Turk. J. Math. Comput. Sci.
13(1)(2021) 174–181
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DOI : 10.47000/tjmcs.839168

On Gaussian Balancing and Gaussian Cobalancing Quaternions

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Received: 11-12-2020 • Accepted: 22-04-2021

Abstract. In this paper, we present and study new kinds of sequence of quaternion numbers called as Gaussian Balancing and Gaussian Cobalancing Quaternions involving some interesting results, Binet formula and generating functions. We show matrix representations for these quaternions. Thus, we have carried the quaternions to the complex space.

2010 AMS Classification: 11B37, 11R52

Keywords: Balancing numbers, cobalancing numbers, gaussian balancing numbers, gaussian cobalancing numbers, gaussian balancing quaternions, gaussian cobalancing quaternions, matrix representations.

1. Introduction

Behera and Panda in [1] introduced Balancing numbers $n \in \mathbb{Z}^+$ as solutions of the equation

$$1 + 2 + ... + (n-1) = (n+1) + (n+2) + ... + (n+r) \tag{1.1}$$

They call $n \in \mathbb{Z}^+$ a Balancing number and $r \in \mathbb{Z}^+$ the balancer corresponding to n. For example; the corresponding of the Balancing numbers are $6, 35$ and 204 with $2, 14$ and 84, respectively.

The Balancing sequence B_n is defined in [1, 2, 11–13, 15, 16] by the initial conditions $B_1 = 1$, $B_2 = 6$ and by the second order linear recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 2$.

G.K. Panda and P.K. Ray [13, 16] slightly modified (1.1) and introduced $n \in \mathbb{Z}^+$ a cobalancing number if

$$1 + 2 + ... + n = (n+1) + (n+2) + ... + (n+r)$$

for some $r \in \mathbb{Z}^+$. Here, they call r the cobalancer corresponding to the cobalancing number n. The first three cobalancing numbers are $2, 14$ and 84 with cobalancers $1, 6$ and 35, respectively. The Cobalancing sequence in [11, 13–16] is $b_1 = 0, b_2 = 2$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$.

Gaussian Balancing numbers GB_n were defined by Tasci in [21] with the initial conditions $GB_0 = i$, $GB_1 = 1$ and the second order linear recurrence relation

$$GB_{n+1} = 6GB_n - GB_{n-1}$$

for all $n \geq 1$. We can see easily

$$GB_n = B_n - iB_{n-1}$$

where B_n is the *nth* Balancing number.

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This work is supported by the Scientific Research Project (BAP) 2020FEBE003, Pamukkale University, Denizli, Turkey.

Asci and Yilmaz defined Gaussian Cobalancing numbers in [23] as

$$Gb_{n+1} = 6Gb_n - Gb_{n-1} + 2 - 2i$$

with the initial conditions $Gb_0 = -2i, Gb_1 = 0$.

One can see easily that

$$Gb_n = b_n - ib_{n-1}$$

where b_n is nth Cobalancing number.

Quaternion arithmetic has been used in many fields such as computer sciences, physics, applied mathematics and differential geometry. One can use in [4], for quaternion analysis.

Irish Mathematician William Rowan Hamilton first introduced the real quaternions in 1843 in [6]. The set of real quaternions can be defined as

$$H = \{q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 : q_i \in \mathbb{R}, i = 0, 1, 2, 3\}$$

as the four-dimensional vector space over $\mathbb{R}$ having a basis $\{e_0, e_1, e_2, e_3\}$ which satisfies the following multiplication rules:

TABLE 1. The multiplication table for the basis of H

$\times$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

A quaternion is a hyper-complex number and is shown by the following equation;

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 = \sum_{i=0}^{3} q_ie_i \in H.$$

The quaternion consists of two parts. The first part is called a scalar part as $S_q = q_0e_0$ and second is called vector part as $\overrightarrow{V_q} = q_1e_1 + q_2e_2 + q_3e_3 = \sum_{i=1}^{3} q_ie_i$. Then we can write $q = S_q + \overrightarrow{V_q}$. The conjugate of q is defined by

$$\overline{q} = S_q - \overrightarrow{V_q} = q_0e_0 - q_1e_1 - q_2e_2 - q_3e_3 = q_0e_0 - \sum_{i=1}^{3} q_ie_i.$$

Let q and p be two quaternions such that $q = S_q + \overrightarrow{V_q} = q_0e_0 + \sum_{i=1}^{3} q_ie_i$ and $p = S_p + \overrightarrow{V_p} = p_0e_0 + \sum_{i=1}^{3} p_ie_i$. The equality, addition, subtraction and multiplication by scalar are defined by the following:

- Equality: $q = p$ if and only if $q_0 = p_0, q_1 = p_1, q_2 = p_2, q_3 = p_3$
- Addition: $q + p = (q_0 + p_0)e_0 + (q_1 + p_1)e_1 + (q_2 + p_2)e_2 + (q_3 + p_3)e_3$
  $= \sum_{i=0}^{3}(q_i + p_i)e_i$
- Subtraction: $q - p = (q_0 - p_0)e_0 + (q_1 - p_1)e_1 + (q_2 - p_2)e_2 + (q_3 - p_3)e_3$
  $= \sum_{i=0}^{3}(q_i - p_i)e_i$
- Multiplication by scalar: If $k \in \mathbb{R}$, $k \cdot q = kq_0e_0 + kq_1e_1 + kq_2e_2 + kq_3e_3$
  $= \sum_{i=0}^{3} kq_ie_i$

The multiplication of q and p is defined by

$$q \cdot = S_qS_p + S_q\overrightarrow{V_p} + \overrightarrow{V_q}S_p - \overrightarrow{V_q} \cdot \overrightarrow{V_p} + \overrightarrow{V_q} \times \overrightarrow{V_p}$$

where $\overrightarrow{V_q} \cdot \overrightarrow{V_p} = q_1p_1 + q_2p_2 + q_3p_3 = \sum_{i=1}^{3} q_ip_i$ and $\overrightarrow{V_q} \times \overrightarrow{V_p} = (q_2p_3 - q_3p_2)e_1 - (q_1p_3 - q_3p_1)e_2 + (q_1p_2 - q_2p_1)e_3$.

The norm of q is defined as

$$\|q\| = N(q) = q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = \sum_{i=0}^{3} q_i^2.$$

For results on quaternions theory, we refer to [4, 6, 22].

A.F. Horadam in [7] introduced *nth* Fibonacci and Lucas quaternions in 1963 and examined [8] the recurrence relations of quaternion in 1993 and referred to defining Pell quaternions and generalized Pell quaternions. In [9] many interesting properties can be given about Fibonacci and Lucas quaternions. Halici in [5] examined Binet's formulas, generating functions and some properties about Fibonacci and Lucas quaternions. In [3] Cimen and Ipek introduced new kinds of sequences of quaternion number called as Pell quaternions and Pell-Lucas quaternions. Liana and Wloch in [20] defined the Jacobsthal quaternions and Jacobsthal-Lucas quaternions and gave some properties. In [10] Keçilioglu and Akkus introduced Fibonacci octanions. In addition, "Bi-periodic Balancing Quaternions" were studied by Sevgi and Tasci in [19] and various properties were given.

In this paper we define and study the Gaussian Balancing quaternions and Gaussian Cobalancing quaternions. We give Binet's formulas, generating functions, some properties and describe matrix representations about Gaussian Balancing and Gaussian Cobalancing quaternions.

2. Gaussian Balancing Quaternions and Gaussian Cobalancing Quaternions

Definition 2.1. The *nth* Gaussian Balancing quaternion QGB_n and the *nth* Gaussian Cobalancing quaternion QGb_n are defined respectively as shown

$$QGB_n = GB_n e_0 + GB_{n+1} e_1 + GB_{n+2} e_2 + GB_{n+3} e_3 \tag{2.1}$$

and

$$QGb_n = Gb_n e_0 + Gb_{n+1} e_1 + Gb_{n+2} e_2 + Gb_{n+3} e_3 \tag{2.2}$$

where GB_n and Gb_n are *nth* Gaussian Balancing and Gaussian Cobalancing quaternions.

Let QGB_n and QGB_m be two Gaussian Balancing quaternions such that $QGB_n = GB_n e_0 + GB_{n+1} e_1 + GB_{n+2} e_2 + GB_{n+3} e_3$ and $QGB_m = GB_m e_0 + GB_{m+1} e_1 + GB_{m+2} e_2 + GB_{m+3} e_3$. The scalar part of Gaussian Balancing quaternions QGB_n and QGB_m are denoted by $S_{QGB_n} = GB_n e_0$ and $S_{QGB_m} = GB_m e_0$, respectively. Also, $\overrightarrow{V_{QGB_n}} = GB_{n+1} e_1 + GB_{n+2} e_2 + GB_{n+3} e_3$ and $\overrightarrow{V_{QGB_m}} = GB_{m+1} e_1 + GB_{m+2} e_2 + GB_{m+3} e_3$ are called vectorial part of Gaussian Balancing Quaternions QGB_n and QGB_m. The equality, addition, subtraction and multiplication by scalar of Gaussian Balancing Quaternions are defined by the following:

- Equality: $QGB_n = QGB_m$ if and only if $GB_n = GB_m$, $GB_{n+1} = GB_{m+1}$, $GB_{n+2} = GB_{m+2}$, $GB_{n+3} = GB_{m+3}$
- Addition: $QGB_n + QGB_m = (GB_n + GB_m)e_0 + (GB_{n+1} + GB_{m+1})e_1 + (GB_{n+2} + GB_{m+2})e_2 + (GB_{n+3} + GB_{m+3})e_3$
- Subtraction: $QGB_n - QGB_m = (GB_n - GB_m)e_0 + (GB_{n+1} - GB_{m+1})e_1 + (GB_{n+2} - GB_{m+2})e_2 + (GB_{n+3} - GB_{m+3})e_3$
- Multiplication by scalar $k \in \mathbb{R}$: $kQGB_n = kGB_n e_0 + kGB_{n+1} e_1 + kGB_{n+2} e_2 + kGB_{n+3} e_3$
- The multiplication of QGB_n and QGB_m is defined by

$$\begin{aligned} QGB_n.QGB_m &= S_{QGB_n} S_{QGB_m} + S_{QGB_n}.\overrightarrow{V_{QGB_m}} + \overrightarrow{V_{QGB_n}}.S_{QGB_m} \\ &\quad - \overrightarrow{V_{QGB_n}}.\overrightarrow{V_{QGB_m}} + \overrightarrow{V_{QGB_n}} \times \overrightarrow{V_{QGB_m}}. \end{aligned}$$

Definition 2.2. The conjugates of QGB_n and QGb_n are defined by

$$\overline{QGB_n} = GB_n e_0 - GB_{n+1} e_1 - GB_{n+2} e_2 - GB_{n+3} e_3, \tag{2.3}$$

$$\overline{QGb_n} = Gb_n e_0 - Gb_{n+1} e_1 - Gb_{n+2} e_2 - Gb_{n+3} e_3.$$

Definition 2.3. The norms of QGB_n and QGb_n are defined by

$$\|QGB_n\| = N_{QGB_n} = GB_n^2 + GB_{n+1}^2 + GB_{n+2}^2 + GB_{n+3}^2,$$

$$\|QGb_n\| = N_{QGb_n} = Gb_n^2 + Gb_{n+1}^2 + Gb_{n+2}^2 + Gb_{n+3}^2.$$

Proposition 2.4. *For $n \geq 2$, we have the following properties;*

$$\begin{aligned} QGB_n + \overline{QGB_n} &= 2GB_n \\ &= 2(B_n - iB_{n-1}) \end{aligned} \tag{2.4}$$

$$QGB_n^2 + QGB_n \cdot \overline{QGB_n} = 2GB_n.QGB_n \tag{2.5}$$

$$QGB_n.\overline{QGB_n} = GB_n^2 + GB_{n+1}^2 + GB_{n+2}^2 + GB_{n+3}^2$$

where GB_n and B_n are nth Gaussian Balancing and Balancing number, respectively.

Proof. From (2.1) and (2.3), we get

$$\begin{aligned} QGB_n + \overline{QGB_n} &= \sum_{i=0}^{3} GB_{n+i}e_i + GB_n e_0 - \sum_{i=1}^{3} GB_{n+i}e_i \\ &= 2GB_n \\ &= 2\left(B_n - iB_{n-1}\right) \end{aligned}$$

which gives (2.4). Also, from (2.4) we have

$$\begin{aligned} QGB_n^2 &= QGB_n \cdot QGB_n \\ &= QGB_n \cdot (2GB_n - \overline{QGB_n}) \\ &= 2GB_n \cdot QGB_n - QGB_n \cdot \overline{QGB_n}. \end{aligned}$$

We obtain (2.5)

$$QGB_n^2 + QGB_n \cdot \overline{QGB_n} = 2GB_n \cdot QGB_n.$$

From (2.1), (2.3) and Table 1,

$$\begin{aligned} QGB_n \cdot \overline{QGB_n} &= \left(\sum_{i=0}^{3} GB_{n+i}e_i\right) \times \left(GB_n e_0 - \sum_{i=1}^{3} GB_{n+i}e_i\right) \\ &= GB_n^2 + GB_{n+1}^2 + GB_{n+2}^2 + GB_{n+3}^2. \end{aligned}$$

□

Proposition 2.5. *For $n \geq 2$,we have the following identities;*

$$QGb_n + \overline{QGb_n} = 2Gb_n$$

$$QGb_n^2 + QGb_n \cdot \overline{QGb_n} = 2Gb_n \cdot QGb_n$$

$$QGb_n \cdot \overline{QGb_n} = Gb_n^2 + Gb_{n+1}^2 + Gb_{n+2}^2 + Gb_{n+3}^2$$

where Gb_n and b_n are nth Gaussian Cobalancing and Cobalancing number, respectively.

Theorem 2.6. *The Gaussian Balancing and Gaussian Cobalancing quaternions are the second order linear recurrence sequence as for $n \geq 0$,*

$$QGB_{n+2} = 6QGB_{n+1} - QGB_n \tag{2.6}$$

$$QGb_{n+2} = 6QGb_{n+1} - QGb_n + (2-2i)(e_0 + e_1 + e_2 + e_3).$$

Proof. From (2.1), we get

$$6QGB_{n+1} - QGB_n = 6\left(\sum_{i=0}^{3} GB_{n+1+i}e_i\right) - \left(\sum_{i=0}^{3} GB_{n+i}e_i\right)$$

and since from the recurrence relation of Gaussian Balancing numbers

$$GB_{n+2} = 6GB_{n+1} - GB_n$$

we can obtain in (2.6).

Now, we find recurrence relation of Gaussian Cobalancing quaternions. From (2.2), we get

$$\begin{aligned} 6QGb_{n+1} - QGb_n + (2-2i)(e_0+e_1+e_2+e_3) &= 6\left(\sum_{i=0}^{3} Gb_{n+1+i}e_i\right) \\ &\quad -\left(\sum_{i=0}^{3} Gb_{n+i}e_i\right) \\ &\quad +(2-2i)\left(\sum_{i=0}^{3} e_i\right) \end{aligned}$$

and since from the recurrence relation of Gaussian Cobalancing numbers

$$Gb_{n+2} = 6Gb_{n+1} - Gb_n + 2 - 2i$$

we can obtain the recurrence relation of Gaussian Cobalancing quaternions. □

Theorem 2.7. *For QGB_n Gaussian Balancing quaternion and QGb_n Gaussian Cobalancing quaternion; we have the following identities;*

$$QGB_n = \frac{QGb_{n+1} - QGb_n}{2}$$

$$QGB_{n+1} - QGB_n = 2QGb_{n+1} + (1-i)(e_0+e_1+e_2+e_3)$$

$$QGB_n - \sum_{i=1}^{3} QGB_{n+i}e_i = \sum_{i=0}^{3} QGB_{n+2i}.$$

Definition 2.8. Since $B_{-n} = -B_n$ in [17, 18] and $b_{-n} = (-1)^{n+1}b_{n+1}$ in [14, 16], the Gaussian Balancing and Gaussian Cobalancing quaternions with negative subscripts are defined by

$$\begin{aligned} QGB_{-n} &= \sum_{i=0}^{3} GB_{-n+i}e_i \\ &= -\sum_{i=0}^{3} GB_{n+i}e_i \\ &= -QGB_n \end{aligned}$$

and

$$\begin{aligned} QGb_{-n} &= \sum_{k=0}^{3} Gb_{-n+i}e_i \\ &= \sum_{k=0}^{3} (-1)^{n+1-k}.i.Gb_{n+2-k}e_k. \end{aligned}$$

Corollary 2.9. *The following relations are easily seen from the definition of Gaussian Balancing and Gaussian Cobalancing quaternions with negative subscripts:*

$$QGB_{-n} + \overline{QGB_{-n}} = -2GB_n$$

$$QGb_{-n} + \overline{QGb_n} = 2.i.(-1)^{n+1}Gb_{n+2}.$$

Now we give Binet's formulas for Gaussian Balancing and Gaussian Cobalancing quaternions.

Theorem 2.10. *[Binet's Formula] For $n \geq 0$, the Binet's formula for the Gaussian Balancing quaternion is as follows*

$$QGB_n = \frac{1}{\alpha - \beta}\left[\left(\alpha^{'}\alpha^n - \beta^{'}\beta^n\right) - i\left(\alpha^{'}\alpha^{n-1} - \beta^{'}\beta^{n-1}\right)\right]$$

where $\alpha = 3 + \sqrt{8}$, $\beta = 3 - \sqrt{8}$, $\alpha^{'} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$ and $\beta^{'} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$.

Proof. The characteristic equation of recurrence relation for Gaussian Balancing quaternion (2.6) is

$$r^2 - 6r + 1 = 0.$$

The roots of this equation are $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$. Using recurrence relation and initial conditions $QGB_0 = (i, 1, 6 - i, 35 - 6i)$, $QGB_1 = (1, 6 - i, 35 - 6i, 204 - 35i)$ the Binet's Formula for QGB_n is get as follows;

$$QGB_n = A\alpha^n + B\beta^n.$$

The values of $n = 0$ and $n = 1$ are substituted for QGB_n and the unknown A and B are found. Finally, the Binet's formula is obtained as follows;

$$QGB_n = \frac{1}{\alpha - \beta}\left[\left(\alpha^{'}\alpha^n - \beta^{'}\beta^n\right) - i\left(\alpha^{'}\alpha^{n-1} - \beta^{'}\beta^{n-1}\right)\right].$$

□

Theorem 2.11. *[Binet's Formula] For $n \geq 0$, the Binet's formula for Gaussian Cobalancing quaternion is as follows*

$$\begin{aligned} QGb_n &= \frac{1}{4\sqrt{2}}\left[\left(\alpha^{'}\alpha^{2n-1} - \beta^{'}\beta^{2n-1}\right) - i\left(\alpha^{'}\alpha^{2n-3} - \beta^{'}\beta^{2n-3}\right)\right] \\ &\quad -\frac{1}{2}(1 - i)(e_0 + e_1 + e_2 + e_3) \end{aligned}$$

where $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$, $\alpha^{'} = e_0 + \alpha^2 e_1 + \alpha^4 e_2 + \alpha^6 e_3$ and $\beta^{'} = e_0 + \beta^2 e_1 + \beta^4 e_2 + \beta^6 e_3$.

Theorem 2.12. *The generating function for the Gaussian Balancing quaternions QGB_n is*

$$\begin{aligned} G(x,t) &= \frac{QGB_0 + t\,(QGB_1 - 6QGB_0)}{1 - 6t + t^2} \\ &= \frac{(t + i(1 - 6t))\,e_0 + (1 - it)\,e_1 + ((6 - t) - i)\,e_2 + ((35 - 6t) + i(-6 + t))\,e_3}{1 - 6t + t^2}. \end{aligned}$$

Proof. Let

$$G(x,t) = \sum_{n=0}^{\infty} QGB_n(x)t^n$$

be the generating function of the Gaussian Balancing quaternions.

$$\begin{aligned} G(x,t) &= QGB_0 + QGB_1 t + QGB_2 t^2 + \sum_{n=3}^{\infty} QGB_n t^n \\ &= QGB_0 + QGB_1 t + QGB_2 t^2 + \sum_{n=3}^{\infty}\left[6QGB_{n-1} - QGB_{n-2}\right]t^n \\ &= QGB_0 + QGB_1 t + QGB_2 t^2 + 6\sum_{n=3}^{\infty} QGB_{n-1}t^n - \sum_{n=3}^{\infty} QGB_{n-2}t^n \\ &= QGB_0 + QGB_1 t + QGB_2 t^2 + 6t\sum_{n=3}^{\infty} QGB_{n-1}t^{n-1} - t^2\sum_{n=3}^{\infty} QGB_{n-2}t^{n-2} \\ &= QGB_0 + QGB_1 t + QGB_2 t^2 + 6t\left[\sum_{n=2}^{\infty} QGB_n t^n\right] - t^2\left[\sum_{n=1}^{\infty} QGB_n t^n\right] \\ &= QGB_0 + QGB_1 t + QGB_2 t^2 + 6t\left[G(x,t) - QGB_0 - QGB_1 t\right] \\ &\quad -t^2\left[G(x,t) - QGB_0\right] \end{aligned}$$

by making necessary arrangement, the generating function of Balancing quaternion is found as follows.

$$\begin{aligned} G(x,t) &= \frac{QGB_0 + t\,(QGB_1 - 6QGB_0)}{1 - 6t + t^2} \\ &= \frac{(t + i(1 - 6t))\,e_0 + (1 - it)\,e_1 + ((6 - t) - i)\,e_2 + \begin{pmatrix} (35 - 6t) \\ +i(-6 + t) \end{pmatrix} e_3}{1 - 6t + t^2}. \end{aligned}$$

□

Theorem 2.13. *The generating function of Gaussian Cobalancing quaternion QGb_n is*

$$G(x,t)=\frac{(1-t)\left[QGb_0+t(QGb_1-6QGb_0)+t^2(QGb_2-6QGb_1+QGb_0)\right]+t^3(2-2i)(e_0+e_1+e_2+e_3)}{(1-t)(1-6t+t^2)},$$

where $QGb_0=(-2i,0,2,14-2i)$, $QGb_1=(0,2,14-2i,84-14i)$ *and* $QGb_2=(2,14-2i,84-14i,492-84i)$.

3. Matrix Representations of Gaussian Balancing and Gaussian Cobalancing Quaternions

In this section, we give the matrix representation of Gaussian Balancing and Gaussian Cobalancing quaternions. Throughout this section, v_0 is a 2×1 matrix defined by $v_0=\begin{bmatrix}0\\1\end{bmatrix}$, A is a 2×2 matrix defined by $A=\begin{bmatrix}0&1\\-1&6\end{bmatrix}$ and Q is a 2×2 matrix defined by $Q=\begin{bmatrix}-QGB_0&QGB_1\\-QGB_1&QGB_2\end{bmatrix}$.

Theorem 3.1. *Let $n\geq 1$ be integer. Then*

$$\begin{bmatrix}-QGB_{n-1}&QGB_n\\-QGB_n&QGB_{n+1}\end{bmatrix}=\begin{bmatrix}-QGB_0&QGB_1\\-QGB_1&QGB_2\end{bmatrix}.\begin{bmatrix}0&1\\-1&6\end{bmatrix}^{n-1}.$$

Proof. The proof is based on mathematical induction. For $n=1$,

$$\begin{bmatrix}-QGB_{n-1}&QGB_n\\-QGB_n&QGB_{n+1}\end{bmatrix}=\begin{bmatrix}-QGB_0&QGB_1\\-QGB_1&QGB_2\end{bmatrix}.\begin{bmatrix}0&1\\-1&6\end{bmatrix}^{n-1}$$

is obvious. Assume that

$$\begin{bmatrix}-QGB_{n-1}&QGB_n\\-QGB_n&QGB_{n+1}\end{bmatrix}=\begin{bmatrix}-QGB_0&QGB_1\\-QGB_1&QGB_2\end{bmatrix}.\begin{bmatrix}0&1\\-1&6\end{bmatrix}^{n-1}$$

for $1\leq n\leq k$. Then

$$\underbrace{\begin{bmatrix}-QGB_0&QGB_1\\-QGB_1&QGB_2\end{bmatrix}.\begin{bmatrix}0&1\\-1&6\end{bmatrix}^{n-1}}.\begin{bmatrix}0&1\\-1&6\end{bmatrix}=\begin{bmatrix}-QGB_k&6QGB_k-QGB_{k-1}\\-QGB_{k+1}&6QGB_{k+2}-QGB_k\end{bmatrix}$$

showing that the assertion is true for $n=k+1$. This completes the proof of the theorem. □

Corollary 3.2. *Let S be 2×2 matrix defined by $S=Q.A^{n-1}$. Then*

$$QGB_{n+1}=v_0^T.S.v_0.$$

Theorem 3.3. *Let B be 2×2 matrix defined by $B=\begin{bmatrix}0&1\\-1&5\end{bmatrix}$. Then*

$$\begin{bmatrix}-QGB_{n-1}&QGB_n\\-QGB_n&QGB_{n+1}\end{bmatrix}.\begin{bmatrix}0&1\\-1&5\end{bmatrix}=\begin{bmatrix}-QGB_n&2QGb_{n+1}+(1-i)(e_0+e_1+e_2+e_3)\\-QGB_{n+1}&2QGb_{n+2}+(1-i)(e_0+e_1+e_2+e_3)\end{bmatrix}.$$

Corollary 3.4. *Let F be 2×2 matrix defined by $F=S.B$. Then*

$$QGb_{n+2}=\frac{1}{2}\left[v_0^T.F.v_0-(1-i)(e_0+e_1+e_2+e_3)\right].$$

4. Conclusion

In this paper we study Gaussian Balancing and Gaussian Cobalancing quaternions involving some interesting results. The generating functions, Binet's formulas are given and proved. After obtaining some properties of the Gaussian Balancing and Gaussian Cobalancing quaternions we define Q,S and F matrices to get QGB_n and QGb_n quaternions. Thus, we have carried the quaternions to the complex space.

Acknowledgement

The authors thank to the anonymous referees for his/her comments and valuable suggestions that improved the presentation of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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