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Solitary and Periodic Wave Solutions of the Space-Time Fractional Extended Kawahara Equation

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Abstract: The extended Kawahara (Gardner Kawahara) equation is the improved form of the Korteweg–de Vries (KdV) equation, which is one of the most significant nonlinear evolution equations in mathematical physics. In that research, the analytical solutions of the conformable fractional extended Kawahara equation were acquired by utilizing the Jacobi elliptic function expansion method. The given expansion method was applied to different fractional forms of the extended Kawahara equation, such as the fraction that occurs in time, space, or both time and space by suitably changing the variables. In addition, various types of fractional problems are exhibited to expose the realistic application of the given method, and some of the obtained solutions were illustrated in two- or three-dimensional graphics as proof of the visualization.

Keywords: Jacobi elliptic function; expansion method; fractional partial differential equation; extended Kawahara equation

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1. Introduction

The famous Korteweg–de Vries (KdV) equation has been known since 1895 when it was first obtained by Korteweg and de Vries in their research on long waves in shallow water [1]. The KdV equation and its variations play a remarkable and operational role in modelling and explicating many facts that appear in many subdivisions of science, such as fluids, Bose–Einstein condensates (BECs), plasma physics, shallow water waves, capillary–gravity water waves, quark–gluon plasma waves (like solitons), nuclear waves (like soliton), and in electrical networks, etc. [2]. Therefore, solving fractional KdV equations in the sense of different fractional derivatives has attracted scientists, and they have found solutions for the time, space, or space-time fractional KdV equation utilizing different techniques and methods [3–16]. The general form of the fractional KdV equation is

$$D_t^{\alpha} u + cuD_x^{\beta} u + bD_x^{\beta} D_x^{\beta} D_x^{\beta} u = 0, \quad 0 < \alpha, \beta \leq 1$$

where b and c are the coefficients of dispersion and nonlinear terms, respectively. Since KdV equations and its variations are used for the modelling of weakly nonlinear and dispersive long waves, the balance between the dispersion term (wave broadening) and the nonlinear term (wave steepening) leads to the origination of *solitons* (*solitary waves*). Therefore, when a higher order of nonlinearity is considered to identify the solitons at critical values, the following fractional modified KdV equation (mKdV equation) and combined fractional KdV-mKdV equation or fractional extended Korteweg de Vries (eKdV) equation, respectively, arise [2]:

$$D_t^{\alpha} u + au^2 D_x^{\beta} u + bD_x^{\beta} D_x^{\beta} D_x^{\beta} u = 0, \quad 0 < \alpha, \beta \leq 1$$

$$D_t^{\alpha} u + \left(cu + au^2\right) D_x^{\beta} u + bD_x^{\beta} D_x^{\beta} D_x^{\beta} u = 0, \quad 0 < \alpha, \beta \leq 1.$$

In these equations, if $\alpha = 1$ and $0 < \beta < 1$, the equations are space fractional; if $0 < \alpha < 1$ and $\beta = 1$, the equations are time fractional or they are named space-time fractional equations. There are several ways to solve these types of fractional evolution equations in the literature [17–26]. Up to this point, we have reviewed the effects of nonlinearity terms on wave construction; however, in some circumstances, a higher order dispersion effect could be needed. For that purpose, a higher-order KdV equation with an additional derivative term of the fifth order was first introduced by Kawahara in 1972 to give an equation which describes solitary wave propagation in media. After that, the modified Kawahara equation was obtained, while the derivative of the fifth order for the dispersion was added to the modified KdV equation. For the explained values of α and β, the following equation is named the space-time (sometimes space or sometimes time) fractional Kawahara equation

$$D_t^{\alpha} u + cuD_x^{\beta} u + bD_x^{\beta} D_x^{\beta} D_x^{\beta} u - \lambda D_x^{\beta} D_x^{\beta} D_x^{\beta} D_x^{\beta} D_x^{\beta} u = 0, \quad 0 < \alpha, \beta \leq 1$$

and the following equation is called as the space-time (sometimes space or sometimes time) fractional modified Kawahara equation

$$D_t^{\alpha} u + au^2 D_x^{\beta} u + bD_x^{\beta} D_x^{\beta} D_x^{\beta} u - \lambda D_x^{\beta} D_x^{\beta} D_x^{\beta} D_x^{\beta} D_x^{\beta} u = 0, \quad 0 < \alpha, \beta \leq 1$$

where a, b, and λ are arbitrary constants that occurs in many branches of physics, such as shallow water waves, plasma waves, capillary–gravity water waves, and water waves with surface tension. Numerous types of methods have been used for resolving these equations for the different values of the fractional derivatives α and β [27–33].

Furthermore, by combining the modified Kawahara and the Kawahara equations, we obtain the Extended Kawahara equation (sometimes the Gardner Kawahara equation), which could be employed for examining various nonlinear structures in optical fibers, the physics of plasma, etc., close to the decisive values of the appropriate physical arguments that make the coefficients of dispersions and nonlinearity closed to zero [2]. So far, the solutions of the extended Kawahara equation have been obtained by using the traditional tanh method, the Jacobian elliptic function method, the sech square method, and Weierrtrass elliptic function method [2], ansatz method [34–36], the septic B-spline collocation method [36], and the method of lines [36]. On the other hand, solutions for the fractional extended Kawahara equation have never been researched so far, as indicated in the paper by El-Tantawy et al. [2].

Consequently, in this research paper, we develop an expansion method based upon the JEFs for analytical solutions of the conformable time-space fractional extended Kawahara equation in the general form

$$D_t^{\alpha} u + (cu + au^2) D_x^{\beta} u + bD_x^{\beta} D_x^{\beta} D_x^{\beta} u - \lambda D_x^{\beta} D_x^{\beta} D_x^{\beta} D_x^{\beta} D_x^{\beta} u = 0, \quad 0 < \alpha, \beta \leq 1 \qquad (1)$$

where a, c, and λ are nonzero constants and b is arbitrary constant. D_t^{α} and D_x^{β} typify the fractional derivative of the two-variable function $u(x,t)$ in the conformable sense with respect to time variable t and space variable x, respectively. The equation given in (1) is the most generalized structure of the fractional extended Kawahara equation in the researched literature, and solutions for this equation are investigated for the first time using the Jacobian elliptic function expansion method. Using this method, a large number of solutions have been researched since Jacobi elliptic functions comprise different types of functions, such as trigonometric, hyperbolic, complex, and rational functions. When the fractional orders α and β are both equal to one, the fractional equation transforms into the integer order ordinary differential equation, and therefore, the method given in this paper also involves solutions for this equation.

The remainder of the paper is systematized as indicated here: In the second section, the Jacobian elliptic functions (JEFs) and their useful properties are discussed; the definition and rudimentary features of the fractional derivative in the conformable sense are also presented. In the third section, the expansion method based upon Jacobi elliptic functions is introduced and used to attain analytical solutions for the space-time fractional extended Kawahara equation in the conformable sense, and these solutions are listed and exhibited in a table. In the fourth section, variable types of problems are provided to testify the applicability of the given method, and a number of the solutions obtained are illustrated by both two- and three-dimensional graphics. The paper is concluded in the last section.

2. Preliminaries

In this part of the paper, we give necessary definitions and theorems that are utilised within the process of solving the presented conformable fractional extended Kawahara equation.

First, we begin with introducing elementary Jacobi elliptic functions (JEFs), which are given as

$$\text{sn}\xi = \text{sn}(\xi; m),\ \text{cn}\xi = \text{cn}(\xi; m),\ \text{dn}\xi = \text{dn}(\xi; m).$$

Here the variable m symbolizes the modulus of the elliptic function, and it takes a value between 0 and 1. Jacobi elliptic functions are doubly periodic functions, and they have their own relationships, similar to the trigonometric and hyperbolic functions:

$$\text{sn}^2\xi + \text{cn}^2\xi = 1,\ \text{dn}^2\xi + m^2\text{sn}^2\xi = 1,$$
$$\text{sn}'\xi = \text{cn}\xi\ \text{dn}\xi,\ \text{cn}'\xi = -\text{sn}\xi\ \text{dn}\xi,\ \text{dn}'\xi = -m^2\text{cn}\xi\ \text{sn}\xi$$

Moreover, there are nine more elliptic functions that are formed by the basic ones, namely, nc, ns, nd, sd, sc, cd, ds, dc, and cs. Another explanation for the notation can be obtained from the definition stated in [37]. Furthermore, the differential properties of JEFs are also shown in Table 1.

Table 1. The derivatives of twelve JEFs.

1	$(\text{cn})'(\xi) = -\text{sn}\xi\text{dn}\xi$	$(\text{sn})'(\xi) = \text{cn}\xi\text{dn}\xi$	$(\text{dn})'(\xi) = -m^2\text{sn}\xi\text{dn}\xi$
2	$(\text{cd})'(\xi) = (m^2-1)\text{sd}\xi\text{nd}\xi$	$(\text{sd})'(\xi) = \text{cd}\xi\text{nd}\xi$	$(\text{nd})'(\xi) = m^2\text{sd}\xi\text{cd}\xi$
3	$(\text{nc})'(\xi) = \text{sc}\xi\text{dc}\xi$	$(\text{sc})'(\xi) = \text{dc}\xi\text{nc}\xi$	$(\text{dc})'(\xi) = (1-m^2)\text{sc}\xi\text{nc}\xi$
4	$(\text{cs})'(\xi) = -\text{ds}\xi\text{ns}\xi$	$(\text{ns})'(\xi) = -\text{cs}\xi\text{ds}\xi$	$(\text{ds})'(\xi) = -\text{cs}\xi\text{ns}\xi$

In addition, the similarity mentioned above is not coincidental since Jacobian elliptic functions convert into trigonometric functions when the elliptic modulus $m \to 0$, and they turn into hyperbolic functions when $m \to 1$ as it can be seen in Table 2 explicitly [38].

Table 2. The behavior of JEFs when $m \to 0$ and $m \to 1$.

	JEF	$m\to 0$	$m\to 1$		**JEF**	$m\to 0$	$m\to 1$
1	snξ	sin ξ	tanhξ	7	dcξ	sec ξ	1
2	cnξ	cos ξ	sechξ	8	ncξ	sec ξ	cosh ξ
3	dnξ	1	sechξ	9	scξ	tan ξ	sinhξ
4	cdξ	cos ξ	1	10	nsξ	csc ξ	cothξ
5	sdξ	sin ξ	sinhξ	11	dsξ	csc ξ	cschξ
6	ndξ	1	cosh ξ	12	csξ	cot ξ	cschξ

Since Khalil et al. [39] brought the definition for "the conformable fractional derivative" into the literature, this uncomplicated fractional derivative has become very popular among mathematicians, physicians, and other scientists due to its dependence just on the well-known definition of the usual derivative. Therefore, in this paper, the conformable fractional derivative is integrated into the extended Kawahara equation. Now, in the final part of this section, we introduce this new fractional derivative:

Definition 1 ([39]). *Suppose $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a function, then the fractional derivative of the α-th order in conformable sense of the function f is specified by the limit*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \qquad t > 0, \quad \alpha \in (0, 1].$$

When f is differentiable of the α-th order in some $(0, \alpha)$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ occurs, then we can identify $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

Theorem 1 ([39]). *Suppose $\alpha \in (0, 1]$ and f, g are conformable differentiable of order α at the point $t > 0$. Then, the following 6 expressions are satisfied for all f and g:*

1. *Linearity: $T_\alpha(kf + lg) = kT_\alpha(f) + lT_\alpha(g)\ \forall\, k, l \in \mathbb{R}$.*
2. $T_\alpha(f) = 0\ if\ f\ is\ a\ constant\ function.$
3. $T_\alpha\left(t^k\right) = kt^{k-\alpha}\ \forall k \in \mathbb{R}$.
4. *If the function g is α-differentiable, then $T_\alpha(g)(t) = t^{1-\alpha} \frac{dg}{dt}$.*

Theorem 2 ([40]). *Suppose that $g, h : \mathbb{R}^+ \to \mathbb{R}$ are conformable differentiable functions of the order α, where $\alpha \in (0, 1]$ and $f(t) = h(g(t))$. Then the composite function $h(t)$ is conformable differentiable of order α and for all t with $t \neq 0$ and $g(t) \neq 0$, we get*

$$T_\alpha(f)(t) = T_\alpha(h)(g(t)).T_\alpha(g)(t).g(t)^{\alpha-1}$$

If $t = 0$, we have

$$T_\alpha(f)(0) = \lim_{t \to 0} T_\alpha(h)(g(t)).T_\alpha(g)(t).g(t)^{\alpha-1}.$$

This theorem is called the chain rule for the fractional derivative of the conformable type.

Additionally, since the definition of the conformable fractional derivative is very accustomed to the definition of the usual integer order derivative, there is an apparent correlation between the two definitions. In the definition of the conformable fractional derivative, when the fractional order α equals one, the fractional derivative converts into the first order usual derivative.

3. The Solution Method (JEF Expansion Method)

In this part of the paper, we take into consideration the time-space conformable fractional extended (Gardner) Kawahara Equation (1). To obtain analytical solutions for this equation, we use the well-known Jacobi elliptic function (JEF) expansion method. Literally, in the Jacobi elliptic function (JEF) expansion method, the solutions of the given problems are investigated in terms of these functions. The JEF expansion method has been used several times to attain solutions for multifarious classes of equations either of the integer order or the fractional order [13,26,31,41–44]. In the solution process, before applying the mentioned expansion method, the fractional extended Kawahara equation is transformed into an integer order ordinary differential equation in a new variable by using a suitable transformation to change the variables.

Now, using the transformation

$$\xi = k\frac{t^\alpha}{\alpha} + l\frac{x^\beta}{\beta}$$

such that k and l arbitrary nonzero constants, and utilizing the statement of the Theorem 2, fractional extended Kawahara Equation (1) is modified into an integer order ordinary differential equation (ODE) given by

$$k\frac{du}{d\xi} + clu\frac{du}{d\xi} + alu^2\frac{du}{d\xi} + bl^3\frac{d^3u}{d\xi^3} - \lambda l^5\frac{d^5u}{d\xi^5} = 0. \tag{2}$$

The principal notion of the expansion method based upon elliptic functions of the Jacobi type is to attain the solutions $u(\xi)$ formed as

$$u(\xi) = \sum_{j=0}^{N} c_j F^j(\xi).$$

Here, the constants N and c_j for $j = 0, 1, \ldots,\ N$ are supposed to be selected and decided according to the situation. The function F is the solution for the following nonlinear ODE (Jacobi elliptic equation)

$$(dF/d\xi)^2(\xi) = PF^4(\xi) + QF^2(\xi) + R. \tag{3}$$

where ξ is a variable depending on both x and t, where $P,\ Q,$ and R are constants. Further information about the solutions of the elliptic Equation (3) can be found in Ref. [45].

Firstly, since balancing wave broadening (dispersion term) and wave steepening (the nonlinear term) leads to the creation of solitons (solitary waves), we attain the balance as $N = 2$ (the homogeneous balance between the term $u^2\frac{du}{d\xi}$ and the term $\frac{d^5u}{d\xi^5}$,). Hence, second-degree solutions of the ODE (2) are given explicitly as

$$u(\xi) = c_0 + c_1F(\xi) + c_2F^2(\xi).$$

By differentiating this function five times and replacing the necessary expressions by Equation (3), we have

$$\begin{aligned}
&u'(\xi) = (c_1 + 2c_2F)F',\\
&u'''(\xi) = \left(c_1Q + 6c_1PF^2 + 8c_2QF + 24c_2PF^3\right)F',\\
&u^{(5)}(\xi) = \Big(c_1Q^2 + 12c_1PR + \left(32c_2Q^2 + 144c_2PR\right)F + 60c_1PQF^2 + \\
&\qquad +480c_2PQF^3 + 120c_1P^2F^4 + 720c_2P^2F^5\Big)F'.
\end{aligned}$$

After that, by substituting these expressions into Equation (2), two possibilities occur: in the former case $F' = 0$; therefore, the first solution is obtained for $c_0 =$ arbitrary constant, $c_0 = c_1 = 0$. In the latter case, a polynomial of the fifth order in the function F is attained, and then since the RHS of the obtained equation is zero, by adjusting the coefficients of each order to be zero also, the following nonlinear equations system is found

$$\begin{aligned}
kc_1 + clc_0c_1 + alc_0^2c_1 + bl^3c_1Q - \lambda l^5c_1Q^2 - 12\lambda l^5c_1PR &= 0\\
2kc_2 + clc_1^2 + 2clc_0c_2 + 2alc_0c_1^2 + 2alc_0^2c_2 + 8bl^3c_2Q - 32\lambda l^5c_2Q^2 - 144\lambda l^5c_2PR &= 0\\
3clc_1c_2 + alc_1^3 + 6alc_0c_1c_2 + 6bl^3c_1P - 60\lambda l^5c_1PQ &= 0\\
2clc_2^2 + 4alc_1^2c_2 + 4alc_0c_2^2 + 24bl^3c_2P - 480\lambda l^5c_2PQ &= 0\\
5alc_1c_2^2 - 120\lambda l^5c_1P^2 &= 0\\
2alc_2^3 - 720\lambda l^5c_2P^2 &= 0
\end{aligned}$$

Finally, solving this algebraic system yields $c_0 = \mp(4AQ - B) - C,\ c_1 = 0,\ c_2 = \mp 12AP$, and $c_0 =$ arbitrary constant, $c_1 = c_2 = 0$, such that

$$24\lambda l^5\left(3PR - Q^2\right) = k + \left(b^2l/10\lambda\right) - cl\left[\mp(4AQ - B) + \frac{C}{2}\right]. \tag{4}$$

Here, $A = l^2\sqrt{5\lambda/2a}$, $B = b/\sqrt{10\lambda a}$, and $C = c/2a$. Therefore, solutions for Equation (2) become

$$u = \mp(4AQ - B) - C \mp 12APF^2 \text{ and } u = c_0.$$

In the final step, since we have solutions for Equation (2), we use the solution table of the Jacobi elliptic equation to exhibit some of the solutions of this equation in a table [13,31], and we present these solutions in Table 3. Using Table 3 and by taking the inverse transformation from ξ to the variables x and t, we obtain JEF solutions for the time-space conformable fractional extended Kawahara Equation (1). For further solutions, the reader can check the extended solution table (Table 2) in [31].

Table 3. JEF Solutions of Equation (2) for particular values of P, Q, and R.

	P	Q	R	Solutions
1	m^2	$-(1+m^2)$	1	$u_{1,1} = \pm B \pm 4A(1+m^2) - C \mp 12Am^2\mathrm{sn}^2\xi$ $u_{1,2} = \pm B \pm 4A(1+m^2) - C \mp 12Am^2\mathrm{cd}^2\xi$
2	$-m^2$	$2m^2-1$	$1-m^2$	$u_2 = \mp(4A(2m^2-1) - B) - C \pm 12Am^2\mathrm{cn}^2\xi$
3	-1	$2-m^2$	m^2-1	$u_3 = \mp 4A(2-m^2) \pm B - C \pm 12A\mathrm{dn}^2\xi$
4	1	$-(1+m^2)$	m^2	$u_{4,1} = \pm B \pm 4A(1+m^2) - C \mp 12A\mathrm{ns}^2\xi$ $u_{4,2} = \pm B \pm 4A(1+m^2) - C \mp 12A\mathrm{dc}^2\xi$
5	$1-m^2$	$2m^2-1$	$-m^2$	$u_5 = \mp(4A(2m^2-1) - B) - C \mp 12A(1-m^2)\mathrm{nc}^2\xi$
6	m^2-1	$2-m^2$	-1	$u_6 = \mp(4A(2-m^2) - B) - C \pm 12A(1-m^2)\mathrm{nd}^2\xi$
7	$1-m^2$	$2-m^2$	1	$u_7 = \mp(4A(2-m^2) - B) - C \mp 12A(1-m^2)\mathrm{sc}^2\xi$
8	m^4-m^2	$2m^2-1$	1	$u_8 = \mp(4A(2m^2-1) - B) - C \mp 12A(m^4-m^2)\mathrm{sd}^2\xi$
9	1	$2-m^2$	$1-m^2$	$u_9 = \pm B \pm 4A(m^2-2) - C \mp 12A\mathrm{cs}^2\xi$
10	1	$2m^2-1$	$-m^2+m^4$	$u_{10} = \pm B \mp 4A(2m^2-1) - C \mp 12A\mathrm{ds}^2\xi$
11	$-\frac{1}{4}$	$\frac{1+m^2}{2}$	$-\frac{(1-m^2)^2}{4}$	$u_{11} = \pm B \mp 2A(1+m^2) - C \pm 3A(m\mathrm{cn}\xi \mp \mathrm{dn}\xi)^2$
12	$\frac{1}{4}$	$\frac{-2m^2+1}{2}$	$\frac{1}{4}$	$u_{12} = \pm B \mp 2A(1-2m^2) - C \mp 3A(\mathrm{ns}\xi \mp \mathrm{cs}\xi)^2$
13	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$u_{13} = \pm B \mp 2A(1+m^2) - C \mp 3A(1-m^2)(\mathrm{nc}\xi \mp \mathrm{sc}\xi)^2$
14	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$	$u_{14} = \pm B \mp 2A(m^2-2) - C \mp 3A(\mathrm{ns}\xi \mp \mathrm{ds}\xi)^2$
15	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$u_{15,1} = \pm B \mp 2A(m^2-2) - C \mp 3Am^2(\mathrm{sn}\xi \mp i\mathrm{cn}\xi)^2$ $u_{15,2} = \pm B \mp 2A(m^2-2) - C \mp 3A\frac{m^2\mathrm{dn}^2\xi}{1-m^2\mathrm{sn}\xi \mp \mathrm{cn}\xi}$
16	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$u_{16,1} = \pm B \mp 2A(1-2m^2) - C \mp 3A(m\mathrm{cn}\xi \mp i\mathrm{dn}\xi)^2$ $u_{16,2} = \pm B \mp 2A(1-2m^2) - C \mp 3A\left(\frac{\mathrm{sn}\xi}{1\mp\mathrm{cn}\xi}\right)^2$
17	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$u_{17} = \pm B \mp 2A(m^2-2) - C \mp 3Am^2\left(\frac{\mathrm{sn}\xi}{1\mp\mathrm{dn}\xi}\right)^2$
18	$\frac{m^2-1}{4}$	$\frac{1+m^2}{2}$	$\frac{m^2-1}{4}$	$u_{18} = \pm B \mp 2A(1+m^2) - C \mp 3A(m^2-1)\left(\frac{\mathrm{dn}\xi}{1\mp m\mathrm{sn}\xi}\right)^2$
19	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$u_{19} = \pm B \mp 2A(1+m^2) - C \mp 3A(1-m^2)\left(\frac{\mathrm{cn}\xi}{1\mp\mathrm{sn}\xi}\right)^2$
20	$\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1}{4}$	$u_{20} = \pm B \mp 2A(1+m^2) - C \mp 3A(1-m^2)^2\left(\frac{\mathrm{sn}\xi}{\mathrm{dn}\xi\mp\mathrm{cn}\xi}\right)^2$
21	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$u_{21} = \pm B \mp 2A(m^2-2) - C \mp 3Am^2\frac{\mathrm{cn}^2\xi}{1-m^2\mp\mathrm{dn}\xi}$

Moreover, by utilizing the outcomes of Table 2, we can obtain the well-known trigonometric and hyperbolic function solutions of Equation (2) in Table 4.

Table 4. The solutions of Equation (2) for the values of the elliptic modulus m.

	$m\to 0$	$m\to 1$
1	$u = -C \pm B \pm 4A$	$u = -C \pm B \pm 8A \mp 12A\tanh^2\xi$ $u = -C \pm B \mp 4A$
2	$u = -C \mp B \mp 4A$	$u = -C \mp (4A - B) \pm 12A\mathrm{sech}^2\xi$
3	$u = -C \pm B \pm 4A$	$u = -C \mp 4A \pm B \pm 12A\mathrm{sech}^2\xi$

Table 4. *Cont.*

	$m\to 0$	$m\to 1$
4	$u=-C\pm B\pm 4A\mp 12A\csc^2\xi$ $u=-C\pm B\pm 4A\mp 12A\sec^2\xi$	$u=-C\pm B\pm 8A\mp 12A\coth^2\xi$ $u=-C\pm B\mp 8A$
5	$u=-C\pm B\pm 4A\mp 12A\sec^2\xi$	$u=-C\mp(4A-B)$
6	$u=-C\pm B\pm 4A$	$u=-C\mp(4A-B)$
7	$u=-C\mp(8A-B)\mp 12A\tan^2\xi$	$u=-C\mp(4A-B)$
8	$u=-C\pm(4A+B)$	$u=-C\mp(4A-B)$
9	$u=-C\pm B\mp 8A\mp 12A\cot^2\xi$	$u=-C\pm B\mp 4A\mp 12A\text{csch}^2\xi$
10	$u=-C\pm B\pm 4A\mp 12A\csc^2\xi$	$u=-C\pm B\mp 4A\mp 12A\text{csch}^2\xi$
11	$u=-C\pm B\pm A$	$u=-C\pm B\mp 4A\pm 12A\text{sech}^2\xi$ $u=-C\pm B\mp 4A$
12	$u=-C\pm B\mp 2A\mp 3A(\csc\xi\mp\cot\xi)^2$	$u=-C\pm B\pm 2A\mp 3A(\coth\xi\mp\text{csch}\xi)^2$
13	$u=-C\pm B\mp 2A\mp 3A(\sec\xi\mp\tan\xi)^2$	$u=-C\pm B\mp 4A$
14	$u=-C\pm B\pm 4A\mp 12A\csc^2\xi$ $u=-C\pm B\pm 4A$	$u=-C\pm B\pm 2A\mp 3A(\coth\xi\mp\text{csch}\xi)^2$
15	$u=-C\pm B\pm 4A$	$u=-C\pm B\pm 2A\mp 3A(\tanh\xi\mp i\text{sech}\xi)^2$ $u=-C\pm B\pm 2A\mp 3A\frac{\text{sech}^2\xi}{1-\tanh\xi\mp\text{sech}\xi}$
16	$u=-C\pm B\pm A$ $u=-C\pm B\mp 2A\mp 3A\left(\frac{\sin\xi}{1\mp\cos\xi}\right)^2$	$u=-C\pm B\pm 2A\mp 3A((1\mp \text{i})\text{sech}\xi)^2$ $u=-C\pm B\pm 2A\mp 3A\left(\frac{\sinh\xi}{1\mp\cosh\xi}\right)^2$
17	$u=-C\pm B\pm 4A$	$u=-C\pm B\pm 2A\mp 3A\left(\frac{\sinh\xi}{1\mp\cosh\xi}\right)^2$
18	$u=-C\pm B\pm A$	$u=-C\pm B\mp 4A$
19	$u=-C\pm B\mp 2A\mp 3A\left(\frac{\cos\xi}{1\mp\sin\xi}\right)^2$	$u=-C\pm B\mp 4A$
20	$u=-C\pm B\mp 4A\mp 3A\left(\frac{\sin\xi}{1\mp\cos\xi}\right)^2$	$u=-C\pm B\mp 4A$
21	$u=-C\pm B\pm 8A$	$u=-C\pm B\pm 2A+3A\text{sech}\xi$

4. Demonstrations and Applications

In this part of the paper, we present distinctive sorts of examples of conformable fractional extended Kawahara Equation (1), which are either time, space, or space-time fractional. The solutions of these examples are supported by tables, and some of them will be illustrated using 2D or 3D graphics, which have been sketched using Mathematica 13.2.

Example 1. *Let us think about the conformable time fractional extended Kawahara Equation (1) for the coefficients $a=-10$, $b=20$, $c=40$, $\lambda=-1$, $\alpha=0.2$, and $\beta=1$; that is*

$$D_t^{1/5}u-10u^2u_x+20u_{xxx}+u_{xxxxx}=0 \tag{5}$$

When $m\to 0$, on the left-hand side of the condition (4) $3PR-Q^2$ is equal to -1 for the different values of P, Q, and R in some cases in Table 3, and Q is either -1 or 2 for this case. This condition is satisfied for $k=-56$ and $l=1$; then, the transformation given in Section 3 becomes $\xi=-280\sqrt[5]{t}+x$. Therefore, the solutions of the Equation (5) are

$$u=\mp(2Q-2)+2\mp 6PF^2$$

such that $A=1/2$, $B=2$, and $C=-2$. When $m\to 0$ in Table 4, the solution u_7 becomes $u_7=2\mp\left(2+6\tan^2\xi\right)$.

In this case, we demonstrate these solutions for $0\le t\le 0.0001$ and $0\le x\le 0.0001$ in Figures 1 and 2, respectively.

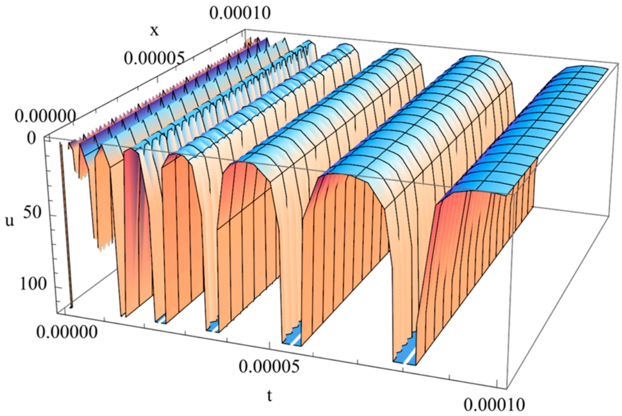

Figure 1. Three-dimensional graph of obtained solution $u_7(x,t) = 6\tan^2\left(-x - 280\sqrt[5]{t}\right)$ when $m \to 0$.

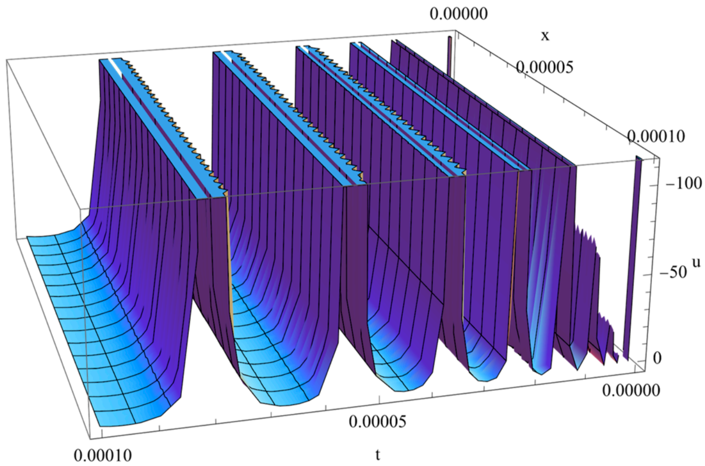

Figure 2. Three-dimensional graph of the solution $u_7(x,t) = 4 - 6\tan^2\left(-x - 280\sqrt[5]{t}\right)$ when $m \to 0$.

Moreover, Figures 3 and 4 exemplify the referred solutions in the 2-dimensional graph for the space variable changing between 0 and 20 at the fixed time $t = 1$. By analyzing Figures 3 and 4, we can observe that the wavelengths do not change, and the wave amplitudes reach up to infinity when x approaches infinity.

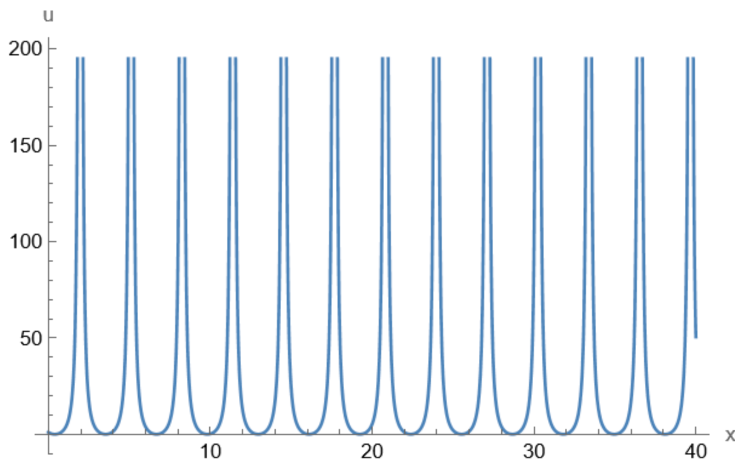

Figure 3. Two-dimensional graph of the exact solution $u_7(x,1) = 6\tan^2(-x - 280)$.

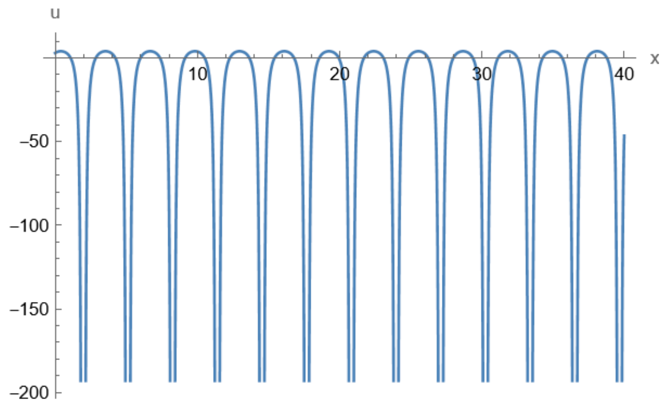

Figure 4. Two-dimensional graph of the exact solution $u_7(x,1) = 4 - 6\tan^2(-x - 280)$.

Example 2. *Let us think about the conformable space fractional extended Kawahara Equation (1) for the determined coefficients* $a = -10$, $b = 20$, $c = 40$, $\lambda = -1$, $\alpha = 0.5$ *and* $\beta = 1$;

$$u_t + \left(40u - 10u^2\right) D_x^{1/2} u + 20 D_x^{1/2} D_x^{1/2} D_x^{1/2} u + D_x^{1/2} D_x^{1/2} D_x^{1/2} D_x^{1/2} D_x^{1/2} u = 0. \quad (6)$$

When $m \to 1$, on the left-hand side of the condition (4), $3PR - Q^2$ is equal to -1 for the different values of $P,\ Q$, and R in some cases in Table 3, and Q is either 1 or -2 for this case. This condition is satisfied for $k = -216$ and $l = 1$; then, the transformation given in Section 3 becomes $\xi = -216t + 2\sqrt{x}$. Thus, the analytical solutions of Equation (6) are in the form

$$u = \mp(2Q - 2) + 2 \mp 6PF^2$$

such that $A = 1/2$, $B = 2$, and $C = -2$. When $m \to 1$ in Table 4, the solutions u_4 become $u_4 = 2 \mp \left(6 + 6\coth^2\xi\right)$. In that case, we exemplify given solutions for $x \in [0,\ 0.1]$ at $0 \le t \le 0.1$ and at $0 \le t \le 0.005$ in Figures 5 and 6, separately. Furthermore, Figures 7 and 8 exemplify the referred solutions in the 2-dimensional graph for space variable changing of $0 \le x \le 1$ at $t = 0.01$.

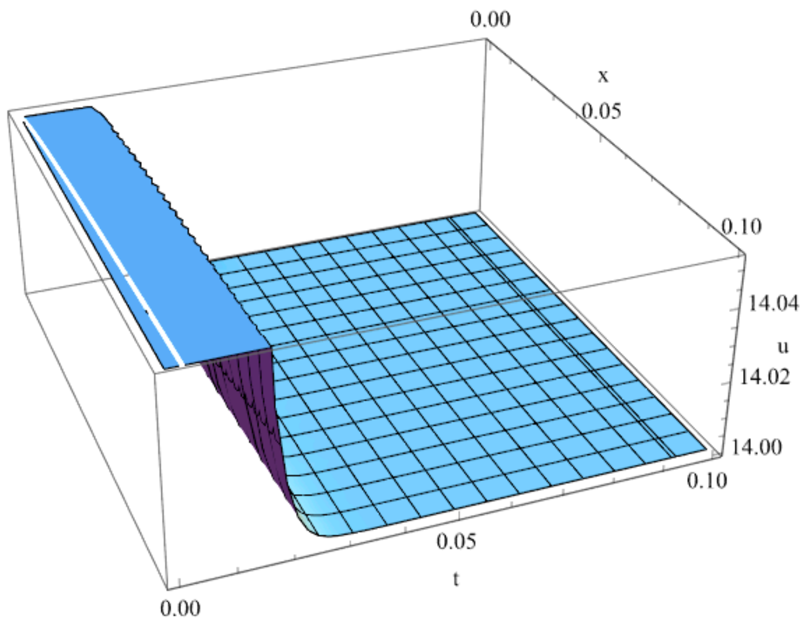

Figure 5. Three-dimensional graph of the solution $u_4(x,t) = 8 + 6\coth^2(2\sqrt{x} - 216t)$ when $m \to 1$.

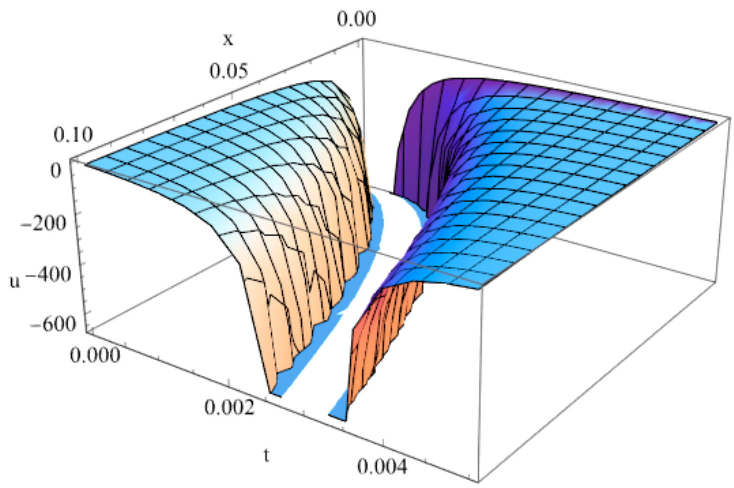

Figure 6. Three-dimensional graph of the solution $u_4(x,t) = 8 - 6\coth^2(2\sqrt{x} - 216t)$ when $m \to 1$.

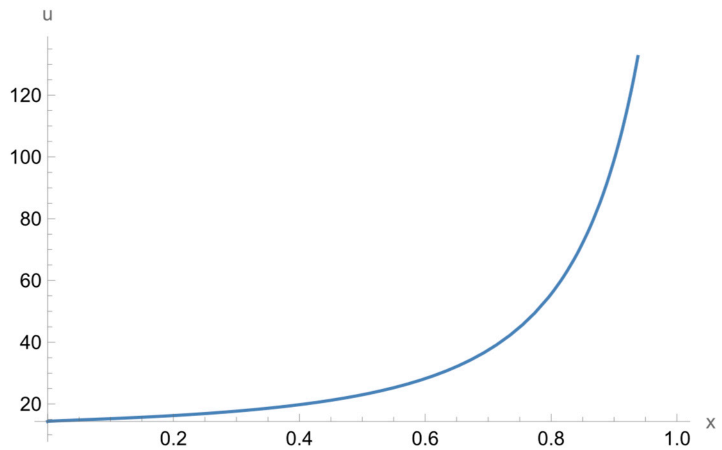

Figure 7. Two-dimensional graph of the exact solution $u_4(x, 0.01) = 8 + 6\coth^2(-2.16 + 2\sqrt{x})$.

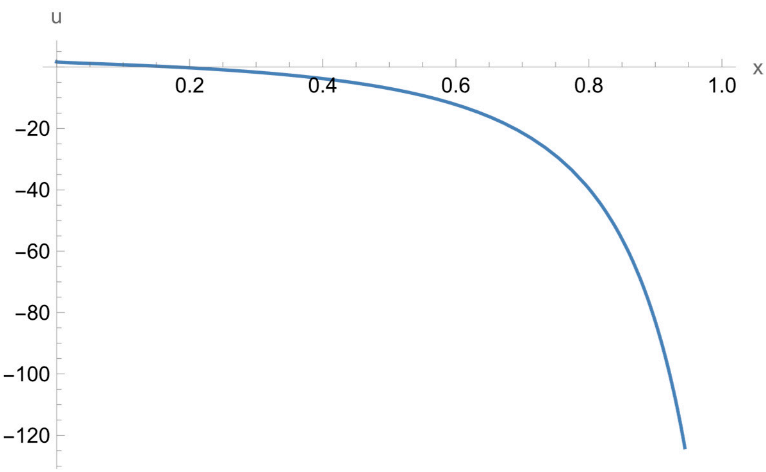

Figure 8. Two-dimensional graph of the exact solution $u_4(x, 0.01) = 8 - 6\coth^2(-2.16 + 2\sqrt{x})$.

Example 3. *Take the conformable time-space fractional extended Kawahara Equation (1) under the consideration for the coefficients* $a = 10$, $b = 10$, $c = 20$, $\lambda = 1$, $\alpha = 0.5$ *and* $\beta = 0.25$;

$$D_t^{1/2}u + 10\left(2u + u^2\right)D_x^{1/4}u + 10D_x^{1/4}D_x^{1/4}D_x^{1/4}u - D_x^{1/4}D_x^{1/4}D_x^{1/4}D_x^{1/4}D_x^{1/4}u = 0. \quad (7)$$

When $m \to 0$, on the left-hand side of the condition (4), $3PR - Q^2$ is equal to -1 for the different values of P, Q, and R in some cases in Table 3, and Q is either -1 or 2 for this case. This condition given by (4) is satisfied for $k = 36$ and $l = 1$; then, the transformation becomes $\xi = 72\sqrt{t} + 4\sqrt[4]{x}$. Therefore, the solutions for Equation (7) are

$$u = \mp(-1 + 2Q) - 1 \mp 6PF^2$$

such that $A = 1/2$, $B = 1$, and $C = 1$. When $m \to 0$ in Table 4, the solution u_9 becomes $u_9 = -1 \mp \left(3 + 6\cot^2\xi\right)$. In this situation, we demonstrate these solutions for $x \in [0,\ 0.05]$ and $t \in [0,\ 0.1]$ in Figures 9 and 10, separately. Moreover, Figures 11 and 12 illustrate the referred solutions in the 2-dimensional graph for the space variable changing of $0 \leq x \leq 500$ at time $t = 1$. By analyzing Figures 11 and 12, we can clearly observe that the wavelengths increase, and the wave amplitudes reach up to infinity when x approaches infinity. Thus, the wave frequency increases for values of x close to zero.

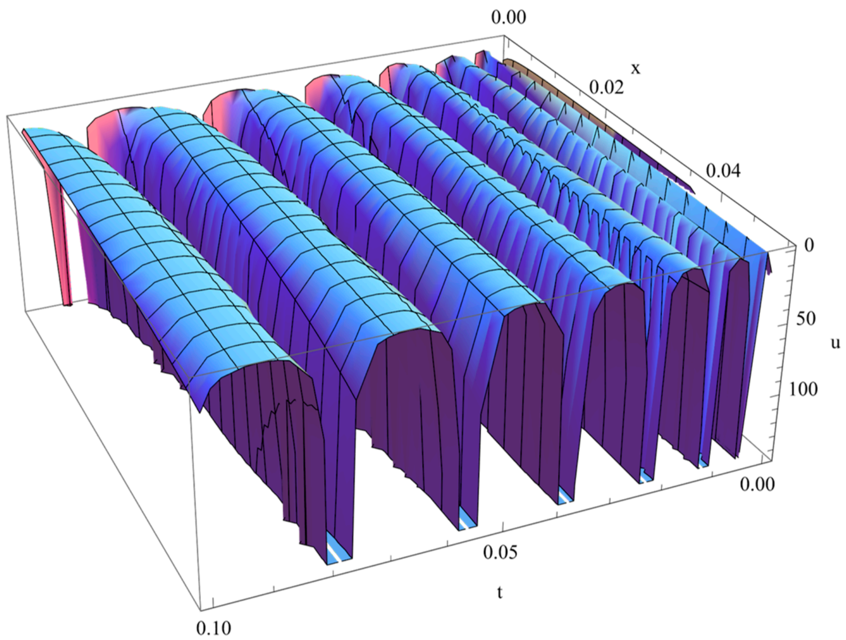

Figure 9. Three-dimensional graph of the solution $u_9(x,t) = 2 + 6\cot^2\left(4\sqrt[4]{x} + 72\sqrt{t}\right)$ when $m \to 0$.

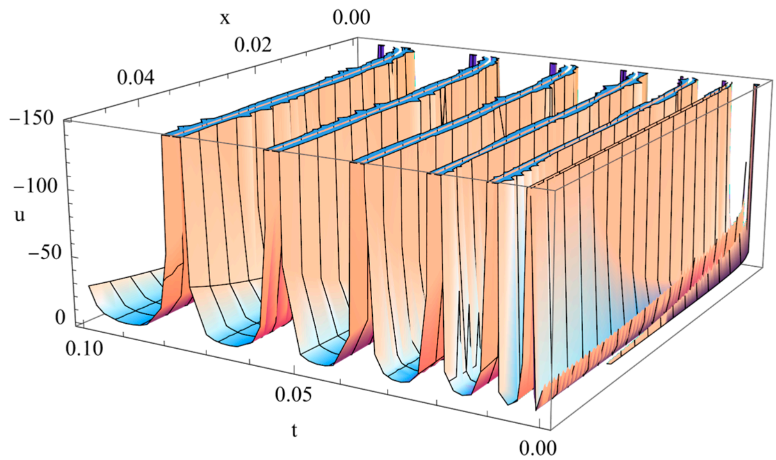

Figure 10. Three-dimensional graph of the solution $u_9(x,t) = -4 - 6\cot^2\left(4\sqrt[4]{x} + 72\sqrt{t}\right)$ when $m \to 0$.

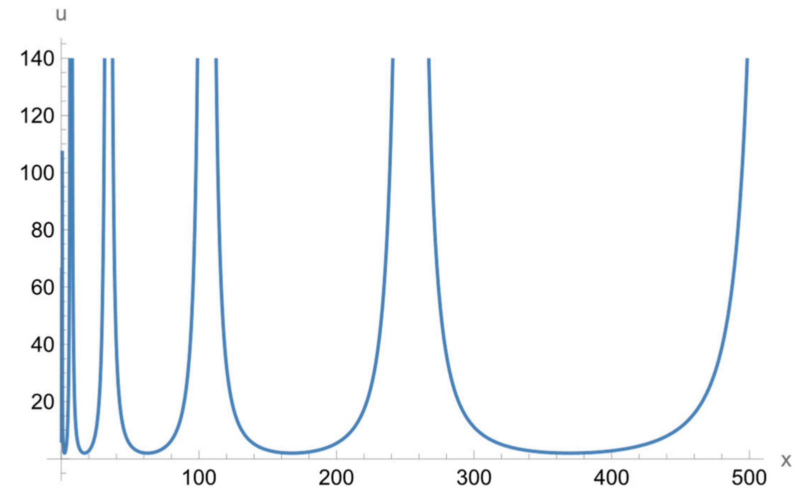

Figure 11. Two-dimensional graph of the exact solution $u_9(x,1) = 2 + 6\cot^2(72 + 4\sqrt[4]{x})$.

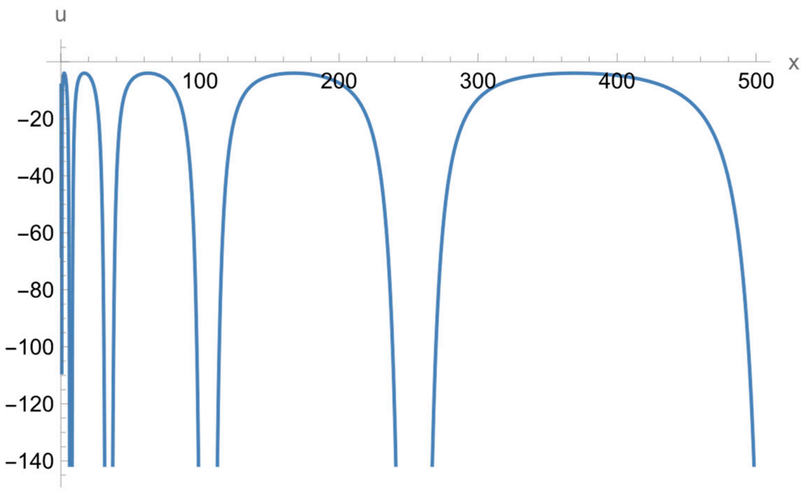

Figure 12. Two-dimensional graph of the solution $u_9(x,1) = \left(-4 - 6\cot^2(72 + 4\sqrt[4]{x})\right)$.

Example 4. *Let us take the conformable time-space fractional extended Kawahara Equation (1) under the consideration for* $\lambda = 1$, $a = 10$, $b = 10$, $c = 20$, $\beta = 0.5$ *and* $\alpha = 0.5$;

$$D_t^{1/2}u + 10\left(2u + u^2\right)D_x^{1/2}u + 10D_x^{1/2}D_x^{1/2}D_x^{1/2}u - D_x^{1/2}D_x^{1/2}D_x^{1/2}D_x^{1/2}D_x^{1/2}u = 0. \quad (8)$$

When $m \to 1$, on the left-hand side of the condition (4), $3PR - Q^2$ is equal to -1 for the different values of P, Q, and R in some cases in Table 3, and Q is either 1 or -2 for this case. This condition is satisfied for $k = -4$ and $l = 1$; then, the transformation given in Section 3 becomes $\xi = -8\sqrt{t} + 2\sqrt{x}$. Hence, the solutions of Equation (8) are

$$u = \mp(2Q - 1) - 1 \mp 6PF^2$$

such that $A = 1/2$, $B = 1$, and $C = 1$. When $m \to 1$ in Table 4, the solution u_3 becomes $u_3 = -1 \mp \left(1 + \text{sech}^2\xi\right)$. In that case, we demonstrate some of these solutions for $0 \le x \le 10$ and $0 \le x \le 50$ at $0 \le t \le 1$ in Figures 13 and 14, separately. Furthermore, Figures 15 and 16 exemplify the referred solutions in the 2-dimensional graph for $x \in [0,\ 100]$ at time $t = 1$.

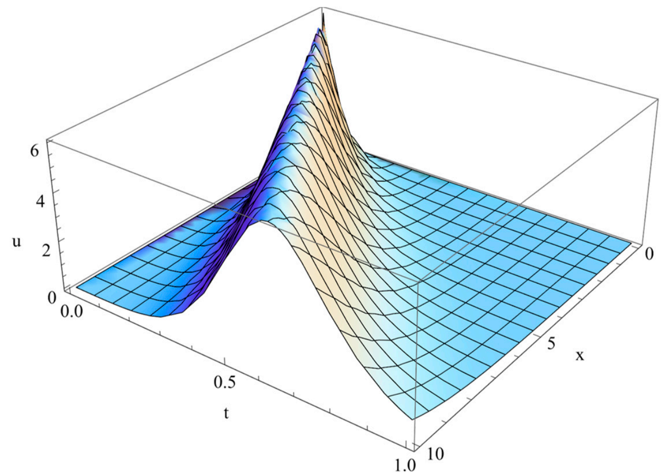

Figure 13. Three-dimensional graph of the solution $u_3(x,t) = 6\text{sech}^2\left(-8\sqrt{t}+2\sqrt{x}\right)$ when $m \to 1$.

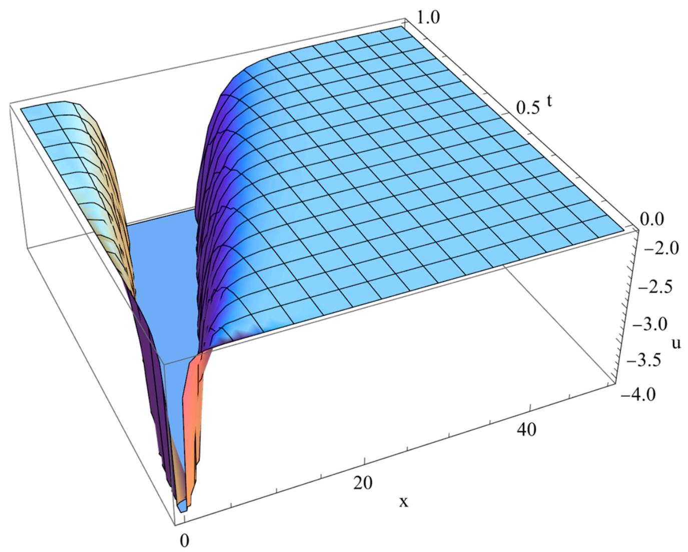

Figure 14. Three-dimensional graph of the solution $u_3(x,t) = -2 - 6\text{sech}^2\left(-8\sqrt{t}+2\sqrt{x}\right)$ when $m \to 1$.

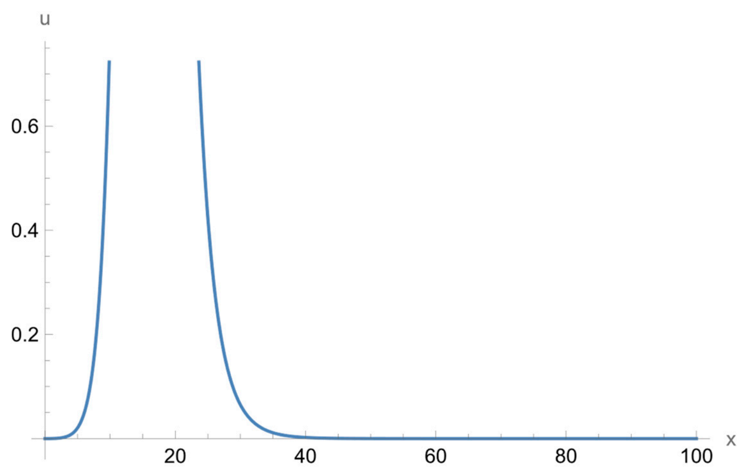

Figure 15. Two-dimensional graph of the exact solution $u_3(x,1) = 6\text{sech}^2(-8+2\sqrt{x})$.

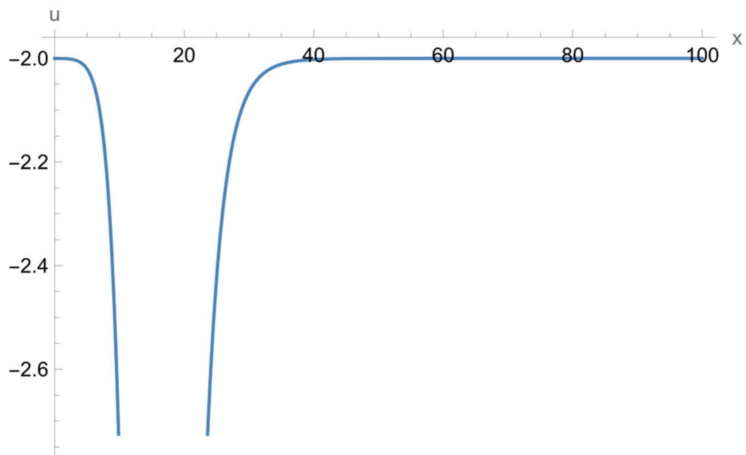

Figure 16. Two-dimensional graph of the exact solution $u_3(x,1) = -2 - 6\mathrm{sech}^2\left(-8 + 2\sqrt{x}\right)$.

Eventually, in this section, four problems of distinguishable types (either space fractional or time fractional or both) have been solved, and some of the solutions have been illustrated both in 2-dimensional and 3-dimensional graphics for different values of the fractional orders α and β. Naturally, the effects of the values of the fractional orders on the solutions could be seen as the changes in the wave width, wave length, and wave amplitude together with the positions of the waves with respect to time.

5. Conclusions

In this full paper, an expansion method based upon JEFs is introduced to acquire analytical solutions for all the time, space, and space-time conformable fractional extended (Gardner) Kawahara equations. Here, the fractional extended Kawahara Equation (1) is given in its most general form in the literature, and hence, this presented method is the first method used for obtaining analytical solutions for the equation. The proposed method has numerous benefits: it is direct, quick, and simple. The primary benefit of the proposed method is because of the fact that the solutions are composed of 12 JEFs, solutions are discovered in a comprehensive structure, which contains the well-known functions, such as trigonometric, hyperbolic, complex, and rational functions. Furthermore, because of the relationship between JEFs and these functions, the solutions of a variety of methods, such as tanh, sech, and sine-cosine ansatz methods, are handled at the same time by using this single method. Moreover, it is obviously clear that the solutions can represent the solitary waves in some of the examples.

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