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THE HORIZONTAL LIFTS OF TENSOR FIELDS TO SECOND ORDER EXTENDED MANİFOLD

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Abstract: The aim of this study is to obtain the restrictions to the second order extended manifold of the horizontal lifts of the basic tensor fields on the tangent bundle obtained in terms of vertical, complete and horizontal lifts.

Key Words: Second order tangent bundle, second order extended manifold, second order lifts of tensor fields

AMS Classification: 55R65, 58A30, 53A45.

1 Introduction

The mathematicians interested the bundle geometry has introduced with the concept of the second order tangent bundle of a differentiable manifold in the last years of 1960.

The motion of a mechanical system which has n-particles is characterized by a space of curves including trajectory each particles. If all particles in system move without changing the distance between two different particles, all particles in system has the same velocity and the same acceleration.

The curves which have the same initial points, velocities and accelerations in an n-dimensional manifold are equivalent. The representative curve, trajectory of a particle in system, of a equivalent class of the equivalence relation is represented by 3n-coordinates. The first n-components describe the position coordinates of the particle in space, the second n-components describe the velocity of the particle and third n-components describe the acceleration of the particle. These structures with 3n-coordinates are considered as the second order extended tangent bundle of a manifold with 3n-dimensional by mathematicians.

When a point in double tangent bundle of an n-dimensional manifold has 4n-coordinates, a point in second order extended manifold, sub-bundle of double tangent bundle, has 3n-coordinates.

The lifts of the differential geometric objects from a manifold to extended space as second order tangent bundle, double tangent bundle and second order extended manifold are important because geometric structures of extended space of a manifold has coincided more knowledge than geometric structures of a manifold.

The lifts from a manifold to its second order tangent bundle or lifts to double tangent bundle are found in literature. Moreover, the vertical and complete lifts from a manifold to its second order extended manifold are also found. So far the horizontal lifts from a manifold to its second order extended manifold hasn't been considered.

Yano and Ishihara [9] defined the second order tangent bundle of a manifold and they considered new lifts of the tensor field in the second order tangent bundles.

Esin and Civelek [6] and [7] found out the vertical and complete lifts of the basic tensor fields from a manifold to its the double tangent bundle and its second order extended manifold.

Aycan [1] considered the vertical and complete lifts of the basic tensor fields from a manifold to its the higher order extension.

Ayhan et al. [2] obtained the horizontal lifts of the basic tensor fields from a manifold to its the double tangent bundle by using the Sasaki Riemann metric on the tangent bundle.

In this study, we studied on the restriction of the horizontal lifts of the basic tensor field in tangent bundle TM of a manifold M obtained in terms of the vertical, complete and horizontal lifts to second order extended manifold ^{2}M by using the pseudo- Riemann metric g^C on TM.

2 The Second Order Extended Manifolds

In this section, we will consider the concepts as the double tangent bundle, the second order tangent bundle or the second order extended manifold. Moreover, we will show that the second order extended manifold of a n-dimensional manifold is 3n-dimensional manifold which consisted of sprays.

Definition 1 Let M be a n-dimensional differentiable manifold and $(x^1, x^2,..., x^n)$ be the local coordinates at a point $p \in U \subset M$. T_pM is called tangent vector space at a point p of M and $(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n})$ is a base of T_pM . T_p^*M is called co-tangent vector space at a point p of M and $(dx^1,..., dx^n)$ is a base of T_p^*M .

Definition 2 The disjoint union of each tangent vector space at all point of M is called the tangent bundle of M and represented by

$$TM = \bigcup_{\forall p \in M} T_pM. \tag{1}$$

Any point in TM is represented by (p, X) where p is a point in M and X is a tangent vector at p. $\pi_1 : TM \to M$ is called as canonical projection map. Let $\widetilde{p}$ be a point in $\pi_1^{-1}(U) \subset TM$. Then $\left(x^1,..., x^n, y^1,..., y^n\right)$ is a local coordinates of a point $\widetilde{p}$ in TM where $\{x^i\}, i = 1,..., n$ is the local components of a point $\pi_1(\widetilde{p}) = p$ and $\{y^i\}, i = 1,..., n$ is the local coordinate function of the tangent vector X providing $y^i = dx^i(X) = X[x^i]$. Thus TM has 2n dimensional manifold structure [9].

Theorem 3 Let M be a Riemann manifold with g Riemann metric and TM be a pseudo-Riemann manifold with g^C called complete lift of g. Let Γ_{ij}^k be connection coefficients of Levi Civita connection ∇ of M with respect to the Riemann metric g and R_{ijk}^h be the coefficients of Riemann curvature tensor R . Then the connection coefficients of Levi Civita connection ∇^C of TM with respect to the pseudo-Riemann metric g^C are

$$\nabla^C_{\partial_i}\partial_j = \Gamma_{ij}^h\partial_h + y^k\partial_k\Gamma_{ij}^h\dot{\partial}_h\,,\ \nabla^C_{\partial_i}\dot{\partial}_j = \Gamma_{ij}^h\dot{\partial}_h\ ,\ \nabla^C_{\dot{\partial}_i}\partial_j = \Gamma_{ij}^h\dot{\partial}_h\,,\ \nabla^C_{\dot{\partial}_i}\dot{\partial}_j = 0\,, \tag{2}$$

and the value on dual base one forms of TM of ∇^C are

$$\nabla^C_{\partial_i}dx^j = -\Gamma_{ih}^j dx^h\,,\ \nabla^C_{\partial_i}dy^j = -y^k\partial_k\Gamma_{ih}^j dx^h - \Gamma_{ih}^j dy^h\ ,\ \nabla^C_{\dot{\partial}_i}dx^j = 0\,,\ \nabla^C_{\dot{\partial}_i}dy^j = -\Gamma_{ih}^j dx^h. \tag{3}$$

where $\partial_i = \partial / \partial x^i$ and $\dot{\partial}_i = \partial / \partial y^i$ [10].

Theorem 4 Let f be a differentiable function, $X = X^h\frac{\partial}{\partial x^h}$ be a vector field and $\omega = \omega_i dx^i$ be a one form on M. The vertical and complete lifts of these basic tensor fields have the local expressions:

$$\begin{aligned} &\textbf{i)}\quad f^V = f o \tau_M\,, \qquad f^C = y^i(\frac{\partial f}{\partial x^i})^V, \\ &\textbf{ii)}\quad X^V = X^h\frac{\partial}{\partial y^h}, \quad X^C = X^h\frac{\partial}{\partial x^h} + y^i\frac{\partial X^h}{\partial x^i}\frac{\partial}{\partial y^h}, \\ &\textbf{iii)}\quad \omega^V = \omega_i dx^i, \qquad \omega^C = y^j\frac{\partial \omega_i}{\partial x^j}dx^i + \omega_i dy^i, \end{aligned} \tag{4}$$

[10].

Definition 5 The disjoint union of each tangent vector space at all point of TM is called the tangent bundle of TM or double tangent bundle of M and represented by

$$TTM = \bigcup_{\forall(p,X)\in TM} T_{(p,X)}TM. \tag{5}$$

Any point in TTM is created by $(p,X,\alpha_{(p,X)})$ where (p,X) is a point in TM and $\alpha_{(p,X)}$ is a tangent vector at a point (p,X). $\pi_2 : TTM \to M$ is called as canonical projection map providing $\pi_2(p,X,\alpha_{(p,X)}) = p$. Moreover, the projection $\pi_{12} : TTM \to TM$ can be defined such that $\pi_2 = \pi_1 \circ \pi_{12}$ and $\pi_{12}(p,X,\alpha_{(p,X)}) = (p,X)$. Let $\tilde{\tilde{p}}$ be a point in $\pi_2^{-1}(U) \subset TTM$. Then $(x^1,...,x^n,y^1,...,y^n,z^1,...,z^n,t^1,...,t^n)$ is a local coordinates of a point $\tilde{\tilde{p}}$ in TTM where $\{x^i\}, i=1,...,n$ is the local components of a point $\pi_2(\tilde{\tilde{p}}) = p$, $\{y^i\}, i=1,...,n$ is the local coordinate function of the tangent vector X_P providing $y^i = dx^i(X_p) = X_p[x^i]$, $\{z^i\}, i=1,...,n$ is the local coordinate function of the tangent vector $\alpha_{(p,X)}$ providing $z^i = dx^i(\alpha_{(p,X)}) = \alpha_{(p,X)}[x^i]$ and $\{t^i\}, i=1,...,n$ is the local coordinate function of the tangent vector $\alpha_{(p,X)}$ providing $t^i = dy^i(\alpha_{(p,X)}) = \alpha_{(p,X)}[y^i]$. Thus TTM has 4n dimensional manifold structure [2].

Theorem 6 Let f be a differentiable function in M, $\frac{\partial}{\partial x^i}$ be a base vectors and dx^i be a dual base vectors in the tangent and co-tangent vector space at any point p of M. The vertical and complete lifts of these basic tensor fields to double tangent bundle of M have the local expressions as follow:

i)
$$\begin{aligned} &f^{VV} = fo\pi_1 o\pi_{12}, \qquad f^{CV} = y^i\left(\frac{\partial f}{\partial x^i}\right)^{VV}, \\ &f^{VC} = z^i\left(\frac{\partial f}{\partial x^i}\right)^{VV}, \quad f^{CC} = y^i z^j\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)^{VV} + t^i\left(\frac{\partial f}{\partial x^i}\right)^{VV}, \end{aligned} \tag{6}$$

ii)
$$\begin{aligned} &\left(\frac{\partial}{\partial x^i}\right)^{VV} = \frac{\partial}{\partial t^i}, \quad \left(\frac{\partial}{\partial x^i}\right)^{VC} = \frac{\partial}{\partial y^i}, \\ &\left(\frac{\partial}{\partial x^i}\right)^{CV} = \frac{\partial}{\partial z^i}, \quad \left(\frac{\partial}{\partial x^i}\right)^{CC} = \frac{\partial}{\partial x^i}, \end{aligned} \tag{7}$$

iii)
$$\begin{aligned} &(dx^i)^{VV} = dx^i, \quad (dx^i)^{VC} = dz^i, \\ &(dx^i)^{CV} = dy^i, \quad (dx^i)^{CC} = dt^i, \end{aligned} \tag{8}$$

[6] and [7].

Definition 7 Let M be an n-dimensional differentiable manifold and R be real line. If two mappings $F : R \to M$ and $G : R \to M$ satisfy the conditions,

$$F(0) = G(0), \quad \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, \quad \frac{d^2F^h(0)}{dt^2} = \frac{d^2G^h(0)}{dt^2}, \tag{9}$$

where $F(0) = p \in U \subset M$ and $x^h \circ F(t) = F^h(t)$, we say that the mapping F is equivalent to G. Each equivalence class determined by this equivalence relation is called a 2-jet of M. The set of all 2-jets of M is called second order tangent bundle of M. Let (U, x^h) be a coordinate neighbourhood of M. Then 2-jet which is a point in second order tangent bundle has the induced coordinate functions

$$x^h = F^h(0), \quad y^h = \frac{dF^h(0)}{dt}, \quad z^h = \frac{1}{2}\frac{d^2F^h(0)}{dt^2}, \tag{10}$$

[9].

Definition 8 If M is n-dimensional differentiable manifold and its tangent bundle TM must be 2n-dimensional manifold and its double tangent bundle TTM must be 4n-dimensional manifold. By helping π_1, π_2 and π_{12} maps, following commutative diagram can be define:

$$\begin{array}{ccc} TTM & \xrightarrow{(\pi_1)_*} & TM \\ \pi_{12}\downarrow & & \downarrow \pi_1 \\ TM & \xrightarrow[\pi_1]{} & M \end{array} \tag{11}$$

If we take

$$^2M = \{A : A \in TTM, \pi_{12}(A) = (\pi_1)_*(A)\} \tag{12}$$

then ^2M is an imbedded sub-manifold of TTM. ^2M is called as the second order extended manifold [3].

Theorem 9 ^2M is 3n-dimensional manifold consisted of sprays, which are the horizontal vector field on TM.

Proof. Let $c : t \in I \subset R \to c(t) \in M$ be curve on M and Z be parallel vector field along the curve c(t). Namely $\nabla_X Z = 0$ where $X = c'(0)$. Then the curve $Z : t \to Z(t)$ defines a horizontal curve in TM. Since $Z : M \to TM$ is a transformation that assigns to each point of M is a tangent vector Z_p, $Z_* : TM \to TTM$ must a transformation that assigns to each point of TM is a horizontal vector $Z_* X$ expressed by

$$Z_* X = X^i A_i\big|_{(p,Z)} + X^k \frac{\partial Z^i}{\partial x^k} A_{n+i}\big|_{(p,Z)} \in T_{(p,Z)}TM \tag{13}$$

where $A_i\big|_{(p,Z)}$ and $A_{n+i}\big|_{(p,Z)}$ horizontal and vertical base vectors of $T_{(p,Z)}TM$. The images of these base vectors under the $(\pi_1)_*$ are

$$(\pi_1)_*(A_i\big|_{(p,Z)}) = \frac{\partial}{\partial x^i}\bigg|_p, \quad (\pi_1)_*(A_{n+i}\big|_{(p,Z)}) = 0_p. \tag{14}$$

Thus we get

$$(\pi_1)_*(Z_* X) = X^i \frac{\partial}{\partial x^i} = X_p \tag{15}$$

The induced coordinates of the point $Z_* X$ in TTM are

$$(x^1(p),...,x^n(p), Z^1(p),...,Z^n(p), X^1(p),...,X^n(p), X^k \frac{\partial Z^1}{\partial x^k},...,X^k \frac{\partial Z^n}{\partial x^k}). \tag{16}$$

S, a vector field on TM, is called as a geodesic spray if $(\pi_1)_*(S_Z) = Z$, where S_Z is horizontal vector in $T_{(p,Z)}TM$ which has the local expression

$$S_Z = y^i(p,Z) A_i\big|_{(p,Z)} + (\Gamma^i_{jk} \circ \pi_1) y^j(p,Z) y^k(p,Z) A_{n+i}\big|_{(p,Z)} \tag{17}$$

where $y^i(p,Z) = dx^i(Z_p) = Z_p[x^i] = Z^i(p)$. The projections of the integral curves Z of the vector field S on TM under the π_1 correspond to geodesics in M [5].

Thus the components corresponding the coordinate functions y^i and z^i of the point (p, Z, S_Z) in TTM are equal. For that reason ^2M can be represented by 3n-dimensional differentiable manifold.

3 Lifted functions in second order extended manifold

In this section, we will define the horizontal lift of a function $\tilde{f}$ on TM to ^2M. Then we will obtain the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a function f on M to ^2M.

Theorem 10 Let $\tilde{f}$ be a function of TM and ∇^C be Levi Civita connection with respect to pseudo Riemann metric g^C. The horizontal lift of $\tilde{f}$ is

$$\widetilde{f}^H = \widetilde{f}^C - \nabla^C_{\widetilde{\gamma}}\widetilde{f} = 0. \tag{18}$$

Proof Since $\nabla^C_{\widetilde{\gamma}}\widetilde{f} = \widetilde{\gamma}\left(\nabla^C\widetilde{f}\right)$, $\nabla^C\widetilde{f} = \left(\nabla^C_{\partial_i}\widetilde{f}\right)dx^i + \left(\nabla^C_{\dot{\partial}_i}\widetilde{f}\right)dy^i$ and the linear transformation $\widetilde{\gamma}$ is given by

$$\begin{array}{lcl} \widetilde{\gamma}: \quad \Im^1_1(TM) & \to & \Im^1_0(TTM) \\ A = A^j_i dx^i \otimes \dfrac{\partial}{\partial y^j} & \to & \widetilde{\gamma}(A) = z^i A^j_i \left(\dfrac{\partial}{\partial y^j}\right)^V \\ B = B^j_i dy^i \otimes \dfrac{\partial}{\partial x^j} & \to & \widetilde{\gamma}(B) = t^i B^j_i \left(\dfrac{\partial}{\partial x^j}\right)^V \end{array} \tag{19}$$

$\widetilde{f}^H$ has the local expression

$$\widetilde{f}^H = z^i\partial_i\widetilde{f} + t^i\dot{\partial}_i\widetilde{f} - \gamma\left(\partial_i\widetilde{f}dx^i + \dot{\partial}_i\widetilde{f}dy^i\right) = 0. \tag{20}$$

Definition 11 Let $\widetilde{f}$ be a function of TM. The restriction to ${}^2M$ of the horizontal lift of $\widetilde{f}$ has the local expression

$$\widetilde{f}^H\Big|_{{}^2M} = 0, \tag{21}$$

$\widetilde{f}^H\Big|_{{}^2M}$ are called as the horizontal lift of the function $\widetilde{f}$ to ${}^2M$.

Theorem 12 Let f^V, f^C, f^H be vertical, complete and horizontal lift of a function of f on M. Then the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a function f on M to ${}^2M$ are

i) $f^{VH}\Big|_{{}^2M} = 0,$

ii) $f^{CH}\Big|_{{}^2M} = 0,$

iii) $f^{HH}\Big|_{{}^2M} = 0.$

4 Lifted vector fields in second order extended

In this section, we will define the horizontal lift of a vector field $\widetilde{X}$ on TM to ${}^2M$. Then we will obtain the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a vector field X on M to ${}^2M$.

Theorem 13 Let $\widetilde{X}$ be a vector field on TM and ∇^C be Levi Civita connection with respect to g^C pseudo Riemann metric. The horizontal lift of $\widetilde{X}$ to TTM is

$$\widetilde{X}^H = \widetilde{X}^C - \nabla^C_{\widetilde{\gamma}}\widetilde{X} \tag{22}$$

and $\widetilde{X}^H$ has the local expression

$$\widetilde{X}^H = \begin{pmatrix} \alpha^h \\ \alpha^{\bar{h}} \\ -z^i\alpha^h\Gamma^k_{ih} \\ -z^i\alpha^{\bar{h}}\Gamma^k_{ih} - t^i\alpha^h\Gamma^k_{ih} \end{pmatrix}. \tag{23}$$

where $\widetilde{X}$ has the matrix representation $\begin{pmatrix} \alpha^h \\ \alpha^{\bar{h}} \end{pmatrix}$.

Proof In terms of the local expressions of the vector fields $\widetilde{X}^C$ and $\nabla^C_{\widetilde{\gamma}}\widetilde{X}$

$$\tilde{X}^C = \begin{pmatrix} \alpha^h \\ \alpha^{\bar{h}} \\ z^i\partial_i\alpha^h + t^i\dot{\partial}_i\alpha^h \\ z^i\partial_i\alpha^{\bar{h}} + t^i\dot{\partial}_i\alpha^{\bar{h}} \end{pmatrix}, \quad \nabla^C_{\tilde{\gamma}}\tilde{X} = \begin{pmatrix} 0 \\ 0 \\ z^i\partial_i\alpha^h + z^i\alpha^h\Gamma^k_{ih} + t^i\dot{\partial}_i\alpha^h \\ z^i\partial_i\alpha^{\bar{h}} + z^i\alpha^{\bar{h}}\Gamma^k_{ih} + t^i\dot{\partial}_i\alpha^{\bar{h}} + t^i\alpha^h\Gamma^k_{ih} \end{pmatrix}, \tag{24}$$

it is seen that the claim of the theorem is true.

Definition 14 Let $\tilde{X}$ be any vector field in TM. $\tilde{X}^H\big|_{{}^2M}$ is called as the restriction of the horizontal lift of $\tilde{X}$ to ${}^2M$ and has the local expression

$$\tilde{X}^H\big|_{{}^2M} = \begin{pmatrix} \alpha^h \\ \alpha^{\bar{h}} - z^i\alpha^h\Gamma^k_{ih} \\ -z^i\alpha^{\bar{h}}\Gamma^k_{ih} - t^i\alpha^h\Gamma^k_{ih} \end{pmatrix}, \tag{25}$$

we are called $\tilde{X}^H\big|_{{}^2M}$ as the horizontal lift of the vector field $\tilde{X}$ to ${}^2M$.

Theorem 15 Let X^V, X^C, X^H be vertical, complete and horizontal lift of a vector field of X on M. Then the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a vector field X on M to ${}^2M$ are

i) $X^{VH}\big|_{{}^2M} = \begin{pmatrix} 0 \\ X^h \\ -z^iX^h\Gamma^k_{ih} \end{pmatrix}$,

ii) $X^{CH}\big|_{{}^2M} = \begin{pmatrix} X^h \\ y^i\partial_iX^h - y^iX^j\Gamma^h_{ij} \\ -y^ky^i\partial_k\left(X^j\Gamma^h_{ij}\right) - t^kX^j\Gamma^h_{kj} \end{pmatrix}$,

iii) $X^{HH}\big|_{{}^2M} = \begin{pmatrix} X^h \\ -2y^iX^j\Gamma^h_{ij} \\ -y^iy^j\Gamma^l_{kj}\Gamma^h_{il}X^k - y^iy^j\partial_j\left(\Gamma^h_{ik}\right)X^k - t^iX^k\Gamma^h_{ik} \end{pmatrix}$.

Proof i) Taking $X^V = \begin{pmatrix} 0 \\ X^h \end{pmatrix}$ instead of $\tilde{X}$ in (25), it is seen to be correct the claim of the theorem (i).

ii) Taking $X^C = \begin{pmatrix} X^h \\ y^i\partial_iX^h \end{pmatrix}$ instead of $\tilde{X}$ in (25), it is seen to be correct the claim of the theorem (ii).

iii) Taking $X^H = \begin{pmatrix} X^h \\ -y^iX^j\Gamma^h_{ij} \end{pmatrix}$ instead of $\tilde{X}$ in (25), it is seen to be correct the claim of the theorem (iii) easily.

Theorem 16 Let X be a vector field defined on M. As $X^{VH} \neq X^{HV}$ in double tangent bundle, there is the equality $X^{VH}\big|_{{}^2M} = X^{HV}\big|_{{}^2M}$ in ${}^2M$.

Proof Let $X = X^h\partial_h$ be a vector field on M. The horizontal-vertical lift of X to the TTM is

$$\begin{aligned} X^{HV} &= \left(X^H\right)^V = \left(X^h\partial_h - y^iX^j\Gamma^h_{ij}\dot{\partial}_h\right)^V \\ &= X^h\ddot{\partial}_h - y^iX^j\Gamma^h_{ij}\dddot{\partial}_h \end{aligned} \tag{26}$$

where $\ddot{\partial}_h = \dfrac{\partial}{\partial z^h}$ and $\dddot{\partial}_h = \dfrac{\partial}{\partial t^h}$. Taking $X^V = \begin{pmatrix} 0 \\ X^h \end{pmatrix}$ instead of $\tilde{X}$ in (23), we get

$$X^{VH} = X^h\ddot{\partial}_h - y^iX^j\Gamma^h_{ij}\dddot{\partial}_h \tag{27}$$

From the equalities (26) and (27) we conclude that $X^{HV} \neq X^{VH}$. But the restrictions of these vector fields to ${}^2M$ are equal.

Theorem 17 Let X be a vector field on M. The complete-horizontal lifts or horizontal-complete lifts of X to TTM or ${}^2M$ is equal.

Proof Taking $X^H = \begin{pmatrix} X^h \\ -y^i X^j \Gamma_{ij}^h \end{pmatrix}$ instead of $\widetilde{X}$ in the left equality of (24), we get

$$X^{HC} = \begin{pmatrix} X^h \\ y^i \partial_i X^h \\ -z^i \Gamma_{ij}^h X^j \\ y^i z^j \partial_{ij}^2 X^h - t^i y^k \Gamma_{ij}^h \partial_k X^j \end{pmatrix} \tag{28}$$

and taking $X^C = \begin{pmatrix} X^h \\ y^i \partial_i X^h \end{pmatrix}$ instead of $\widetilde{X}$ in (23), it is seen to be $X^{HC} = X^{CH}$. As consequence this equality, it is true that $X^{HC}\big|_{^2M} = X^{CH}\big|_{^2M}$.

5 Lifted 1-forms in second order extended manifold

In this section, we will define the horizontal lift of a one form $\widetilde{\omega}$ on TM to ^2M . Then we will obtain the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a one form ω on M to ^2M .

Theorem 18 Let $\widetilde{\omega}$ be a vector field on TM and ∇^C be Levi Civita connection with respect to g^C pseudo Riemann metric. The horizontal lift of $\widetilde{\omega}$ to TTM is

$$\widetilde{\omega}^H = \widetilde{\omega}^C - \nabla_{\widetilde{\gamma}}^C \widetilde{\omega} \tag{29}$$

and $\widetilde{\omega}^H$ has the local expression

$$\widetilde{\omega}^H = \begin{pmatrix} z^i \theta_k \Gamma_{ih}^k + z^i y^j \partial_j \Gamma_{ih}^k \theta_{\bar{k}} + t^i \theta_{\bar{k}} \Gamma_{ih}^k & z^i \theta_{\bar{k}} \Gamma_{ih}^k & \theta_h & \theta_{\bar{h}} \end{pmatrix} \tag{30}$$

where $\widetilde{\omega}$ has the matrix representation $\begin{pmatrix} \theta_h & \theta_{\bar{h}} \end{pmatrix}$.

Proof In terms of the following local expressions of the one forms $\widetilde{\omega}^C$ and $\nabla_{\widetilde{\gamma}}^C \widetilde{\omega}$:

$$\widetilde{\omega}^C = \begin{pmatrix} z^i \partial_i \theta_h + t^i \dot{\partial}_i \theta_h & z^i \partial_i \theta_{\bar{h}} + t^i \dot{\partial}_i \theta_{\bar{h}} & \theta_h & \theta_{\bar{h}} \end{pmatrix}, \tag{31}$$

and

$$\nabla_{\widetilde{\gamma}}^C \widetilde{\omega} = \begin{pmatrix} z^i \left(\partial_i \theta_h - \theta_k \Gamma_{ih}^k - y^j \partial_j \Gamma_{ih}^k \theta_{\bar{k}} \right) + t^i \left(\dot{\partial}_i \theta_h - \theta_{\bar{k}} \Gamma_{ih}^k \right) & z^i \left(\partial_i \theta_{\bar{h}} - \theta_{\bar{k}} \Gamma_{ih}^k \right) + t^i \dot{\partial}_i \theta_{\bar{h}} & 0 & 0 \end{pmatrix} \tag{32}$$

it is seen that the claim of the theorem is true.

Definition 19 Let $\widetilde{\omega}$ be any one form in TM. $\widetilde{\omega}^H\big|_{^2M}$ is called as the restriction of the horizontal lift of $\widetilde{\omega}$ to ^2M and has the local expression

$$\widetilde{\omega}^H\big|_{^2M} = \begin{pmatrix} y^i \theta_k \Gamma_{ih}^k + y^i y^j \partial_j \Gamma_{ih}^k \theta_{\bar{k}} + t^i \theta_{\bar{k}} \Gamma_{ih}^k & \theta_h + y^i \theta_{\bar{k}} \Gamma_{ih}^k & \theta_{\bar{h}} \end{pmatrix}, \tag{33}$$

we are called $\widetilde{\omega}^H\big|_{^2M}$ as the horizontal lift of the one form $\widetilde{\omega}$ to ^2M .

Theorem 20 Let $\omega^V, \omega^C, \omega^H$ be vertical, complete and horizontal lift of a one form of $\omega = \omega_h dx^h$ on M. Then the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of the one form ω on M to ^2M are

i) $\omega^{VH}\big|_{^2M} = \begin{pmatrix} y^i \omega_k \Gamma_{ih}^k & \omega_h & 0 \end{pmatrix}$,

ii) $\omega^{CH}\big|_{^2M} = \begin{pmatrix} y^i y^j \partial_j \left(\omega_k \Gamma_{ih}^k \right) + t^i \omega_k \Gamma_{ih}^k & y^i \left(\partial_i \omega_h + \omega_k \Gamma_{ih}^k \right) & \omega_h \end{pmatrix}$,

iii) $\omega^{HH}\big|_{^2M} = \begin{pmatrix} y^i y^j \left(\partial_j \left(\Gamma_{ih}^k \right) + \Gamma_{jk}^h \Gamma_{ih}^k \right) \omega_k & 2 y^i \omega_k \Gamma_{ih}^k & y^j \omega_k \Gamma_{jh}^k \end{pmatrix}$.

Proof i) Taking $\omega^V = \begin{pmatrix} \omega_h & 0 \end{pmatrix}$ instead of $\widetilde{\omega}$ in (33), it is seen to be correct the claim of the theorem (i).

ii) Taking $\omega^C = \left(y^j \partial_j \omega_h \quad \omega_h\right)$ instead of $\widetilde{\omega}$ in (33), it is seen to be correct the claim of the theorem (ii).

iii) Taking $\omega^H = \left(y^j \omega_k \Gamma^k_{jh} \quad \omega_h\right)$ instead of $\widetilde{\omega}$ in (33), it is seen to be correct the claim of the theorem (iii) easily.

Theorem 21 Let ω be a one form defined on M. As $\omega^{VH} \neq \omega^{HV}$ in double tangent bundle, there is the equality $\omega^{VH}\big|_{{}^2M} = \omega^{HV}\big|_{{}^2M}$ in ${}^2M$.

Proof Let $\omega = \omega_h dx^h$ be a one form on M. The horizontal-vertical lift of ω to the TTM is

$$\begin{aligned} \omega^{HV} &= \left(\omega^H\right)^V = \left(y^i \omega_h \Gamma^h_{ik} dx^k + \omega_h dy^h\right)^V \\ &= y^i \omega_h \Gamma^h_{ik} dx^k + \omega_h dy^h . \end{aligned} \tag{34}$$

Taking $\omega^V = \omega_i dx^i$ instead of $\widetilde{\omega}$ in (30), we get

$$\omega^{VH} = z^i \omega_k \Gamma^k_{ih} dx^h + \omega_h dz^h \tag{35}$$

From the equalities (34) and (35) we conclude that $\omega^{HV} \neq \omega^{VH}$. But the restriction of these vector fields to ${}^2M$ are equal.

Theorem 22 Let ω be a one form on M. As $\omega^{CH} \neq \omega^{HC}$ in double tangent bundle, there is the equality $\omega^{CH}\big|_{{}^2M} = \omega^{HC}\big|_{{}^2M}$ in ${}^2M$.

Proof Taking $\omega^H = \left(y^i \omega_k \Gamma^k_{ih} \quad \omega_h\right)$ instead of $\widetilde{\omega}$ in the left equality of (31), we get

$$\omega^{HC} = \left(z^k y^i \partial_k\left(\Gamma^h_{ij}\omega_h\right) + t^i \Gamma^h_{ij}\omega_h \quad z^k \partial_k\left(\omega_j\right) \quad y^i \Gamma^h_{ij}\omega_h \quad \omega_j\right) \tag{36}$$

and taking $\omega^C = \left(y^j \partial_j \omega_h \quad \omega_h\right)$ instead of $\widetilde{\omega}$ in (28), we get

$$\omega^{HC} = \left(z^i y^k \Gamma^h_{ij} \partial_k\left(\omega_h\right) + t^i \partial_i\left(\omega_j\right) \quad y^k \partial_k\left(\omega_j\right) + t^i \Gamma^h_{ij}\omega_h \quad y^k \partial_k\left(\omega_j\right) \quad \omega_j\right) \tag{37}$$

From (36) and (37) it is seen to be $\omega^{HC} \neq \omega^{CH}$ and $\omega^{HC}\big|_{{}^2M} \neq \omega^{CH}\big|_{{}^2M}$.

6 Concluding Remarks

In this paper, we defined the horizontal lifts of a function, a vector field and one form on TM to ${}^2M$. Then we obtained the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of these tensor fields on M to ${}^2M$.

By using the same operation, second order lifts of different tensor fields on M to ${}^2M$ can be considered.

REFERENCES.

[1] Aycan, C., *On Higher Order Lifts on Extended Jet Bundles*, Int. Jour. of Pure and Appl. Math., 3(21), 75-84, 2005.
[2] Ayhan, I., Coken, A. C., Civelek, S., *Lifts on Double Tangent Bundle,* SDU Journal of Sci. (e-journ.), 3(1), 60-71, 2008.
[3] Bowman, R.,H., *Second Order Connections*, Jour. of Diff. Geom., 7, 549-561, 1972.
[4] Civelek, S., *The Extended Vector Bundles*, Jour. of Geom., 62, 40-47, 1998.
[5] Domdrowski P., *On the Geometry of the Tangent Bundle*, Jour. für die reine und ang. Math., 73-88, 1961.
[6] Esin, E., Civelek, S., *The Lifts on the Second Order Tangent Bundles*, Jour. Math. Stat. Fac. Art. Sc. Gazi Univ., 2, 117-135, 1989.
[7] Esin, E., Civelek, S., *The Lifts on the Second Order Extended Manifold*, Jour. Math. Stat. Fac. Art. Sc. Gazi Univ., 2, 137-152, 1989.
[8] Sounders, D.J., *The Geometry of Jet Bundles*, Cambridge University Press, Cambridge, 1989.
[9] Yano, K., Ishihara, S., *Differential Geometry of Tangent Bundle of order 2*, Kodai Math., Sem., Rep., 20, 318-354, 1968.
[10] Yano, K., Ishihara, S., *Tangent and Cotangent Bundles*, Marcel Dekker Inc., New York, 1973.