(2)D_{n+1}(2) - D_n(2)D_{m+1}(2) = m - n}$$

And let us take $c = -2$. The formula $D_m(-2)D_{n+1}(-2) - D_n(-2)D_{m+1}(-2)$ is

$(-1)^m(m+1)(-1)^{n+1}(n+2)-(-1)^n(n+1)(-1)^{m+1}(m+2).$

Thus, we have

$D_m(-2)D_{n+1}(-2)-D_n(-2)D_{m+1}(-2)\ =(-1)^{m+n+1}(m-n).$

Thus, the proof is completed.

The formula for integer sequences was given by E. Gelin and proved by E. Cesaro in 1880. In the following theorem, we have given the Gelin-Cesaro formula for the elements of the sequence $\{D_k(c)\}$, which are defined and studied for the first time.

Teorem 5 (Gelin-Cesaro identity). For $n \geq 2$, we have

$D_{n-2}(c)D_{n-1}(c)D_{n+1}(c)D_{n+2}(c)-{D_n}^4(c)$ is

$$\begin{cases} c^2-(1-c^2){D_n}^2(c); & c \neq \pm 2 \\ -5n^2-10n-1; & c=2\,, c=-2 \end{cases}$$

Proof. Let us first examine cases c ≠2 and c ≠ - 2 to show the accuracy of the equation. If we use the Catalan identity for this purpose, then we write

$D_{n-1}(c)D_{n+1}(c)-D_n(c)^2=-D_0(c)^2$

$D_{n-2}(c)D_{n+2}(c)-D_n(c)^2=-D_1(c)^2$

We write these equations in the following equation

$D_{n-2}(c)D_{n-1}(c)D_{n+1}(c)D_{n+2}(c)-{D_n}^4(c)$

then, we have

$(D_n(c)^2-D_1(c)^2)(D_n(c)^2-D_0(c)^2)-{D_n}^4(c)$

$={D_n}^4(c)-{D_n}^2(c){D_0}^2(c)-{D_1}^2(c){D_n}^2(c)+{D_1}^2(c){D_0}^2(c)-{D_n}^4(c).$

Here, the values $D_0(c)=1,\ D_1(c)=c$ are written the following equation is obtained.

$-{D_n}^2(c){D_0}^2(c)-{D_1}^2(c){D_n}^2(c)+{D_1}^2(c){D_0}^2(c)\ =-{D_n}^2(c)-c^2{D_n}^2(c)+c^2.$

If necessary arrangements are made

$D_{n-2}(c)D_{n-1}(c)D_{n+1}(c)D_{n+2}(c)-{D_n}^4(c)=c^2-(1+c^2){D_n}^2(c)$

is obtained which is the desired result. If we write $c=2$, then we get

$D_{n-2}(2)D_{n-1}(2)D_{n+1}(2)D_{n+2}(2)-{D_n}^4(2)=(n-1)n(n+2)(n+3)-(n+1)^4$

$D_{n-2}(2)D_{n-1}(2)D_{n+1}(2)D_{n+2}(2)-{D_n}^4(2)=-5n^2-2n-1$

which is the desired result. If we write $c=-2$, then we get

$$D_{n-2}(-2)D_{n-1}(-2)D_{n+1}(-2)D_{n+2}(-2)-{D_n}^4(-2)$$
$$=(-1)^{n-2}(n-1)(-1)^{n-1}(-1)^{n+1}(n+2)(-1)^{n+2}(n+3)-((-1)^n(n+1))^4,$$
$$=(n^4+4n^3+n^2-6n)-(n^4+4n^3+6n^2+4n+1)=-5n^2-2n-1.$$

Thus, the proof is completed.

We have given the Honsberger's identity provided by the terms of the sequence we are working on in the following theorem.

Theorem 6. For $k \geq 1,\ n \geq 0$ the value $D_{k-1}(c)D_n(c)+D_k(c)D_{n+1}(c)$ is

$$\begin{cases} \dfrac{\alpha^{n+1}(\alpha^k+\alpha^{k+2}-2\beta^k)+\beta^{n+1}(\beta^k+\beta^{k+2}-2\alpha^k)}{(\alpha-\beta)^2}; & c \neq \pm 2 \\ k(2n+3)+(n+2); & c=2 \\ (-1)^{k+n+1}k(2n+3)+(n+2); & c=-2 \end{cases}$$

Proof. For $c \neq \pm 2$, the following equation can be written for the right side of the claim.

$$D_{k-1}(c)D_n(c)+D_k(c)D_{n+1}(c)=\left(\frac{\alpha^{k+n+1}+\beta^{k+n+1}+\alpha^{k+n+3}+\beta^{k+n+3}-2\alpha^k\beta^{n+1}-2\beta^k\alpha^{n+1}}{(\alpha-\beta)^2}\right),$$

$$D_{k-1}(c)D_n(c)+D_k(c)D_{n+1}(c)=\frac{\alpha^{n+1}(\alpha^k+\alpha^{k+2}-2\beta^k)+\beta^{n+1}(\beta^k+\beta^{k+2}-2\alpha^k)}{(\alpha-\beta)^2}.$$

And for $c=2$, we write

$$D_{k-1}(2)D_n(2)+D_k(2)D_{n+1}(2)=k(n+1)+(k+1)(n+2)=k(2n+3)+(n+2)$$

For $c=-2$, $D_{k-1}(-2)D_n(-2)+D_k(-2)D_{n+1}(-2)$ is follows:

$$D_{k-1}(-2)D_n(-2)+D_k(-2)D_{n+1}(-2)=(-1)^{k-1}k(-1)^n(n+1)+(-1)^k(k+1)(-1)^{n+1}(n+2)$$

$$D_{k-1}(-2)D_n(-2)+D_k(-2)D_{n+1}(-2)=(-1)^{k+n}(-kn-k-kn-2k-n-2).$$

Thus, we get

$$D_{k-1}(-2)D_n(-2)+D_k(-2)D_{n+1}(-2)=(-1)^{k+n+1}k(2n+3)+(n+2).$$

Thus, the proof is completed.

Theorem 7. For $k\geq 1$, $c=\pm 2$, the following equality is satisfied.

$$D_{k+1}(c)+D_{k-1}(c)=cD_k(c).$$

Proof. If we write,

$$\alpha^{k+2}-\beta^{k+2}=(\alpha+\beta)(\alpha^{k+1}-\beta^{k+1})+\beta^k-\alpha^k$$

and write this equality in the formula, then we obtain

$$D_{k+1}(c)+D_{k-1}(c)=\left(\frac{(\alpha+\beta)(\alpha^{k+1}-\beta^{k+1})+\beta^k-\alpha^k}{\alpha-\beta}\right)+\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right),$$

$$D_{k+1}(c)+D_{k-1}(c)=\frac{(\alpha+\beta)(\alpha^{k+1}-\beta^{k+1})}{\alpha-\beta}+\left(\frac{\beta^k-\alpha^k}{\alpha-\beta}\right)+\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right),$$

$$D_{k+1}(c)+D_{k-1}(c)=\frac{(\alpha+\beta)(\alpha^{k+1}-\beta^{k+1})}{\alpha-\beta},$$

$$D_{k+1}(c)+D_{k-1}(c)=cD_k(c).$$

Thus, the proof is completed.

Theorem 8. For $k\geq n$, the following equality is satisfied.

$$D_{k+n}(c)+D_{k-n}(c)=c\big(D_{k-1+n}(c)+D_{k-1-n}(c)\big)-\big(D_{k-2+n}(c)+D_{k-2-n}(c)\big).$$

Proof. If we write the following facts

$$\alpha^{k+n+1}-\beta^{k+n+1}=(\alpha+\beta)(\alpha^{k+n}-\beta^{k+n})+\beta^{k+n-1}-\alpha^{k+n-1}.$$

and

$$\alpha^{k-n+1}-\beta^{k-n+1}=(\alpha+\beta)(\alpha^{k-n}-\beta^{k-n})+\beta^{k-n-1}-\alpha^{k-n-1}$$

in the formula $D_{k+n}(c)+D_{k-n}(c)=\left(\frac{\alpha^{k+n+1}-\beta^{k+n+1}}{\alpha-\beta}\right)+\left(\frac{\alpha^{k-n+1}-\beta^{k-n+1}}{\alpha-\beta}\right)$,

then we get

$$\frac{(\alpha+\beta)\big(\alpha^{k+n}-\beta^{k+n}\big)+\beta^{k+n-1}-\alpha^{k+n-1}}{\alpha-\beta}+\frac{(\alpha+\beta)\big(\alpha^{k-n}-\beta^{k-n}\big)+\beta^{k-n-1}-\alpha^{k-n-1}}{\alpha-\beta}.$$

And if we make the necessary simplifications, then we have

$$(\alpha+\beta)\frac{\alpha^{k+n}-\beta^{k+n}}{\alpha-\beta}-\frac{\alpha^{k+n-1}-\beta^{k+n-1}}{\alpha-\beta}+(\alpha+\beta)\frac{\alpha^{k-n}-\beta^{k-n}}{\alpha-\beta}-\frac{\alpha^{k-n-1}-\beta^{k-n-1}}{\alpha-\beta}.$$

From here, we get

$$D_{k+n}(c)+D_{k-n}(c)=c\big(D_{k+n-1}(c)+D_{k-n-1}(c)\big)-D_{k+n-2}(c)-D_{k-n-2}(c)$$

which the proof is completed.

CONCLUSION

In this study, using the studies on the central Delannoy numbers $D(k)$ in the literature, we have examined the sequence formed by the Cauchy products of these numbers and some basic properties of this sequence. We have also derived some important identities, such as Cassini's, Catalan's, d'Ocagne's identities that contain elements of this sequence. Further different properties of the Central Delannoy numbers and their Cauchy products can be derived using this work in subsequent studies.

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Conflict of Interest

The author declares that there is no conflict of interest.

Author's Contributions

The entire work belongs to the author herself.

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