Possible effects of space-time nonmetricity on neutrino oscillations

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The contribution of gravitational neutrino oscillations to the solar neutrino problem is studied by constructing a Dirac Hamiltonian and calculating the corresponding dynamical phase in the vicinity of the Sun in a non-Riemann background Kerr space-time with torsion and nonmetricity. We show that certain components of nonmetricity and the axial as well as nonaxial components of torsion may contribute to neutrino oscillations. We also note that the rotation of the Sun may cause a suppression of transitions among neutrinos. However, the observed solar neutrino deficit could not be explained by any of these effects because they are of the order of Planck scale.

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I. INTRODUCTION

Neutrinos have always attracted a lot of attention in high energy physics [1]. A major problem of interest at present is the solar neutrino problem. The Sun is a strong source of electron neutrinos ν_e because of the thermonuclear reactions taking place in its core. According to the standard solar model, the number of ν_e to be emitted from the Sun can be predicted. At the same time, the flux of electron neutrinos coming from the Sun can be measured on Earth. The measured amount of ν_e is approximately one-third of the predicted amount. Essentially, this is the so-called *solar neutrino problem*. One well-known solution to this problem is provided by the assumption of *neutrino oscillations* [1,2]. Briefly stated, the neutrino oscillations imply that the electron neutrinos coming out of the Sun may be converted to other neutrino species, muon ν_μ and tau ν_τ, during their journey towards the Earth, assuming neutrinos to have a mass, whereas the standard electroweak model asserts zero mass for them. It should also be noted that all of the above arguments have been cast in Minkowski space-time. However, we know that we live in a curved space-time—perhaps even in a curved space-time with torsion and nonmetricity. Therefore, in more recent years, physicists have turned their attention to specifically gravitational contributions to neutrino oscillations—see [3–7] and references therein. We recently investigated the effects of space-time torsion on neutrino oscillations [8]—see also [9,10]. The essence of this work is to calculate the dynamical phase of neutrinos by finding the form of the Hamiltonian H from the Dirac equation in a non-Riemannian space-time. The phase then follows from the formula

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi \Rightarrow \psi(t)=\exp\left(-\frac{i}{\hbar}\int Hdt\right)\psi(0), \qquad (1)$$

where ψ is a Dirac 4-spinor and H is a 4×4 matrix. The Hamiltonian H will depend, for example, on momentum $\vec{p}$, and this is expressed not as a differential operator but simply as a vector.[1] In this paper we investigate within the same approach the possible effects of space-time nonmetricity on neutrino oscillations.

II. SPACE-TIME GEOMETRY

Space-time is denoted by the triple $\{M,g,\nabla\}$ where M is a four-dimensional differentiable manifold, equipped with a Lorentzian metric g which is a (0,2)-type covariant, symmetric, nondegenerate tensor, and ∇ is a connection which defines parallel transport of vectors (or more generally tensors). We shall give a coordinate system set up at a point $p\in M$ by coordinate functions (or independent variables) $\{x^\alpha(p)\}$, $\alpha=\hat{0},\hat{1},\hat{2},\hat{3}$. This coordinate system forms a set of *natural* (or *coordinate*) *reference frame* at p as $\{(\partial/\partial x^\alpha)(p)\}$, with shorthand notation $\partial_\alpha\equiv\partial/\partial x^\alpha$. This natural reference frame is a basis vector set for the tangent space at p, denoted by $T_p(M)$. Similarly, differentials $\{dx^\alpha(p)\}$ of coordinate functions $\{x^\alpha(p)\}$ at p form a *natural* (or *coordinate*) *reference coframe* in the cotangent space at p, denoted by $T^*_p(M)$. Interior product of the basis vectors with the basis covectors is defined by the Kroenecker symbol:

$$dx^\alpha\left(\frac{\partial}{\partial x^\beta}\right)\equiv\imath_{\partial_\beta}dx^\alpha=\delta^\alpha_\beta. \qquad (2)$$

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[1]The exponential of $\exp[-(i/\hbar)\int Hdt]$ is defined by its power series expansion.

In general, any set of linearly independent vectors in tangent space, $T_p(M)$, can be taken as basis vectors and these vectors can be orthonormalized by, for example, the Gram-Schmidt process. We denote a set like this by $\{X_a\}$, $a=0,1,2,3$, and call it an *orthonormal reference frame*. In this case the metric defined on M satisfies the relation

$$g(X_a,X_b)=\eta_{ab}, \tag{3}$$

where η_{ab} is known as the Minkowski metric which is a matrix whose diagonal terms are $-1,1,1,1$ and off-diagonal terms are zero. The basis set dual to the orthonormal reference frame is denoted by $\{e^a\}$, $a=0,1,2,3$, and called the *orthonormal reference coframe*. $\{X_a\}$, and its dual $\{e^a\}$ satisfies the following set of equalities that is another manifestation of Eq. (2):

$$e^a(X_b)\equiv \imath_{X_b}(e^a)=\delta^a_b. \tag{4}$$

Here we adhere to the following conventions: indices denoted by Greek letters $\alpha, \beta, \ldots=\hat{0},\hat{1},\hat{2},\hat{3}$ and $\mu, \nu, \ldots=\hat{1},\hat{2},\hat{3}$ are holonomic or coordinate indices, and $a, b, \ldots=0,1,2,3$ and $i, j, \ldots=1,2,3$ are anholonomic or frame indices. In terms of the local coordinate frame $\partial_\alpha(p)$, the orthonormal frame $X_a(p)$ can be expanded via the so-called vierbein (or tetrad) $h^\alpha{}_a(p)$ as

$$X_a(p)=h^\alpha{}_a(p)\partial_\alpha(p). \tag{5}$$

In order for X_a to serve as an anholonomic basis, the $h^\alpha{}_a(p)$ are required to be nondegenerate—i.e., $\det h^\alpha{}_a(p)\neq 0$. In $T^*_p(M)$ an orthonormal coframe $e^a(p)$ can be expanded in terms of the local coordinate coframe $dx^\alpha(p)$ as

$$e^b(p)=h^b{}_\beta(p)dx^\beta(p). \tag{6}$$

The inverse vierbein $h^b{}_\beta(p)$ has to be nondegenerate as well. Moreover, the duality of the frame and the coframe requires for the vierbein and its inverse to satisfy

$$\imath_{X_a}e^b=h^\alpha{}_a(p)h^b{}_\alpha(p)=\delta^b_a. \tag{7}$$

We set the space-time orientation by the choice $\epsilon_{0123}=1$. The nonmetricity 1-forms, torsion 2-forms, and curvature 2-forms are defined by the Cartan structure equations

$$2Q_{ab}=-D\eta_{ab}:=\Lambda_{ab}+\Lambda_{ba}, \tag{8}$$

$$T^a=De^a:=de^a+\Lambda^a{}_b\wedge e^b, \tag{9}$$

$$R^a{}_b=D\Lambda^a{}_b:=d\Lambda^a{}_b+\Lambda^a{}_c\wedge\Lambda^c{}_b. \tag{10}$$

d, D, $\imath_a$, and $*$ denote the exterior derivative, the covariant exterior derivative, the interior derivative, and the Hodge star operator, respectively. The linear connection 1-forms can be decomposed in a unique way according to [11]

$$\Lambda^a{}_b=\omega^a{}_b+K^a{}_b+q^a{}_b+Q^a{}_b, \tag{11}$$

where $\omega^a{}_b$ are the Levi-Cività connection 1-forms,

$$\omega^a{}_b\wedge e^b=-de^a, \tag{12}$$

$K^a{}_b$ are the contortion 1-forms,

$$K^a{}_b\wedge e^b=T^a, \tag{13}$$

and $q^a{}_b$ are the antisymmetric tensor 1-forms,

$$q_{ab}=-(\imath_aQ_{bc})e^c+(\imath_bQ_{ac})e^c. \tag{14}$$

It is cumbersome to take into account all components of nonmetricity and torsion in gravitational models. Therefore we will be content with dealing only with certain irreducible parts of them to gain physical insight. The irreducible decompositions of torsion and nonmetricity invariant under the Lorentz group are summarily given below. For details one may consult Ref. [12]. The nonmetricity 1-forms Q_{ab} can be split into their trace-free $\overline{Q}_{ab}$ and trace parts as

$$Q_{ab}=\overline{Q}_{ab}+\frac{1}{4}\eta_{ab}Q, \tag{15}$$

where the Weyl 1-form $Q=Q^a{}_a$ and $\eta^{ab}\overline{Q}_{ab}=0$. Let us define

$$\Lambda_b:=\imath_a\overline{Q}^a{}_b,\ \ \Lambda:=\Lambda_ae^a,$$

$$\Theta_b:=*(\overline{Q}_{ab}\wedge e^a),\ \ \Theta:=e^b\wedge\Theta_b,$$

$$\Omega_a:=\Theta_a-\frac{1}{3}\imath_a\Theta, \tag{16}$$

so as to use them in the decomposition of Q_{ab} as

$$Q_{ab}=Q^{(1)}_{ab}+Q^{(2)}_{ab}+Q^{(3)}_{ab}+Q^{(4)}_{ab}, \tag{17}$$

where

$$Q^{(2)}_{ab}=\frac{1}{3}*(e_a\wedge\Omega_b+e_b\wedge\Omega_a), \tag{18}$$

$$Q^{(3)}_{ab}=\frac{2}{9}\left(\Lambda_ae_b+\Lambda_be_a-\frac{1}{2}\eta_{ab}\Lambda\right), \tag{19}$$

$$Q^{(4)}_{ab}=\frac{1}{4}\eta_{ab}Q, \tag{20}$$

$$Q^{(1)}_{ab}=Q_{ab}-Q^{(2)}_{ab}-Q^{(3)}_{ab}-Q^{(4)}_{ab}. \tag{21}$$

We have $\imath^aQ^{(1)}_{ab}=\imath^aQ^{(2)}_{ab}=0$, $\eta^{ab}Q^{(1)}_{ab}=\eta^{ab}Q^{(2)}_{ab}=\eta^{ab}Q^{(3)}_{ab}=0$, and $e^a\wedge Q^{(1)}_{ab}=0$. In a similar way the irreducible decomposition of T^a's invariant under the Lorentz group is given in terms of

$$\alpha=\imath_aT^a,\ \ \sigma=e_a\wedge T^a, \tag{22}$$

so that

$$T^a=T^{a(1)}+T^{a(2)}+T^{a(3)}, \tag{23}$$

where

$$T^{a(2)}=\frac{1}{3}e^a\wedge\alpha, \tag{24}$$

$$T^{a(3)}=\frac{1}{3}\iota^a\sigma, \tag{25}$$

$$t^a:=T^{a(1)}=T^a-T^{a(2)}-T^{a(3)}. \tag{26}$$

Here $\iota_a t^a=\iota_a T^{a(3)}=0$, $e_a\wedge t^a=e_a\wedge T^{a(2)}=0$. To give the contortion components in terms of the irreducible components of torsion, we first write

$$2K_{ab}=\iota_a T_b-\iota_b T_a-(\iota_a\iota_b T_c)e^c \tag{27}$$

from Eq. (13) and then substituting Eq. (23) into above we find

$$2K_{ab}=\iota_a t_b-\iota_b t_a-(\iota_a\iota_b t_c)e^c+\frac{2}{3}(e_a\wedge\iota_b\alpha-e_b\wedge\iota_a\alpha)$$
$$+\frac{2}{3}(\iota_a\iota_b\sigma)-\frac{1}{3}(\iota_a\iota_b\iota_c\sigma)e^c. \tag{28}$$

In components $K_{ab}=K_{c,ab}e^c$, $t_a=\frac{1}{2}t_{bc,a}e^{bc}$, $\alpha=F_a e^a$, $\sigma=(1/3!)\sigma_{abc}e^{abc}$ this becomes

$$K_{c,ab}=\frac{1}{2}(t_{ac,b}-t_{bc,a}+t_{ab,c})+\frac{1}{3}(F_b\eta_{ac}-F_a\eta_{bc})-\frac{1}{6}\sigma_{abc}. \tag{29}$$

III. HAMILTONIAN OF A DIRAC PARTICLE IN ARBITRARY SPACE-TIMES

The Dirac equation in a non-Riemannian space-time with torsion and nonmetricity is written as [13–15]

$$*\gamma\wedge D\psi+M*1\psi=0 \tag{30}$$

in terms of the Clifford algebra $\mathcal{C}\ell_{3,1}$-valued 1-forms $\gamma=\gamma^a e_a$ and $M=mc/\hbar$. We use the Dirac matrices

$$\gamma^0=i\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\ \gamma^i=i\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},$$

where σ^i are the Pauli matrices. ψ is a 4-component complex-valued Dirac spinor whose covariant exterior derivative is given explicitly by

$$D\psi=d\psi+\frac{1}{2}\Lambda^{[ab]}\sigma_{ab}\psi+\frac{1}{4}Q\psi, \tag{31}$$

where

$$\sigma_{ab}=\frac{1}{4}[\gamma_a,\gamma_b] \tag{32}$$

are the spin generators of the Lorentz group. We write it out explicitly as

$$D\psi=\partial_t\psi dx^{\hat{0}}+\partial_\mu\psi dx^\mu+\frac{1}{2}e^c\Omega_{c,ab}\sigma^{ab}\psi+\frac{1}{4}e^cQ_c\psi$$
$$=h^{\hat{0}}{}_c e^c\partial_t\psi+h^\mu{}_c e^c\partial_\mu\psi+\frac{1}{2}e^c\Omega_{c,ab}\sigma^{ab}\psi$$
$$+\frac{1}{4}e^cQ_c\psi, \tag{33}$$

where $\Lambda_{[ab]}:=\Omega_{ab}=\Omega_{c,ab}e^c$ is the antisymmetric part of the full connection 1-form and $Q=Q_a e^a$, and using $*\gamma=\gamma_a*e^a$ and the identity $*e^a\wedge e^b=-\eta^{ab}*1$ we calculate

$$*\gamma\wedge D\psi=\Bigg(-h^{\hat{0}}{}_c\gamma^c\partial_t\psi-h^\mu{}_c\gamma^c\partial_\mu\psi$$
$$-\frac{1}{2}\Omega_{c,ab}\gamma^c\sigma^{ab}\psi-\frac{1}{4}Q_c\gamma^c\psi\Bigg)*1. \tag{34}$$

Putting this into Eq. (30) we obtain

$$h^{\hat{0}}{}_c\gamma^c\partial_t\psi=-h^\mu{}_c\gamma^c\partial_\mu\psi+M\psi-\frac{1}{2}\Omega_{c,ab}\gamma^c\sigma^{ab}\psi$$
$$-\frac{1}{4}Q_c\gamma^c\psi. \tag{35}$$

We multiply this from the left by

$$i\hbar(h^{\hat{0}}{}_a\gamma^a)^{-1}=\frac{-i\hbar}{b^2}(h^{\hat{0}}{}_a\gamma^a), \tag{36}$$

where

$$b^2:=(h^{\hat{0}}{}_0)^2+h^{\hat{0}}{}_i h^{\hat{0}i}. \tag{37}$$

When we compare the result with the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi, \tag{38}$$

we deduce the Dirac Hamiltonian matrix [7,8,15–20]

$$H=\frac{c}{b^2}h^{\hat{0}}{}_a h^\mu{}_b\gamma^a\gamma^b i\hbar\partial_\mu-\frac{imc^2}{b^2}h^{\hat{0}}{}_a\gamma^a$$
$$+\frac{i\hbar c}{2b^2}h^{\hat{0}}{}_d\Omega_{c,ab}\gamma^d\gamma^c\sigma^{ab}+\frac{i\hbar c}{4b^2}h^{\hat{0}}{}_a Q_b\gamma^a\gamma^b. \tag{39}$$

The right-hand side of Eq. (39) need not be a Hermitian matrix in general; e.g., if $h^{\hat{0}}{}_i\neq 0$, then the mass term contains an anti-Hermitian part such as

$$H=H_0+iH_1, \tag{40}$$

where $H_0^+=H_0$ and $H_1^+=H_1$. However, the decomposition (40) is frame dependent. That is, we can always find a local Lorentz frame in which Hamiltonian is fully Hermitian

[17,18]. First we can get rid of the anti-Hermitian part of the mass term by diagonalizing the matrix $h^{\alpha}{}_{a}$ via a frame transformation

$$\partial_{\alpha}(x)\to\partial_{\beta}(x)L^{-1\beta}{}_{\alpha}(x),$$
$$g_{\alpha\beta}(x)\to g_{\gamma\delta}(x)L^{-1\gamma}{}_{\alpha}L^{-1\delta}{}_{\beta}. \tag{41}$$

Thus[2]

$$h^{\alpha}{}_{a}(x) \;\to\; f(x)\delta^{\alpha}_{a}, \tag{42}$$

where x stands for x^{α} and $f(x)$ is composed of $h^{\alpha}{}_{a}(x)$. Under this change Eq. (39) goes over to

$$H\to H=cf_1(x)\gamma^0\gamma^i i\hbar\partial_{\hat{i}}-imc^2f_2(x)\gamma^0 +i\hbar cf_3(x)\Omega_{c,ab}\gamma^0\gamma^c\sigma^{ab}+i\hbar cf_4(x)Q_b\gamma^0\gamma^b, \tag{43}$$

where $f_i(x)$ are composed of $h^{\alpha}{}_{a}(x)$. Putting in the definition

$$\Omega_{c,ab}=\epsilon_{abcd}S^d \tag{44}$$

and using the identity

$$\gamma^a\sigma^{bc}=\frac{1}{2}\eta^{ab}\gamma^c-\frac{1}{2}\eta^{ac}\gamma^b-\frac{1}{2}\epsilon^{abcd}\gamma_d\gamma_5, \tag{45}$$

where $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$, the Hamiltonian matrix becomes

$$H=cf_1(x)\gamma^0\gamma^i i\hbar\partial_{\hat{i}}-imc^2f_2(x)\gamma^0+i\hbar cN_a(x)\gamma^0\gamma^a +i\hbar cf_5(x)S_a\gamma^0\gamma^a\gamma_5, \tag{46}$$

where we introduced

$$N_a:=f_3(x)\Omega^{b,}{}_{ba}+f_4(x)Q_a.$$

If we now define the canonical momenta

$$p_i:=-i\hbar\left(\partial_{\hat{i}}+\frac{N_i(x)}{f_1(x)}\right) \tag{47}$$

and assume

$$p_i^{+}=p_i, \tag{48}$$

Eq. (46) takes the form

$$H=f_1(x)cp^i\gamma_0\gamma_i+imc^2f_2(x)\gamma_0+i\hbar cf_5(x)S^a\gamma_0\gamma_a\gamma_5 -i\hbar cN_0(x). \tag{49}$$

In order eliminate the last term in Eq. (49) one may further perform a locally unitary transformation

$$\psi(x) \;\to\; U^{+}(x)\psi(x),\;\; H \;\to\; U^{+}(x)HU(x) \tag{50}$$

[2] $L\in SO_{+}(1,3)$ where $SO_{+}(1,3)$ is special orthochronous Lorentz group.

and obtain

$$\begin{aligned}H\to H=&f_1(x)cp^iU^{+}(x)\gamma_0\gamma_iU(x)\\&+imc^2f_2(x)U^{+}(x)\gamma_0U(x)\\&+i\hbar cf_5(x)S^aU^{+}(x)\gamma_0\gamma_a\gamma_5U(x)\\&-i\hbar c[f_1(x)U^{+}(x)\gamma_0\gamma_i\partial^{\hat{i}}U(x)+N_0(x)].\end{aligned} \tag{51}$$

Under the solvable matrix equation

$$U^{+}(x)\gamma_0\gamma_i\partial^{\hat{i}}U(x)=-\frac{N_0(x)}{f_1(x)}, \tag{52}$$

we give the final form of our Hermitian Hamiltonian matrix (up to a sign) by the expression

$$H=f_1(x)cp^i\gamma_0\gamma_i+imc^2f_2(x)\gamma_0+i\hbar cf_5(x)S^a\gamma_0\gamma_a\gamma_5. \tag{53}$$

IV. NEUTRINO OSCILLATIONS IN THE KERR BACKGROUND

Here we construct the Hamiltonian matrix of a Dirac particle (i.e., a massive neutrino) of mass m in the background space-time geometry of a heavy, slowly rotating body of mass M such as the Sun. Its exterior gravitational field will be described by weak constant, uniform torsion and nonmetricity fields, together with the Kerr metric [21]:

$$\begin{aligned}ds^2=&-\left(1-\frac{2MGr}{c^2\rho^2}\right)cdt\otimes cdt+\frac{\rho^2}{\Delta}dr\otimes dr+\rho^2d\theta\otimes d\theta\\&+\left(r^2+\frac{a^2}{c^2}+\frac{2MGa^2r}{c^4\rho^2}\sin^2\theta\right)\sin^2\theta d\varphi\otimes d\varphi\\&-\frac{4MGar}{c^2\rho^2}\sin^2\theta dt\otimes d\varphi,\end{aligned} \tag{54}$$

where $\Delta=r^2-(2MG/c^2)r+(a/c)^2$, $\rho^2=r^2+(a/c)^2\cos^2\theta$, $a\equiv J/M=\frac{2}{5}R^2\omega$. The Sun is assumed a uniform sphere of radius R. Here M, J, and ω are the mass, angular momentum, and angular velocity of the Sun, respectively. We choose the orthonormal coframe

$$e^0=\frac{\sqrt{\Delta}}{\rho}\left(cdt-\frac{a}{c}\sin^2\theta d\varphi\right),\quad e^1=\frac{\rho}{\sqrt{\Delta}}dr,$$
$$e^2=\rho d\theta,\quad e^3=\frac{\sin\theta}{\rho}\left[\left(r^2+\left(\frac{a}{c}\right)^2\right)d\varphi-adt\right], \tag{55}$$

and using the definitions

$$de^a+\omega^a{}_b\wedge e^b=0 \;\Leftrightarrow\; \omega_{ab}=-\frac{1}{2}\imath_a de_b+\frac{1}{2}\imath_b de_a+\frac{1}{2}(\imath_a\imath_b de_c)e^c, \tag{56}$$

calculate the Levi-Cività connection 1-forms

$$\omega^0{}_1=\frac{MG[r^2-(a/c)^2\cos^2\theta]}{\rho^4 c}dt+\frac{[(MG/c^2-r)\rho^2-2MGr^2/c^2]a\sin^2\theta}{\rho^4 c}d\varphi,$$

$$\omega^2{}_3=\frac{2MGra\cos\theta}{\rho^4c^2}dt+\frac{\Delta\left(\frac{a}{c}\right)^2\sin^2\theta-[r^2+(a/c)^2]^2}{\rho^4}\cos\theta d\varphi,$$

$$\omega^0{}_2=-\frac{\sqrt{\Delta}a\sin\theta\cos\theta}{\rho^2 c}d\varphi,$$

$$\omega^1{}_3=-\frac{\sqrt{\Delta}r\sin\theta}{\rho^2}d\varphi,$$

$$\omega^0{}_3=\frac{\sqrt{\Delta}a\cos\theta}{\rho^2 c}d\theta-\frac{ar\sin\theta}{\sqrt{\Delta}\rho^2 c}dr,$$

$$\omega^1{}_2=-\frac{a^2\sin\theta\cos\theta}{\rho^2\sqrt{\Delta}c^2}dr-\frac{r\sqrt{\Delta}}{\rho^2}d\theta. \tag{57}$$

To simplify the discussions, we consider only the motion of massive neutrinos restricted to the equatorial plane of the Sun. Thus we set $\theta=\pi/2$ and $d\theta=0$. Furthermore, since the Sun rotates very slowly $[\omega\simeq 3\times 10^{-6}$ (rad/s)$]$ we approximate the metric functions. Therefore, in reasonably far away distances from the Sun, the restricted line element will be taken as

$$ds^2\simeq-\left(1-\frac{2MG}{c^2r}\right)cdt\otimes cdt+\left(1-\frac{2MG}{c^2r}\right)^{-1}dr\otimes dr+rd\varphi\otimes rd\varphi-4\frac{a}{c}\frac{MG}{c^2r^2}cdt\otimes rd\varphi. \tag{58}$$

We also write the orthonormal coframe approximately up to $O(a/rc)$ as

$$e^0=fcdt-\frac{af}{c}d\varphi,\quad e^1=\frac{1}{f}dr,$$

$$e^2=0,\quad e^3=-\frac{a}{r}dt+rd\varphi, \tag{59}$$

where

$$\frac{\Delta}{\rho^2}\equiv f^2\simeq 1-\frac{2MG}{c^2r}. \tag{60}$$

The inverses of these relations to the same order of approximation are

$$cdt=\frac{1}{f}e^0+\frac{a}{rc}e^3,\quad dr=fe^1,\quad d\theta=0,$$

$$d\varphi=\frac{a}{fr^2c}e^0+\frac{1}{r}e^3, \tag{61}$$

which give

$$h^{\hat 0}{}_0=\frac{1}{f},\quad h^{\hat 0}{}_3=\frac{a}{cr},\quad h^{\hat 1}{}_1=f,\quad h^{\hat 3}{}_0=\frac{a}{fcr^2},\quad h^{\hat 3}{}_3=\frac{1}{r}, \tag{62}$$

with all other components neglected. To this order of approximation Eq. (57) gives

$$\omega_{01}\simeq f'e^0+\frac{a}{cr^2}e^3,\;\; \omega_{03}\simeq\frac{a}{cr^2}e^1,\;\; \omega_{31}\simeq\frac{a}{cr^2}e^0+\frac{f}{r}e^3, \tag{63}$$

with the remaining ones neglected. Then the Hamiltonian matrix (39) reads

$$H\simeq f^2cp_r\gamma_0\gamma_1+fcp_\varphi\gamma_0\gamma_3+ifmc^2\gamma_0-\frac{i}{2}\hbar cff'\gamma_0\gamma_1+\frac{3}{2}i\hbar cfS^a\gamma_0\gamma_5\gamma_a-i\hbar cfN^a\gamma_0\gamma_a-\frac{af^3}{r}p_r\gamma_3\gamma_1+\frac{ia\hbar f^2f'}{2r}\gamma_3\gamma_1-\frac{iamcf^2}{r}\gamma_3+\frac{ia\hbar f}{2r^2}\gamma_0\gamma_2\gamma_5+\frac{3ia\hbar f^2}{2r}S^a\gamma_3\gamma_a\gamma_5+\frac{ia\hbar f^2}{r}N^a\gamma_3\gamma_a, \tag{64}$$

where

$$p_r:=-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right), \tag{65}$$

$$p_\theta:=0, \tag{66}$$

$$p_\varphi:=-\frac{i\hbar}{r}\frac{\partial}{\partial\varphi}, \tag{67}$$

$$N_a := \frac{1}{2}\left(t^b{}_{a,b} + F_a + \frac{5}{4} Q_a - \Lambda_a \right). \tag{68}$$

Note that the contributions of axial components of torsion are given by S^a while certain components of nonmetricity and the nonaxial components of torsion occur only in N_a and the rotation effects are given in terms of the parameter a. We rewrite the Hamiltonian 4×4 matrix in terms of 2×2 matrices as follows:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{69}$$

with

$$H_{11} = fmc^2 + iA + B\sigma_1 + \left(C - i\frac{af^3}{r} p_r \right)\sigma_2 + D\sigma_3,$$

$$H_{22} = -fmc^2 + iA + B\sigma_1 + \left(C - i\frac{af^3}{r} p_r \right)\sigma_2 + D\sigma_3,$$

$$H_{12} = F + (f^2 c p_r + iG)\sigma_1 + iH\sigma_2 + \left(fcp_\varphi + \frac{amcf^2}{r} + iK \right)\sigma_3,$$

$$H_{21} = F + (f^2 c p_r + iG)\sigma_1 + iH\sigma_2 + \left(fcp_\varphi - \frac{amcf^2}{r} + iK \right)\sigma_3, \tag{70}$$

where we set

$$A \simeq -\hbar c f N_0 + \frac{a\hbar f^2}{r} N_3,$$

$$B \simeq \frac{3}{2}\hbar c S_1 + \frac{a\hbar f^2}{r} N_2,$$

$$C \simeq \frac{3}{2}\hbar c S_2 - \frac{a\hbar f^2}{2r^2}(1 + rf' + rN_1),$$

$$D \simeq \frac{3}{2}\hbar c S_3 - \frac{3a\hbar f^2}{2r} S_0,$$

$$F \simeq \frac{3}{2}\hbar c S_0 - \frac{3a\hbar f^2}{2r} S_3,$$

$$G \simeq -\frac{\hbar c f f'}{2r} - \hbar c f N_1 + \frac{3a\hbar f^2}{2r} S_2,$$

$$H \simeq -\hbar c f N_2 - \frac{3a\hbar f^2}{2r} S_1,$$

$$K \simeq -\hbar c f N_3 + \frac{a\hbar f^2}{r} N_0. \tag{71}$$

The way we approach the solar neutrino problem starts by writing down the Dirac equation in a rotating, axially symmetric background space-time geometry and finding phases corresponding to neutrino mass eigenstates, then finally calculating the phase differences among them. There are two cases of special interest: the azimuthal motion and the radial motion. The analysis of the azimuthal motion with $\vec{p} = (p_r, p_\theta, p_\varphi) = (0,0,p)$ yields for ultrarelativistic neutrinos, for which $pc \simeq E$ and $cdt \simeq Rd\varphi$, the phase for the spin up state

$$\Phi^\uparrow = \left(fE + \frac{fm^2c^4}{2E} + \sqrt{\Delta_\varphi} + i(A+K) \right)\frac{R\Delta\varphi}{\hbar c} \tag{72}$$

and similarly for the phase of the spin down state

$$\Phi^\downarrow = \left(fE + \frac{fm^2c^4}{2E} - \sqrt{\Delta_\varphi} + i(A+K) \right)\frac{R\Delta\varphi}{\hbar c}, \tag{73}$$

where

$$\Delta_\varphi \simeq B^2 + C^2 + D^2 + F^2 + G^2 + H^2 + 2(DF + BH - CG). \tag{74}$$

These phases alone do not have an absolute meaning; the quantities relevant for the interference pattern at the observation point of the neutrinos are the phase differences $\Delta\Phi = \Phi_2 - \Phi_1$ where Φ_1 and Φ_2 are the absolute phases of the neutrino mass eigenstates ν_1 and ν_2. It is thus seen from Eqs. (72) and (73) that the phase differences can have explicit dependence on nonmetricity in the case of opposite spin polarizations of mass eigenstates for the azimuthal motion via Eq. (74):

$$\Delta\Phi = \Phi_2^\downarrow - \Phi_1^\uparrow = \left(\frac{\Delta m^2 c^4}{2(E/f)} - 2\sqrt{\Delta_\varphi} \right)\frac{R\Delta\varphi}{\hbar c}, \tag{75}$$

$$\Delta\Phi = \Phi_2^\uparrow - \Phi_1^\downarrow = \left(\frac{\Delta m^2 c^4}{2(E/f)} + 2\sqrt{\Delta_\varphi} \right)\frac{R\Delta\varphi}{\hbar c}, \tag{76}$$

where $\Delta m^2 = m_2^2 - m_1^2$.

The Hamiltonian for the radial motion on the other hand is obtained by the assumption $\vec{p} = (p,0,0)$. In this case with the further assumptions $pc \simeq E$ and $cdt \simeq dr$, the phases appropriate to the spin up and spin down particles are, respectively,

$$\Phi^\uparrow = \frac{1}{\hbar c}\int \left(f^2 E + \frac{m^2c^4}{2E} + \sqrt{\Delta_r} + i(A+G) \right) dr, \tag{77}$$

$$\Phi^\downarrow = \frac{1}{\hbar c}\int \left(f^2 E + \frac{m^2c^4}{2E} - \sqrt{\Delta_r} + i(A+G) \right) dr, \tag{78}$$

where

$$\Delta_r \simeq (D-H)^2 + \left(B+F+\frac{amH}{rp}\right)^2 + \left(C+K-\frac{amG}{rp}\right)^2 - \frac{a^2f^4}{r^2}(mc-fp)^2 + \frac{2iaf^2}{r}(mc-fp)\left(C+K-\frac{amG}{rp}\right). \tag{79}$$

In this case the relevant phase differences depending on nonmetricity via N^a and rotation via a come from the opposite spin polarization states

$$\Delta\Phi = \Phi_2^{\downarrow} - \Phi_1^{\uparrow} = \frac{\Delta m^2 c^3}{2\hbar E}\Delta r - \frac{2}{\hbar c}\int \sqrt{\Delta_r}dr, \tag{80}$$

$$\Delta\Phi = \Phi_2^{\uparrow} - \Phi_1^{\downarrow} = \frac{\Delta m^2 c^3}{2\hbar E}\Delta r + \frac{2}{\hbar c}\int \sqrt{\Delta_r}dr. \tag{81}$$

We point out that $\Delta_r = \mathrm{Re}\,\Delta_r + i\,\mathrm{Im}\,\Delta_r$ implies $\sqrt{\Delta_r} = \alpha + i\beta$ and hence the rotation of the Sun would suppress the transitions among the neutrinos via the phase difference equations (80),(81) in opposite spin polarizations.

V. CONCLUSION

We have here extended our recent study of gravitationally induced neutrino oscillations [8] by including the effects of rotation of the Sun and space-time nonmetricity and as well as components of torsion other than the axial ones. The rotation of the Sun implies a damping of neutrino oscillations. However, this result is frame dependent as we explained in Sec. III in general. We have shown that there are contributions coming from nonaxial components of space-time torsion and definite components of space-time nonmetricity depending on the polarizations of the spin states of the mass eigenstates. If we set the rotation parameter $a=0$, then Eq. (79) gives

$$\sqrt{\Delta_r} = \frac{3}{2}\hbar c\left[(S_0+S_1)^2 + \left(S_2 - \frac{2}{3}fN_3\right)^2 + \left(S_3 + \frac{2}{3}fN_2\right)^2\right]^{1/2}, \tag{82}$$

which means that there is no suppression among the neutrinos and only N_2 and N_3 components of N^a contribute to the oscillations. If we further set $N^a = 0$, we reach agreement with our previous results in [8]. It should be clear that the above scheme only works if the neutrino masses are different from each other and hence, in general, different from zero. This means there are right-handed neutrinos as well as left-handed ones which, however, must interact with matter very weakly as they have not yet been observed. Finally, we note that all possible contributions discussed here so far would be of the order of the Planck scale, and hence do not suffice to account for the observed solar neutrino deficit.

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