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L-DUAL LIFTED TENSOR FIELDS BETWEEN THE TANGENT AND COTANGENT BUNDLES OF A LAGRANGE MANIFOLD

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Abstract: The aim of this study is to obtain images on the cotangent bundle of some tensor fields obtained by means of vertical, complete and horizontal lifts on the tangent bundle of a Lagrange manifold by using the Legendre transformation.

Key Words: Legendre duality between the tangent and cotangent bundle, Lagrange and Hamilton manifolds

AMS Classification: 57R50, 70H03

1 Introduction

Lifting theory between the tangent and cotangent bundles is one of the important subjects of differential geometry, which has been studied over the last 60 years.

Lifts on the cotangent bundle of some tensor fields on a differentiable manifold can’t be defined as a complete lifted a one form and metric with type II+III in the cotangent bundle. This situation makes greater the lifting theory in the tangent bundles than lifting theory in the cotangent bundle. The re-expression idea on the cotangent bundle of general tensor fields on the tangent bundle is not a new idea. But the re-expression idea in the cotangent bundle of the lifted tensor fields on the tangent bundle is original. The most ideal tool used for the transformation of the tensor fields between the tangent and cotangent bundle is Legendre transformation.

In literature, the tensor fields on the tangent bundle and the cotangent bundle of a Lagrange manifold are obtained by two different methods. When one of the methods is lifting the tensor fields in a Lagrange manifold, the other is the $\mathcal{L}$-duality property between the tangent and cotangent bundles of a Lagrange manifold.

Yano and Ishihara [8] have made extensive studies about vertical, complete and horizontal lifts from a manifold to its tangent bundle or its cotangent bundle.

Crampin [4] defined vertical, complete and horizontal lifts from a Lagrange manifold to its tangent bundle.

Miron [5] obtained the images on the cotangent bundle of the general differential geometric objects on the tangent bundle of a Lagrange manifold under the Legendre transformation.

Oproiu and Papaghiuc [7] found the images on the cotangent bundle of the vertical and horizontal base and dual base vector fields on the tangent bundle of a Lagrange manifold under the Legendre transformation.

Ayhan [3] showed the basic tensor fields as functions, vector fields and one forms obtained by vertical, complete and horizontal lifts from a Lagrange manifold to its tangent bundle how they were transformed on the cotangent bundle by using the Legendre transformation.

In this paper, we will examine the images on the cotangent bundle of the tensor fields with type (1,1), (0,2), (2,0) obtained in terms of vertical, complete and horizontal lifts on the tangent bundle of a Lagrange manifold by using the Legendre transformation. Furthermore, we will express to the components of these tensor fields with respect to induced local coordinate on the cotangent bundle.

2 The Tangent and Cotangent Bundle of a Lagrange Manifold

Let M be an n-dimensional differentiable manifold, TM be a 2n-dimensional differentiable manifold called as the tangent bundle of M and $\tau : TM \to M$ be a natural projection. If $(U, x^1, x^2, ..., x^n)$ is a local chart on M. $(\tau^{-1}(U), x^1 \circ \tau, x^2 \circ \tau, ..., x^n \circ \tau, y^1, y^2, ... y^n)$ is defined on TM where $y^1, y^2, ... y^n$ are vector space coordinate with respect to the natural local frame $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$ defined by the local chart $(U, x^1, x^2, ..., x^n)$.

Let the differentiable function $L : TM \to R$ called as Lagrangian be expressed by

$$L(x^i, y^i) = \frac{1}{2}(g_{ij})^V (x, y) y^i y^j \tag{1}$$

where $(g_{ij})^V(x,y)$ is the vertical lift of $g_{ij}(x)$ on M. Since the Hessian of the function L with respect to y^i

$$(g_{ij})^V(x,y) = g_{ij}(x) = \frac{\partial^2 L}{\partial y^i \partial y^j}, i, j = 1,...,n \tag{2}$$

is non degenerate every all points $U \subset M$ and L is regular when $g_{ij}(x)$ is non singular. A Lagrange manifold is a pair (M,L) formed by differentiable n-dimensional manifold M and a regular Lagrangian $L(x,y)$. In the rest of the paper, M is considered to be a Lagrange manifold.

Let the functional

$$\phi[\gamma] = \int_{t_1}^{t_2} L(x^i, y^i)dt, \quad y^i = \frac{dx^i}{dt} \tag{3}$$

be the space of curves passing through the points $x_0^i = x^i(t_0), x_1^i = x^i(t_1)$). If the curve making minimum of the value of $\phi[\gamma]$ is the curve $\gamma : \{(t,x) : x^i = x^i(t), t_0 \le t \le t_1\}$, then the curve γ is the solution curve of the Euler Lagrange equations defined by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{dt}. \tag{4}$$

Using the composite function differentiation, The Euler Lagrange equations become

$$g_{ik}\frac{d^2x^k}{dt^2} + \frac{\partial^2 L}{\partial x^k \partial y^i}\frac{dx^k}{dt} - \frac{\partial L}{\partial x^i} = 0. \tag{5}$$

If the equation in (5) is operated operating by g^{ih}, the entries of the inverse of the non degenerate matrix $[g_{ik}]$, the following equation is obtained:

$$\frac{d^2x^k}{dt^2} + G^h(x^i, y^i) = 0, \tag{6}$$

where

$$G^h(x^i, y^i) = g^{hi}\left(\frac{\partial^2 L}{\partial x^k \partial y^i} y^k - \frac{\partial L}{\partial x^i}\right), \quad h = 1,...,n. \tag{7}$$

In a Lagrange manifold M, there exist the nonlinear connections which depend on the Lagrangian L. One of which has the coefficients

$$N_j^h = \frac{1}{2}\frac{\partial G^h}{\partial y^j} = y^i \Gamma_{ij}^h, \tag{8}$$

([5], [6]).

Let VTM be vertical distribution on TM defined as the kernel of the tangent mapping $\tau_* : TTM \to TM$ of the natural projection τ. A non linear connection on TM is defined as horizontal distribution HTM, which complementary to VTM in TTM. Thus we get

$$TTM = VTM \oplus HTM.$$

The system of the local vector fields $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, ..., \frac{\partial}{\partial y^n})$ is a local frame of VTM and $(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, ..., \frac{\delta}{\delta x^n})$ is a local frame in HTM, where

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N_j^h \frac{\partial}{\partial y^h}, \quad N_j^h = y^i \Gamma_{ij}^h. \tag{9}$$

Thus the system of the local vector fields $\left(\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}\right)$ is called as the local adapted frame of TTM and the local dual adapted frame of TTM is given by

$$(\delta y^1, ..., \delta y^n, dx^1, ..., dx^n),$$

where

$$\delta y^h = dy^h + N_j^h dx^j. \tag{10}$$

Let T^*M be the cotangent bundle of the Lagrange manifold M and $\pi : T^*M \to M$ be a natural projection. If $(U, x^1, x^2, ..., x^n)$ is a local chart on M, it induces a local chart $(\pi^{-1}(U), q^1, ..., q^n, p_1, ..., p_n)$ on T^*M, where $q^i = x^i \circ \pi$, $i = 1,...,n$ and p_i, $i = 1,...,n$ are vector space coordinate with respect to the natural local frame $\left(dx^1, ..., dx^n\right)$ defined by the local chart $(U, q^1, q^2, ..., q^n)$. Let VT^*M be vertical distribution on T^*M defined as the kernel of the tangent mapping $\pi_* : TT^*M \to TM$ of the natural projection π. A non linear connection on T^*M is defined as the horizontal distribution HT^*M, which complementary to VT^*M in TT^*M. Thus we get

$$TT^*M = VT^*M \oplus HT^*M.$$

The system of the local vector fields $(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, ..., \frac{\partial}{\partial p_n})$ is a local frame of VT^*M and $(\frac{\delta}{\delta q^1}, \frac{\delta}{\delta q^2}, ..., \frac{\delta}{\delta q^n})$ is a local frame in HT^*M, where

$$\frac{\delta}{\delta q^j}=\frac{\partial}{\partial x^j}+N_{jh}\frac{\partial}{\partial p_h},\quad N_{jh}=p_i\Gamma^i_{jh}. \tag{11}$$

Thus the system of the local vector fields $(\frac{\partial}{\partial p_1},...,\frac{\partial}{\partial p_n},\frac{\delta}{\delta q^1},...,\frac{\delta}{\delta q^n})$ is called as the local adapted frame of TT^*M and the local dual adapted frame or briefly local adapted co-frame of TT^*M is given by $(\delta p_1,...,\delta p_n,dq^1,...,dq^n)$, where

$$\delta p_h=dp_h-N_{jh}dx^j. \tag{12}$$

The Legendre transformation is defined by φ a smooth mapping:

$$\varphi : TM\rightarrow T^*M \tag{13}$$

by using the induced local coordinates on TM and T^*M as seen in the equation below:

$$q^i=x^i,\ p_i=\frac{\partial L}{\partial y^i}, i=1,...,n. \tag{14}$$

Since L regular, φ is a diffeomorphism. In fact, φ is a local diffeomorphism with respect to the inverse function theorem. Thus, we can consider the following Hamiltonian

$$H(q^i,p_i)=p_iy^i-L(x^i,y^i), \tag{15}$$

which is a differentiable function on T^*M . Then φ^{-1} is given by

$$x^i=q^i,\ y^i=\frac{\partial H}{\partial p_i}, i=1,...,n, \tag{16}$$

whenever (q^i,p_i) is in the image of φ . From the (2), (14), (15) and (16) it is obtained

$$^V\left(g^{ik}\right)(q,p)=g^{ik}(x)=\frac{\partial^2 H}{\partial p_i\partial p_k}, i,k=1,...,n \tag{17}$$

whenever $(q^i,p_i)=\varphi(x^i,y^i)$ Thus, it is seen to be

$$H(q^i,p_i)=g^{ik}(x)p_ip_k. \tag{18}$$

([7]).

3 $\mathcal{L}$-Dual Lifted General Tensor Field

In this section, we considered the tensor fields on the cotangent bundle, which correspond to some tensor fields on the tangent bundle of a Lagrange manifold and we studied on algebraic operations related with these fields.

Definition 1 *Let $P^{\sim}$ be a tensor field with type (r, s) on TM. The pull-back of $P^{\sim}$ by φ^{-1}, a tensor field with type (r, s) on T^*M, is defined by*

$$^{\sim}P=\left(P^{A_1...A_r}_{B_1...B_s}\circ\varphi^{-1}\right)\varphi_*(\frac{\partial}{\partial x^{A_1}})\otimes...\otimes\varphi_*(\frac{\partial}{\partial x^{A_r}})\otimes\left(\varphi^{-1}\right)^*\left(dx^{B_1}\right)\otimes...\otimes\left(\varphi^{-1}\right)^*\left(dx^{B_s}\right),$$

where indices $A_1,...A_r,B_1,...,B_s$ run over the range $\left(1,2,...,n;\overline{1},\overline{2},...\overline{n}\right)$. Then $^{\sim}P$ and $P^{\sim}$ tensor fields are dual by Legendre transformation. These tensor fields are called as $\mathcal{L}$-dual tensor fields with type (r, s) ([5]).

3.1 $\mathcal{L}$-Dual Lifted a tensor field with type (1,1) between the tangent and cotangent bundle of the Lagrange manifold

If F is a tensor field with type (1,1) on M then its vertical lift F^V, its complete lift F^C and its horizontal lift F^H, can a tensor field with type (1,1) on TM.

Definition 2 *Let $F^{\sim}$ be a tensor field with type (1,1) on TM . The pull-back of $F^{\sim}$ by φ^{-1} , a tensor field with type (1,1) on T^*M, is defined by*

$$^{\sim}F=\left(F^B_A\circ\varphi^{-1}\right)\varphi_*(\frac{\partial}{\partial x^B})\otimes\left(\varphi^{-1}\right)^*\left(dx^A\right),$$

or the pull-back of $^{\sim}F$ by φ , a tensor field with type (1,1) on TM , is defined by

$$F^{\sim}=\left(\widetilde{F}^B_A\circ\varphi\right)\varphi^{-1}_*(\frac{\partial}{\partial q^B})\otimes\varphi^*\left(dq^A\right)$$

where $F^B_A\in C^\infty(TM,R)$, $\widetilde{F}^B_A\in C^\infty(T^*M,R)$ and the indices A, B run over the range $\left(1,2,...,n;\overline{1},\overline{2},...\overline{n}\right)$. Then $^{\sim}F$ and $F^{\sim}$ tensor fields are dual by Legendre transformation. These tensor fields are called $\mathcal{L}$-dual tensor fields with type (1,1) ([5]).

Theorem 3 *If F is a tensor field with type (1,1) on M and the tensor field F^V is a vertical lift of F to TM then ${}^V F$ which is $\mathcal{L}$-dual tensor field of F^V is vertical lift of F to T^*M. Moreover ${}^V F$ has the local expression by*

$$ {}^V F = {}^V\left(F_i^k g_{kj}\right)\frac{\partial}{\partial p_j}\otimes dq^i . $$

Proof. Let F be a tensor field of type (1,1) on M and F^V, the vertical lift to *TM of* F, has the local expression by

$$ F^V = \left(F_i^j\right)^V \frac{\partial}{\partial y^j}\otimes dx^i , $$

with respect to induced coordinate $\left(x^i, y^i\right)$ of TM ([8]).

The pull back of F^V by φ^{-1}, a tensor field of type (1,1) on T^*M, is considered by

$$ {}^V F = \left(\left(F_i^j\right)^V \circ\varphi^{-1}\right)\varphi_*\left(\frac{\partial}{\partial y^j}\right)\otimes\left(\varphi^{-1}\right)^*\left(dx^i\right) $$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of $\mathcal{L}$-dual vertical lift ${}^V F$ as follow:

$$ {}^V F = {}^V\left(F_i^k g_{kj}\right)\frac{\partial}{\partial p_j}\otimes dq^i $$

The values of $\mathcal{L}$-dual lifted tensor field with type (1,1) ${}^V F$ on $\mathcal{L}$-dual lifted vector fields to T^*M are

$$ {}^V F\left({}^V X\right) = 0, $$

$$ {}^V F\left({}^C X\right) = {}^V\left(X^i F_i^k g_{kj}\right)\frac{\partial}{\partial p_j} = {}^V(FX), $$

$$ {}^V F\left({}^H X\right) = {}^V\left(X^i F_i^k g_{kj}\right)\frac{\partial}{\partial p_j} = {}^V(FX), $$

where ${}^V X = {}^V\left(X^i g_{ij}\right)\partial/\partial p_j$, ${}^C X = {}^V(X^i)\delta/\delta q^i + p_i\,{}^V(\nabla_j X^i)\partial/\partial p_j$ and ${}^H X = {}^V(X^i)\delta/\delta q^i$.

Theorem 4 *If F is a tensor field with type (1,1) on M and the tensor field F^C is a complete lift of F to TM then ${}^C F$ which is defined the push-forward F^C by φ is complete lift of F to T^*M, which has the local expression by*

$$ {}^C F = {}^V\left(F_i^j\right)\frac{\delta}{\delta q^j}\otimes dq^i + {}^V\left(g_{jm}F_m^n g^{ni}\right)\frac{\partial}{\partial p_j}\otimes\delta p_i + {}^V\left(g^{km}\nabla_k F_i^n g_{jn}\right)p_m\cdot\frac{\partial}{\partial p_j}\otimes dq^i , $$

with respect to adapted frame and co- frame on T^*M

Proof. Let F be a tensor field of type (1,1) on M and F^C, the complete lift to *TM of* F, has the local expression with respect to adapted frame and co- frame of TM by

$$ F^C = \left(F_i^j\right)^V \frac{\delta}{\delta x^j}\otimes dx^i + \left(F_i^j\right)^V\frac{\partial}{\partial y^j}\otimes\delta y^i + y^k\left(\nabla_k F_i^j\right)^V\frac{\partial}{\partial y^j}\otimes dx^i , $$

where $\nabla_k F_i^j = \partial_k F_i^j + F_i^h\Gamma_{hk}^j - F_h^j\Gamma_{ki}^h$ ([8]). The pull back of F^C by φ^{-1}, a tensor field of type on T^*M, is considered by

$$ {}^C F = \left(\left(F_i^j\right)^V\circ\varphi^{-1}\right)\varphi_*\left(\frac{\delta}{\delta x^j}\right)\otimes\left(\varphi^{-1}\right)^*\left(dx^i\right) + \left(\left(F_i^j\right)^V\circ\varphi^{-1}\right)\varphi_*\left(\frac{\partial}{\partial y^j}\right)\otimes\left(\varphi^{-1}\right)^*\left(\delta y^i\right) + {}^V\left(g^{km}\right)\left(\left(\nabla_k F_i^j\right)^V\circ\varphi^{-1}\right)p_m\varphi_*\left(\frac{\partial}{\partial y^j}\right)\otimes\left(\varphi^{-1}\right)^*\left(dx^i\right) $$

By using appropriate the equalities in theorem 1 and theorem 2, it is seen to be the local expression of $\mathcal{L}$-dual complete lift ${}^C F$ as follow:

$$ {}^C F = {}^V\left(F_i^j\right)\frac{\delta}{\delta q^j}\otimes dq^i + {}^V\left(g_{jm}F_n^m g^{ni}\right)\frac{\partial}{\partial p_j}\otimes\delta p_i + p_m\,{}^V\left(g^{km}\nabla_k F_i^n g_{nj}\right)\frac{\partial}{\partial p_j}\otimes dq^i . $$

The values of $\mathcal{L}$-dual lifted tensor field with type (1,1) ${}^C F$ on $\mathcal{L}$-dual lifted vector fields in T^*M are

$$ {}^C F({}^V X) = {}^V\left(X^i F_i^k g_{kj}\right)\frac{\partial}{\partial p_j} = {}^V(FX), $$

$$ {}^C F({}^C X) = {}^V\left(X^i F_i^k\right)\frac{\delta}{\delta q^j} + p_i\,{}^V\left(g^{ki}\nabla_k\left(X^i F_i^n\right)g_{nj}\right)\frac{\partial}{\partial p_j} = {}^C(FX), $$

$$ {}^C F({}^H X) = {}^V\left(X^i F_i^k\right)\frac{\delta}{\delta q^j} + {}^V\left(g^{km}\nabla_k F_i^n g_{nj}X^i\right)\frac{\partial}{\partial p_j} = {}^C(FX) - \varphi_*\left(F^C\left(\gamma(\nabla X)\right)\right), $$

where $\gamma : \Im_s^r(M) \to \Im_{s-1}^r(TM)$ and $\gamma(\nabla X) = y^k (\nabla_k X^h)^V \dfrac{\partial}{\partial y^h}$.

Theorem 5 *If F is a tensor field with type (1,1) on M and the tensor field F^H is a horizontal lift of F to TM then ${}^H F$ which is defined the push-forward F^H by φ is horizontal lift of F to T^*M, which has the local expression by*

$$ {}^H F = {}^V \left(F_i^j\right) \frac{\delta}{\delta q^j} \otimes dq^i + \left(g_{jm} F_m^n g^{ni}\right)^V \frac{\partial}{\partial p_j} \otimes \delta p_i $$

with respect to adapted frame and co- frame on T^*M

Proof. Let F be a tensor field of type (1,1) on M and F^H, the horizontal lift to *TM of* F, has the local expression with respect to adapted frame and co- frame of TM by

$$ {}^H F = {}^V \left(F_i^j\right) \frac{\delta}{\delta q^j} \otimes dq^i + {}^V \left(g_{jm} F_n^m g^{ni}\right) \frac{\partial}{\partial p_j} \otimes \delta p_i , $$

([8]). The pull back of F^H by φ^{-1}, a tensor field of type on T^*M, is considered by

$$ {}^H F = \left(\left(F_i^j\right)^V \circ \varphi^{-1} \right) \varphi_* \left(\frac{\delta}{\delta x^j}\right) \otimes \left(\varphi^{-1}\right)^* \left(dx^i\right) + \left(\left(F_i^j\right)^V \circ \varphi^{-1} \right) \varphi_* \left(\frac{\partial}{\partial y^j}\right) \otimes \left(\varphi^{-1}\right)^* \left(\delta y^i\right) $$

By using appropriate the equalities in theorem 1 and theorem 2, it is seen to be the local expression of **$\mathcal{L}$**-dual horizontal lift ${}^H F$ as follow:

$$ {}^H F = {}^V \left(F_i^j\right) \frac{\delta}{\delta q^j} \otimes dq^i + {}^V \left(g_{jm} F_n^m g^{ni}\right) \frac{\partial}{\partial p_j} \otimes \delta p_i . $$

The values of **$\mathcal{L}$**-dual lifted tensor field with type (1,1) ${}^H F$ on **$\mathcal{L}$**-dual lifted vector fields in T^*M are

$$ {}^H F({}^V X) = {}^V \left(X^i F_i^k g_{kj}\right) \frac{\partial}{\partial p_j} = {}^V (FX) $$

$$ {}^H F({}^C X) = {}^V \left(X^i F_i^k\right) \frac{\delta}{\delta q^j} + {}^V \left(g_{jm} \nabla_j X^i F_n^m g^{ni}\right) p_i \frac{\partial}{\partial p_j} = {}^C (FX) - \varphi_* \left(\gamma(\nabla F)\left(X^C\right)\right) $$

$$ {}^H F({}^H X) = {}^V \left(X^i F_i^j\right) \frac{\delta}{\delta q^j} = {}^H (FX) $$

3.2 $\mathcal{L}$-dual lifted tensor field with type (0,2) between the tangent bundle and the cotangent bundle of the Lagrange manifold

If G is a tensor field with type (0,2) on M then G^V, G^C, G^H, respectively the vertical, complete and horizontal lifts of G, must tensor fields with type (0,2) on TM.

Definition 6 *Let $G^{\sim}$ be a tensor field with type (0,2)on TM The pull back of $G^{\sim}$ by φ^{-1}, a tensor field with type (0,2)on T^*M is defined by*

$$ {}^{\sim} G = \left(\varphi^{-1}\right)^* G^{\sim} $$

and the pull back of ${}^{\sim} G$ by φ, a tensor field on TM is defined by

$$ G^{\sim} = \varphi^* \left({}^{\sim} G\right) $$

*The tensor fields ${}^{\sim} G$ and $G^{\sim}$ are called **$\mathcal{L}$**-dual tensor fields ([5]).*

Theorem 7 *If G is a tensor field with type (0,2) on M and the tensor field G^V is the vertical lift of G to TM then ${}^V G$ which is defined the pull back G^V by φ^{-1} is the vertical lift of G to T^*M, which has the local expression by*

$$ {}^V G = {}^V \left(g_{ij}\right) dq^i \otimes dq^j , $$

with respect to adapted co-frame on T^*M .

Proof. Let G be a tensor field of type (0,2) on M and G^V, the vertical lift to *TM of* G, has the local expression by

$$ G^V = \left(g_{ij}\right)^V dx^i \otimes dx^j , $$

with respect to induced coordinate $\left(x^i, y^i\right)$ of TM ([8]). The pull back of G^V by φ^{-1}, a tensor field of type (0,2) on T^*M, is considered by

$$ {}^V G = \left(\varphi^{-1}\right)^* \left(G^V\right) = \left(\left(g_{ij}\right)^V \circ \varphi^{-1}\right) \left(\varphi^{-1}\right)^* (dx^i) \otimes \left(\varphi^{-1}\right)^* \left(dx^j\right) $$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of **$\mathcal{L}$**-dual vertical lift ${}^V G$ as follow:

$$ {}^V G = \; = {}^V \left(g_{ij}\right) dq^i \otimes dq^j $$

The values of **$\mathcal{L}$**-dual lifted tensor field with type (0,2) ${}^V G$ on **$\mathcal{L}$**-dual lifted vector fields in T^*M are

${}^V G({}^V X, {}^V Y) = 0$, ${}^V G({}^V X, {}^C Y) = 0$, ${}^V G({}^V X, {}^H Y) = 0$,

$${}^V G({}^C X, {}^C Y) = {}^V G(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\, {}^V(X^i)\, {}^V(Y^j) = {}^V(g_{ij})\, {}^V(X^i)\, {}^V(Y^j) = {}^V(G(X,Y)),$$

$${}^V G({}^C X, {}^H Y) = {}^V G(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\, {}^V(X^i)\, {}^V(Y^j) = {}^V(g_{ij})\, {}^V(X^i)\, {}^V(Y^j) = {}^V(G(X,Y)),$$

$${}^V G({}^H X, {}^H Y) = {}^V G(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\, {}^V(X^i)\, {}^V(Y^j) = {}^V(g_{ij})\, {}^V(X^i)\, {}^V(Y^j) = {}^V(G(X,Y)).$$

Theorem 8 *If G is a tensor field with type (0,2) on M and the tensor field G^C is the complete lift of G to TM then ${}^C G$ which is defined the pull back G^C by φ^{-1} are the complete lift of G to T^*M, which has the local expression by*

$${}^C G = p_l\, {}^V(g^{lk})\, {}^V(\nabla_k g_{ij}) dq^i \otimes dq^j + 2\,\delta p_i \otimes dq^j$$

with respect to adapted co-frame on T^*M .

Proof. Let G be a tensor field of type (0,2) on M and G^H, the complete lift to TM *of* G, has the local expression by

$$G^C = y^k (\nabla_k g_{ij})^V dx^i \otimes dx^j + 2(g_{ij})^V \delta y^i \otimes dx^j,$$

with respect to adapted co-frame on T^*M ([8]). The pull back of G^C by φ^{-1}, a tensor field of type (0,2) on T^*M, is considered by

$${}^C G = (\varphi^{-1})^* (G^C) = p_l((g^{lk} \nabla_k g_{ij})^V \circ \varphi^{-1})(\varphi^{-1})^* (dx^i) \otimes (\varphi^{-1})^* (dx^j) + 2((g_{ij})^V \circ \varphi^{-1})(\varphi^{-1})^* (\delta y^i) \otimes (\varphi^{-1})^* (dx^j)$$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of **$\mathcal{L}$**-dual complete lift ${}^C G$ as follow:

$${}^C G = p_l\, {}^V(g^{lk})\, {}^V(\nabla_k g_{ij}) dq^i \otimes dq^j + 2\,\delta p_i \otimes dq^j$$

The values of **$\mathcal{L}$**-dual lifted tensor field with type (0,2) ${}^C G$ on **$\mathcal{L}$**-dual lifted vector fields in T^*M are

${}^C G({}^V X, {}^V Y) = 0$, ${}^C G({}^V X, {}^C Y) = {}^V(G(X,Y))$, ${}^C G({}^V X, {}^H Y) = {}^V(G(X,Y))$,

$${}^C G({}^C X, {}^H Y) = {}^V(g_{kj})\, {}^V(\nabla_k X^i)\, {}^V(Y^j) p_i = (\varphi^{-1})^* (\gamma(G(\nabla X, Y))), \quad {}^C G({}^H X, {}^H Y) = {}^V(\nabla_k g_{ij})\, {}^V(X^i)\, {}^V(Y^j) p_k = (\varphi^{-1})^* (\gamma(\nabla G(X,Y))),$$

$${}^C G({}^C X, {}^C Y) = {}^V(\nabla_k g_{ij})\, {}^V(X^i)\, {}^V(Y^j) p_k + {}^V(g_{im})\, {}^V(X^i)\, {}^V(\nabla_m Y^j) p_j + {}^V(g_{nj})\, {}^V(\nabla_n X^i)\, {}^V(Y^j) p_i = {}^C(G(X,Y)).$$

Theorem 9 *If G is a tensor field with type (0,2) on M and the tensor field G^H is the horizontal lift of G to TM then ${}^H G$ which is defined the pull back G^H by φ^{-1} are the horizontal lift of G to T^*M, which has the local expression by*

$${}^H G = 2\,\delta p_i \otimes dq^j,$$

with respect to adapted co-frame on T^*M .

Proof. Let G be a tensor field of type (0,2) on M and G^H, the complete lift to TM *of* G, has the local expression by

$$G^H = 2(g_{ij})^V \delta y^i \otimes dx^j,$$

with respect to induced coordinate (x^i, y^i) of TM ([8]). The pull back of G^H by φ^{-1}, a tensor field of type (0,2) on T^*M, is considered by

$${}^H G = (\varphi^{-1})^* (G^H) = ((g_{ij})^V \circ \varphi^{-1})(\varphi^{-1})^* (\delta y^i) \otimes (\varphi^{-1})^* (dx^j)$$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of **$\mathcal{L}$**-dual vertical lift ${}^H G$ as follow:

$${}^H G = 2\,\delta p_i \otimes dq^j$$

The values of **$\mathcal{L}$**-dual lifted tensor field with type (0,2) ${}^H G$ on **$\mathcal{L}$**-dual lifted vector fields in T^*M are

${}^H G({}^V X, {}^V Y) = 0$, ${}^H G({}^V X, {}^C Y) = {}^V(G(X,Y))$, ${}^H G({}^V X, {}^H Y) = {}^V(G(X,Y))$,

$${}^H G({}^C X, {}^H Y) = {}^V(g_{kj})\, {}^V(\nabla_k X^i)\, {}^V(Y^j) p_i = (\varphi^{-1})^* (\gamma(G(\nabla X, Y))), \quad {}^H G({}^H X, {}^H Y) = 0.$$

$${}^H G({}^C X, {}^C Y) = {}^V(g_{im})\, {}^V(X^i)\, {}^V(\nabla_m Y^j) p_j + {}^V(g_{nj})\, {}^V(\nabla_n X^i)\, {}^V(Y^j) p_i = {}^C(G(X,Y)) - (\varphi^{-1})^* (\gamma(\nabla G(X,Y))),$$

3.3 $\mathcal{L}$-dual lifted tensor field with type (2,0) between the tangent bundle and the cotangent bundle of the Lagrange manifold

Let H be a tensor field with type (2,0) on M then H^V, H^C, H^H be tensor fields with type (2,0) on TM .

Definition 10 *Let $H^{\sim}$ be a* tensor field with type (2,0) *on TM. The pull-back $H^{\sim}$ by φ is defined by*

$${}^{\sim} H = \varphi^* (H^{\sim})$$

and the pull-back ${}^{\sim}H$ *by* φ^{-1} *is defined by*

$$H^{\sim} = \left(\varphi^{-1}\right)^{*}\left({}^{\sim}H\right).$$

The tensor fields ${}^{\sim}H$ *and* $H^{\sim}$ *are called* ***£****-dual* tensor field with type (2,0) *([5]).*

Theorem 11 *If* H^{V} *is the vertical lift to* TM *of a* tensor field with type (2,0) H *on* M *then* ${}^{V}H$ *, pull-back of* H^{V} *by* φ^{-1} *is the vertical lift of* H *to* $T^{*}M$ *which has the local expression by*

$${}^{V}H = {}^{V}\left(g_{ki}H^{kl}g_{lj}\right)\frac{\partial}{\partial p_i}\otimes\frac{\partial}{\partial p_j}$$

with respect to adapted frame on $T^{*}M$.

Proof. Let H be a tensor field of type (2,0) on M and H^{V} , the vertical lift to *TM of* H , has the local expression by

$$H^{V} = \left(H^{ij}\right)^{V}\frac{\partial}{\partial y^i}\otimes\frac{\partial}{\partial y^j},$$

with respect to adapted frame of TM ([8]). The pull back of H^{V} by φ, a tensor field of type (2,0) on $T^{*}M$, is considered by

$${}^{V}H = \varphi^{*}\left(H^{V}\right) = (\left(H^{ij}\right)^{V}\circ\varphi^{-1})\varphi_{*}(\frac{\partial}{\partial y^i})\otimes\varphi_{*}(\frac{\partial}{\partial y^j})$$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of ***£***-dual complete lift ${}^{V}H$ as follow:

$${}^{V}H = {}^{V}\left(g_{ki}H^{kl}g_{lj}\right)\frac{\partial}{\partial p_i}\otimes\frac{\partial}{\partial p_j}.$$

The values of ***£***-dual lifted tensor field with type (2,0) ${}^{V}H$ on ***£***-dual lifted one forms in $T^{*}M$ are

${}^{V}H({}^{V}\theta,{}^{V}\eta) = 0$, ${}^{V}H({}^{V}\theta,{}^{C}\eta) = 0$, ${}^{V}H({}^{V}\theta,{}^{H}\eta) = 0$,

${}^{V}H({}^{C}\theta,{}^{H}\eta) = {}^{V}H({}^{C}\theta,{}^{C}\eta) = {}^{V}H({}^{H}X,{}^{H}Y) = {}^{V}\left(H(\theta,\eta)\right)$,

where ${}^{V}\theta = {}^{V}\left(\theta^i\right)dq^i$, ${}^{C}\theta = {}^{V}(g^{ji}\theta_j)\delta p_i + {}^{V}(\nabla_j\theta_i)^{V}(g^{jk})p_k dq^i$ and ${}^{H}\theta = {}^{V}(g^{ij}\theta_j)\delta p_i$.

Theorem 12 *If* H^{C} *is the complete lift to* TM *of a* tensor field with type (2,0) H *on* M *then* ${}^{C}H$ *, pull-back of* H^{C} *by* φ *is the complete lift of* H *to* $T^{*}M$ *which has the local expression by*

$${}^{C}H = {}^{V}\left(g_{kj}H^{ki}\right)\frac{\delta}{\delta q^i}\otimes\frac{\partial}{\partial p_j} + {}^{V}\left(g_{ki}H^{kj}\right)\frac{\partial}{\partial p_i}\otimes\frac{\delta}{\delta q^j} + {}^{V}\left(g_{hj}\nabla_k H^{ih}\right)p_k\frac{\partial}{\partial p_i}\otimes\frac{\partial}{\partial p_j}$$

with respect to dual adapted frame on $T^{*}M$ *, where* $\nabla_k H^{ih} = \partial_k H^{ih} + H^{lh}\Gamma^{i}_{kl} + H^{il}\Gamma^{h}_{kl}$.

Proof. Let H be a tensor field of type (2,0) on M and $H^{C} = H^{H} + \gamma(\nabla H)$, the complete lift to *TM of* H , has the local expression by

$$H^{C} = \left(H^{ij}\right)^{V}\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^j} + \left(H^{ij}\right)^{V}\frac{\partial}{\partial y^i}\otimes\frac{\delta}{\delta x^j} + y^k\left(\nabla_k H^{ij}\right)^{V}\frac{\partial}{\partial y^i}\otimes\frac{\partial}{\partial y^j},$$

with respect to adapted frame of TM ([8]). The pull back of H^{C} by φ, a tensor field of type (2,0) on $T^{*}M$, is considered by

$${}^{C}H = \varphi^{*}\left(H^{C}\right) = (\left(H^{ij}\right)^{V}\circ\varphi^{-1})\varphi_{*}(\frac{\delta}{\delta x^i})\otimes\varphi_{*}(\frac{\partial}{\partial y^j}) + (\left(H^{ij}\right)^{V}\circ\varphi^{-1})\varphi_{*}(\frac{\partial}{\partial y^i})\otimes\varphi_{*}(\frac{\delta}{\delta x^j}) + p_h(\left(\nabla_k H^{ij}\right)^{V}\circ\varphi^{-1})g^{kh}\varphi_{*}(\frac{\partial}{\partial y^i})\otimes\varphi_{*}(\frac{\partial}{\partial y^j}).$$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of ***£***-dual complete lift ${}^{C}H$ as follow:

$${}^{C}H = {}^{V}\left(g_{kj}H^{ki}\right)\frac{\delta}{\delta q^i}\otimes\frac{\partial}{\partial p_j} + {}^{V}\left(g_{ki}H^{kj}\right)\frac{\partial}{\partial p_i}\otimes\frac{\delta}{\delta q^j} + {}^{V}\left(g_{hj}\nabla_k H^{ih}\right)p_k\frac{\partial}{\partial p_i}\otimes\frac{\partial}{\partial p_j}$$

The values of ***£***-dual lifted tensor field with type (2,0) ${}^{C}H$ on ***£***-dual lifted one forms in $T^{*}M$ are

${}^{C}H({}^{V}\theta,{}^{V}\eta) = 0$, ${}^{C}H({}^{V}\theta,{}^{C}\eta) = {}^{V}\left(H(\theta,\eta)\right)$, ${}^{C}H({}^{V}\theta,{}^{H}\eta) = {}^{V}\left(H(\theta,\eta)\right)$,

${}^{C}H({}^{C}\theta,{}^{H}\eta) = {}^{C}\left(H(\theta,\eta)\right) - {}^{C}H({}^{C}\theta,\gamma(\nabla\eta))$,

${}^{C}H({}^{C}\theta,{}^{C}\eta) = {}^{C}\left(H(\theta,\eta)\right)$, ${}^{C}H({}^{H}\theta,{}^{H}\eta) = \left(\varphi^{-1}\right)^{*}\left(\gamma(\nabla H(\theta,\eta))\right)$,

where ${}^{V}\theta = {}^{V}\left(\theta^i\right)dq^i$, ${}^{C}\theta = {}^{V}(g^{ji}\theta_j)\delta p_i + {}^{V}(\nabla_j\theta_i)^{V}(g^{jk})p_k dq^i$ and ${}^{H}\theta = {}^{V}(g^{ij}\theta_j)\delta p_i$.

Theorem 13 *If* H^{H} *is the horizontal lift to* TM *of a* tensor field with type (2,0) H *on* M *then* ${}^{H}H$ *, pull-back of* H^{H} *by* φ *is the complete lift of* H *to* $T^{*}M$ *which has the local expression by*

$${}^{H}H = {}^{V}\left(g_{kj}H^{ki}\right)\frac{\delta}{\delta q^i}\otimes\frac{\partial}{\partial p_j} + {}^{V}\left(g_{ki}H^{kj}\right)\frac{\partial}{\partial p_i}\otimes\frac{\delta}{\delta q^j},$$

with respect to dual adapted frame on T^*M .

Proof. Let H be a tensor field of type (2,0) on M and H^H , the horizontal lift to TM *of* H , has the local expression by

$$H^H = \left(H^{ij}\right)^V \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial y^j} + \left(H^{ij}\right)^V \frac{\partial}{\partial y^i} \otimes \frac{\delta}{\delta x^j},$$

with respect to adapted frame of TM ([8]). The pull back of H^H by φ, a tensor field of type (2,0) on T^*M , is considered by

$${}^H H = \varphi^*\left(H^H\right) = (\left(H^{ij}\right)^V \circ \varphi^{-1})\varphi_*(\frac{\delta}{\delta x^i}) \otimes \varphi_*(\frac{\partial}{\partial y^j}) + (\left(H^{ij}\right)^V \circ \varphi^{-1})\varphi_*(\frac{\partial}{\partial y^i}) \otimes \varphi_*(\frac{\delta}{\delta x^j})$$

By using appropriate the equalities in theorem 1 and theorem 2, it is obtained a local components of $\mathcal{L}$-dual complete lift ${}^C H$ as follow:

$${}^C H = {}^V\left(g_{kj} H^{ki}\right) \frac{\delta}{\delta q^i} \otimes \frac{\partial}{\partial p_j} + {}^V\left(g_{ki} H^{kj}\right) \frac{\partial}{\partial p_i} \otimes \frac{\delta}{\delta q^j} + {}^V\left(g_{hj} \nabla_k H^{ih}\right) p_k \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}$$

The values of $\mathcal{L}$-dual lifted tensor field with type (2,0) ${}^H H$ on $\mathcal{L}$-dual lifted one forms in T^*M are

$${}^H H({}^V\theta, {}^V\eta) = 0 ,$$

$${}^H H({}^V\theta, {}^C\eta) = {}^H H({}^V\theta, {}^H\eta) = {}^H H({}^C\theta, {}^H\eta) = {}^V\left(H(\theta,\eta)\right),$$

$${}^C H({}^C\theta, {}^C\eta) = {}^C\left(H(\theta,\eta)\right), \quad {}^C H({}^H\theta, {}^H\eta) = \left(\varphi^{-1}\right)^*\left(\gamma(\nabla H(\theta,\eta))\right).$$

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