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LIFTS FROM A LAGRANGE MANIFOLD TO ITS COTANGENT BUNDLE

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Abstract

In this paper, it is obtained the image on the cotangent bundle of the basic tensor fields (i.e. functions, vector fields and 1-forms) on the tangent bundle of a Lagrange manifold which is obtained by vertical, complete and horizontal lifts under the Legendre transformation.

Key Words: Lagrange equation, duality

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1. Introduction

The papers about the differential geometry the tangent and cotangent bundle have been attracted by geometricians since 1950. Owing to the development about these subjects some concepts (i.e. motion and relativity) of mechanics and physics have become more understandable.
As known, a Lagrangian mechanical system is given by a manifold, i.e. configuration space and function on its tangent bundle, i.e. the Lagrangian. A curve in configuration space is describes a motion. This curve is obtained by solving the Euler Lagrange equations (see [1], [2]).
The principle relativity, as expressed in Newton's first law of motion is based on the idea of uniform motion in a straight line. A straight line is known as the shortest path between two points, but how can we determine which of the infinitely many paths from any given point to another is the shortest? The answer of this question is related with solving the Euler Lagrange equations (see[9]).
In literature, the tensor fields on the tangent bundle and the cotangent bundle of a lagrange manifold are obtained by two different method. First method is lifting the tensor fields in a lagrange manifold and the other is the property of $\mathcal{L}$-duality of between the Lagrange and Hamilton manifolds. Yano and Ishihara [8] have made extensive studies about vertical, complete and horizontal lifts from a manifold to its tangent bundle or its cotangent bundle.
Crampin [3], [4] defined vertical, complete and horizontal lifts from a Lagrange manifold to its tangent bundle.
Miron [5] obtained the images on the cotangent bundle of the general differential geometric objects on the tangent bundle of a Lagrange manifold under the Legendre transformation.
Oproiu and Papaghiuc [7] found the imagines of the vertical and horizontal base and dual base vector fields on the tangent bundle of a Lagrange manifold under the Legendre transformation.

This paper examines the image on the cotangent bundle of the functions, vector fields and 1-forms on the tangent bundle of a Lagrange manifold which is obtained by vertical, complete and horizontal lifts under the Legendre transformation. In addition the components of this basic tensor fields on the cotangent bundle of a Lagrange manifold are calculate.

2. The tangent and cotangent bundle of a Lagrange manifold

Let M be an n-dimensional differentiable manifold, TM be an 2n-dimensional differentiable manifold called the tangent bundle of M and $\tau : TM \rightarrow M$ be a natural projection. If $(U, x^1, x^2, ..., x^n)$ is a local chart on M the $(\tau^{-1}(U), x^1 \circ \tau, x^2 \circ \tau, ..., x^n \circ \tau, y^1, y^2, ... y^n)$ is defined on TM where $y^1, y^2, ... y^n$ are vector space coordinate with respect to the natural local frame $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$ defined by the local chart $(U, x^1, x^2, ..., x^n)$. Let the differentiable function $L : TM \rightarrow R$ called Lagrangian be expressed by

$$L(x^i, y^i) = \frac{1}{2}(g_{ij})^V(x, y) y^i y^j \tag{1}$$

where $(g_{ij})^V(x, y)$ is the vertical lift of $g_{ij}(x)$ on M . Since the Hessian of the function L with respect to y^i

$$(g_{ij})^V(x, y) = g_{ij}(x) = \frac{\partial^2 L}{\partial y^i \partial y^j}, i, j = 1, ..., n \tag{2}$$

is nondegenerate every all points $U \subset M$ and L is regular when $g_{ij}(x)$ is non singular. A lagrange manifold is a pair (M, L) formed by differentiable n-dimensional manifold M and a regular Lagrangian $L(x, y)$. In the rest of the paper, M is considered to be a Lagrange manifold. The curves on Lagrange manifold are curves providing Euler Lagrange equations, obtained helping by following integral operation. In addition, Euler Lagrange equations determinate non linear connection on TM . This fact is explained as follows. The curve $\gamma : \{(t, x) : x^i = x^i(t), t_0 \leq t \leq t_1\}$ is an extremal of the functional

$$\phi[\gamma] = \int_{t_1}^{t_2} L(x^i, y^i) dt, \quad y^i = \frac{dx^i}{dt} \tag{3}$$

on the space of curves passing through the points $x_0^i = x^i(t_0), x_1^i = x^i(t_1)$ if and only if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{dt} \tag{4}$$

along the curve γ . (4) equations are called Euler Lagrange equations. Using the composite function differentiation, The Euler Lagrange equations become

$$g_{ik}\frac{d^2x^k}{dt^2}+\frac{\partial^2 L}{\partial x^k\partial y^i}\frac{dx^k}{dt}-\frac{\partial L}{\partial x^i}=0. \qquad (5)$$

Next operating by g^{ih} , the entries of the inverse of the nondegenerate matrix $[g_{ik}]$, it is obtained

$$\frac{d^2x^k}{dt^2}+G^h(x^i,y^i)=0 \qquad (6)$$

where

$$G^h(x^i,y^i)=g^{hi}\left(\frac{\partial^2 L}{\partial x^k\partial y^i}y^k-\frac{\partial L}{\partial x^i}\right),\quad h=1,...,n \qquad (7)$$

In a Lagrange manifold M there exist the nonlinear connection which depend on the Lagrangian L . One of them has the coefficients

$$N_j^h=\frac{1}{2}\frac{\partial G^h}{\partial y^j}=y^i\Gamma_{ij}^h, \qquad (8)$$

(see [5], [6]). Let VTM be vertical distribution on TM defined as the kernel of the tangent mapping τ_* : $TTM\to TM$ of the natural projection τ . A non linear connection on TM is defined by a called horizontal distribution HTM complementary to VTM in TTM

$$TTM=VTM\oplus HTM.$$

The system of the local vector fields $(\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2},...,\frac{\partial}{\partial y^n})$ is a local frame of VTM and $(\frac{\delta}{\delta x^1},\frac{\delta}{\delta x^2},...,\frac{\delta}{\delta x^n})$ is a local frame in HTM , where

$$\frac{\delta}{\delta x^j}=\frac{\partial}{\partial x^j}-N_j^h\frac{\partial}{\partial y^h},\quad N_j^h=y^i\Gamma_{ij}^h. \qquad (9)$$

Thus the system of the local vector fields $(\frac{\partial}{\partial y^1},...,\frac{\partial}{\partial y^n},\frac{\delta}{\delta x^1},...,\frac{\delta}{\delta x^n})$ is called local adapted frame of TTM . The corresponding local dual adapted frame of TTM is

$$(\delta y^1,...,\delta y^n,dx^1,...,dx^n)\ ,$$

where

$$\delta y^h=dy^h+N_j^h dx^j. \qquad (10)$$

Let T^*M be the cotangent bundle of the lagrange manifold M and π : $T^*M\to M$ be a natural projection. If $(U,x^1,x^2,...,x^n)$ is a local chart on M it induces a local chart $(\pi^{-1}(U), q^1,...,q^n, p_1,...,p_n)$ on T^*M, where $q^i = x^i \circ \pi$, $i = 1,...,n$ and p_i , $i = 1,...,n$ are the coefficients of the elements in T^*M when they are expressed with the help of $\left(dx^1,...,dx^n\right)$. Let VT^*M be vertical distribution on T^*M defined as the kernel of the tangent mapping $\pi_* : TT^*M \to TM$ of the natural projection π . A non linear connection on T^*M is defined by a called horizontal distribution HT^*M complementary to VT^*M in TT^*M

$$TT^*M = VT^*M \oplus HT^*M.$$

The system of the local vector fields $(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2},..., \frac{\partial}{\partial p_n})$ is a local frame of VT^*M and $(\frac{\delta}{\delta q^1}, \frac{\delta}{\delta q^2},..., \frac{\delta}{\delta q^n})$ is a local frame in HT^*M , where

$$\frac{\delta}{\delta q^j} = \frac{\partial}{\partial x^j} + N_{jh}\frac{\partial}{\partial p_h}, \quad N_{jh} = p_i \Gamma^i_{jh}. \tag{11}$$

Thus the system of the local vector fields $(\frac{\partial}{\partial p_1},..., \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1},..., \frac{\delta}{\delta q^n})$ is called local adapted frame of TT^*M . The corresponding local dual adapted frame of TT^*M is $(\delta p_1,...,\delta p_n, dq^1,...,dq^n)$, where

$$\delta p_h = dp_h - N_{jh}dx^j. \tag{12}$$

The Legendre transformation defined by L is a smooth mapping

$$\varphi : TM \to T^*M \tag{13}$$

given in local coordinates induced on TM , T^*M , by

$$q^i = x^i, \ p_i = \frac{\partial L}{\partial y^i}, i = 1,...,n. \tag{14}$$

Since L regular, φ is a diffeomorphism onto its image. In fact, by the inverse function theorem, φ is a local diffeomorphism. Then consider the Hamiltonian

$$H(q^i, p_i) = p_i y^i - L(x^i, y^i) \tag{15}$$

thought of as function on T^*M . Then φ^{-1} is given by

$$x^i = q^i, \ y^i = \frac{\partial H}{\partial p_i}, i = 1,...,n, \tag{16}$$

whenever (q^i, p_i) is in the image of φ . From the (2), (14), (15) and (16) it is obtained

$$^V\left(g^{ik}\right)(q,p) = g^{ik}(x) = \frac{\partial^2 H}{\partial p_i \partial p_k}, i,k = 1,...,n \tag{17}$$

whenever $(q^i, p_i) = \varphi(x^i, y^i)$ Thus, it is seen to be

$$H(q^i, p_i) = g^{ik}(x) p_i p_k. \tag{18}$$

(see [7]).

3. The lifts from a Lagrange Manifold to its cotangent bundle

In this section, firstly the image of the basic base vector fields and basic dual base 1 -forms on the tangent bundle of a lagrange manifold under the Legendre transformation is obtained. Then the image of the tensor fields of the type (0,0), (1,0), (0,1) on the tangent bundle TM obtained by vertical, complete and horizontal lifts from the tensor fields on M under the legendre transformation is found.

Theorem 1 *Let M be a Lagrange manifold with a regular Lagrangian $L = \left(g_{ij}\right)^V y^i y^j$. Let φ be Legendre transformation and φ^{-1} be inverse of φ . Then the image of the local base vector fields on TM and T^*M under the tangent mappings of φ and φ^{-1} have local expressions by*

$$i)\, \varphi_*\left(\tfrac{\delta}{\delta x^i}\right) = \tfrac{\delta}{\delta q^i}, \qquad ii)\, \varphi_*\left(\tfrac{\partial}{\partial y^i}\right) = g_{ij}\, \tfrac{\partial}{\partial p_j},$$

$$iii)\, \varphi_*^{-1}\left(\tfrac{\delta}{\delta q^i}\right) = \tfrac{\delta}{\delta x^i}, \quad iv)\, \varphi_*^{-1}\left(\tfrac{\partial}{\partial p_i}\right) = g^{ij}\, \tfrac{\partial}{\partial y^j}.$$

Proof (i). Let $C^\infty(T^*M, R)$ be ring of differentiable functions from T^*M to R . For $q^j, p_j \in C^\infty(T^*M, R)$, we get

$$\varphi_*\left(\frac{\delta}{\delta x^i}\right)[q^j] = \frac{\delta x^j}{\delta x^i} = \delta_i^j \tag{19}$$

and

$$\begin{aligned} \varphi_*\left(\tfrac{\delta}{\delta x^i}\right)[p_j] &= \tfrac{\delta}{\delta x^i}\left(\tfrac{\partial L}{\partial y^j}\right) = \tfrac{\delta}{\delta x^i}\left(\left(g_{jk}\right)^V y^k\right) = \tfrac{\delta\left(g_{jk}\right)^V}{\delta x^i} y^k + \left(g_{jk}\right)^V \tfrac{\delta y^k}{\delta x^i} \\ &= \left(\tfrac{\partial g_{jk}}{\partial x^i} - g_{jh}\Gamma_{ik}^h\right)^V y^k = \left(g_{hk}\right)^V y^k \Gamma_{ij}^h = p_h \Gamma_{ij}^h. \end{aligned} \tag{20}$$

By the equalities (19) and (20), the vector field of T^*M, $\varphi_*\left(\frac{\delta}{\delta x^i}\right)$, has the following local expression

$$\varphi_*\left(\frac{\delta}{\delta x^i}\right)=\frac{\partial}{\partial q^i}+p_h\Gamma_{ij}^h\frac{\partial}{\partial p_j}. \tag{21}$$

Obtained vector field at (21) is local base vector field, which is span the horizontal subspace of the tangent space of the cotangent bundle of the Lagrange manifold. As a consequence, we write

$$\varphi_*\left(\frac{\delta}{\delta x^i}\right)=\frac{\delta}{\delta q^i}.$$

(*ii*). For $q^j, p_j \in C^\infty(T^*M,R)$, we get

$$\varphi_*\left(\frac{\partial}{\partial y^i}\right)[q^j]=\frac{\partial x^j}{\partial y^i}=0 \tag{22}$$

and

$$\varphi_*\left(\frac{\partial}{\partial y^i}\right)[p_j]=\frac{\partial}{\partial y^i}\left(\frac{\partial L}{\partial y^j}\right)=\frac{\partial}{\partial y^i}\left((g_{jk})^V y^k\right)=g_{ij}. \tag{23}$$

(*iii*). Let $C^\infty(TM,R)$ be ring of differentiable functions from TM to R. For $x^j, y^i \in C^\infty(TM,R)$, we get

$$\varphi_*^{-1}\left(\frac{\delta}{\delta q^i}\right)[x^j]=\frac{\delta q^j}{\delta q^i}=\delta_i^j \tag{24}$$

and

$$\begin{aligned}\varphi_*^{-1}\left(\tfrac{\delta}{\delta q^i}\right)[y^j]&=\tfrac{\delta}{\delta q^i}\left(\tfrac{\partial H}{\partial p_j}\right)=\tfrac{\delta}{\delta q^i}\left({}^V\left(g^{jk}\right)p_k\right)=\tfrac{\delta}{\delta q^i}\left({}^V\left(g^{jk}\right)\right)p_k+{}^V\left(g^{jk}\right)\tfrac{\delta p_k}{\delta q^i}\\&={}^V\left(\tfrac{\partial g^{jk}}{\partial x^i}+g^{jh}\Gamma_{ih}^k\right)p_k=-g^{hk}p_k\Gamma_{ih}^j=-y^h\Gamma_{ih}^j.\end{aligned} \tag{25}$$

By the equalities (24) and (25),the vector field of TM, $\varphi_*^{-1}\left(\frac{\delta}{\delta x^i}\right)$, has the following local expression

$$\varphi_*^{-1}\left(\frac{\delta}{\delta q^i}\right)=\frac{\partial}{\partial x^i}-y^h\Gamma_{ih}^j\frac{\partial}{\partial y^j}. \tag{26}$$

Obtained vector field at (26) is local base vector field, which is span the horizontal subspace of the tangent space of the tangent bundle of the Lagrange manifold. As a consequence, we write

$$\varphi_*^{-1}\left(\frac{\delta}{\delta q^i}\right)=\frac{\delta}{\delta x^i}.$$

(iv). For $x^j, y^i \in C^\infty(TM,R)$, we get

$$\varphi_*^{-1}\left(\frac{\partial}{\partial p_i}\right)[x^j]=\frac{\partial q^j}{\partial p_i}=0 \tag{27}$$

and

$$\varphi_*^{-1}\left(\frac{\partial}{\partial p_i}\right)[y^j]=\frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial p_j}\right)=\frac{\partial}{\partial p_i}\left({}^V\left(g^{jk}\right)p_k\right)=g^{ij}. \tag{28}$$

By the equalities (27) and (28), the vector field of T^*M , $\varphi_*^{-1}\left(\frac{\partial}{\partial p_i}\right)$, has the following local expression

$$\varphi_*^{-1}\left(\frac{\partial}{\partial p_i}\right)=g^{ij}\frac{\partial}{\partial y^j}.$$

Theorem 2 *The image of the local dual base* 1 *-forms on* TM *the pullback mapping of* φ^{-1}*,* $\left(\varphi^{-1}\right)^*$ *, have local expressions by*

i) $\left(\varphi^{-1}\right)^*(dx^i)=dq^i$,

ii) $\left(\varphi^{-1}\right)^*(\delta y^i)=g^{ij}\delta p_j$.

Proof (i). Since $\varphi^{-1} : T^*M \to TM$ is mapping from the cotangent bundle of a Lagrange manifold to the tangent bundle the pull-back of φ^{-1} is a map which is expressed $\left(\varphi^{-1}\right)^* : T^*TM \to T^*T^*M$. For the local base vector fields of T^*M , by using the equalities in (iii) and (iv) of the theorem 1 , we get

$$\left(\varphi^{-1}\right)^*(dx^i)\left(\frac{\delta}{\delta q^j}\right)=dx^i\left(\varphi_*^{-1}\left(\frac{\delta}{\delta q^j}\right)\right)=\delta_j^i \tag{29}$$

and

$$\left(\varphi^{-1}\right)^{*}(dx^{i})\left(\frac{\partial}{\partial p_{j}}\right)=dx^{i}\left(g^{jk}\frac{\partial}{\partial y^{k}}\right)=0. \qquad (30)$$

From the equalities at (29) and (30), $\left(\varphi^{-1}\right)^{*}(dx^{i})$ has the following local expression

$$\left(\varphi^{-1}\right)^{*}(dx^{i})=dq^{i}.$$

(ii) . Similarly,

$$\left(\varphi^{-1}\right)^{*}(\delta y^{i})\left(\frac{\delta}{\delta q^{j}}\right)=\delta y^{i}\left(\varphi_{*}^{-1}\left(\frac{\delta}{\delta q^{j}}\right)\right)=0, \qquad (31)$$

$$\left(\varphi^{-1}\right)^{*}(\delta y^{i})\left(\frac{\partial}{\partial p_{j}}\right)=\delta y^{i}\left(g^{jk}\frac{\partial}{\partial y^{k}}\right)=g^{jk} \qquad (32)$$

From the equalities at (31) and (32), $\left(\varphi^{-1}\right)^{*}(\delta y^{i})$ has the following local expression

$$\left(\varphi^{-1}\right)^{*}(\delta y^{i})=g^{ij}\delta p_{j}.$$

3.1 $\mathcal{L}$ -dual lifted functions between the tangent bundle and the cotangent bundle of the Lagrange manifold

If f is a differentiable function on M then its vertical lift f^{V} and its complete lift f^{C} can a differentiable functions on TM.

Definition 3 *Let $f^{\sim}$ be a function on TM . The pull-back of $f^{\sim}$ by φ^{-1} , a function on $T^{*}M$, is defined by*

$$^{\sim}f=f^{\sim}\circ\varphi^{-1},$$

or the pull-back of $^{\sim}f$ by φ , a function on TM , is defined by

$$f^{\sim}={}^{\sim}f\circ\varphi.$$

Then the functions $^{\sim}f$ and $f^{\sim}$ are dual by Legendre transformation. These functions is called $\mathcal{L}$ -dual functions (see [5]).

Theorem 4 *Let* $f^V : TM \to R$ *be the vertical lift of* f *to* TM *and* ${}^V f : T^*M \to R$ *be the vertical lift of* f *to* T^*M. *Let* $\varphi : TM \to T^*M$ *be the Legendre transformation and* φ^{-1} *be inverse of* φ. *Then the functions*

$$ {}^V f = f^V \circ \varphi^{-1}, \quad f^V = {}^V f \circ \varphi. $$

f^V *and* ${}^V f$ *are* $\mathcal{L}$ *-dual functions.*

Proof. If a point $\tilde{P} \in \pi^{-1}(U) \subset T^*M$ has induced coordinates (q^h, p_h) , and the image of $\tilde{P}$ under φ^{-1} is $P \in \tau^{-1}(U) \subset TM$ with induced coordinates (x^h, y^h) , then

$$ {}^V f(\tilde{P}) = {}^V f(q,p) = f^V \circ \varphi^{-1}(q,p) = f^V(x,y) = f^V(P). \qquad (33) $$

Theorem 5 *Let* $f^C : TM \to R$ *, complete lift of* f *, be a differentiable function on* TM *and* $\varphi^{-1} : T^*M \to TM$ *be inverse of the Legendre transformation* φ . *Then* $\mathcal{L}$ *-dual function of* f^C *,* ${}^C f = f^C \circ \varphi^{-1}$ *,is the complete lift of* f *to* T^*M *which has the local coordinate expression by*

$$ {}^C f = {}^V \left(g^{ij} \partial_i f\right) p_j . $$

Proof. Let f be a differentiable function on M and f^C be a differentiable function on TM with the local expression

$$ f^C = (\partial_i f)^V y^i $$

with respect to induced coordinate system
(see [8]). Then the local coordinate expression of ${}^C f$ is seen by straightforward calculate

$$ \begin{aligned} {}^C f = f^C \circ \varphi^{-1} &= \left(y^i (\partial_i f)^V\right) \circ \varphi^{-1} = \frac{\partial H}{\partial p_i} . {}^V (\partial_i f) \\ &= {}^V \left(g^{ij}\right)^V (\partial_i f) p_j = {}^V \left(g^{ij} \partial_i f\right) p_j . \end{aligned} \qquad (34) $$

3.2. $\mathcal{L}$ -dual lifted vector fields between the tangent bundle and the cotangent bundle of the Lagrange manifold

Let X be a vector field on M then X^V, X^C, X^H, respectively the vertical ,complete and horizontal lifts of X , be vector fields on TM .

Definition 6 *Let $X^{\sim}$ be a vector field on TM The push-forward of $X^{\sim}$ by φ , a vector field on T^*M is defined by*

$$^{\sim}X = \varphi_* \circ X^{\sim} \circ \varphi^{-1}, \tag{35}$$

and The push-forward of $^{\sim}X$ by φ^{-1} , a vector field on TM is defined by

$$X^{\sim} = \varphi_*^{-1} \circ {}^{\sim}X \circ \varphi.$$

The vector fields $^{\sim}X$ and $X^{\sim}$ are called $\mathcal{L}$ -dual vector fields (see [5]).

Theorem 7 *If X is a vector field on M and the vector field X^V is the vertical lift of X to TM then $^V X$ which is defined the push-forward X^V by φ is the vertical lift of X to T^*M, which has the local expression by*

$$^V X = {}^V\left(X^i g_{ij}\right)\frac{\partial}{\partial p_j},$$

with respect to adapted frame on T^*M .

Proof. Let $\tilde{P}$ be a point in open neighborhood $\pi^{-1}(U)$ of T^*M with the induced coordinate (q^i, p_i) . There is a point P on TM providing equality of $\varphi(P) = \tilde{P}$. To obtain the component of $^V X$, if we put X^V instead of $X^{\sim}$ at the composition operation in (33), we get

$$\begin{aligned} {}^V X(\tilde{P}) &= \left(\varphi_* \circ X^V \circ \varphi^{-1}\right)(\tilde{P}) \\ &= \varphi_*\left(X^V(P)\right). \end{aligned}$$

Moreover, we have

$$^V X({}^V f)(\tilde{P}) = \left(\varphi_* \circ X^V\right)({}^V f)(\tilde{P}) = X^V({}^V f \circ \varphi)(P) = X^V\left(f^V\right)(P) = 0 \tag{36}$$

and for all points $\tilde{P}$ and P providing equality of $\varphi(P) = \tilde{P}$, we get

$$ {}^{V}X\left({}^{C}f\right)(\tilde{P}) = \left(\left(\varphi_* \circ X^V\right)({}^{C}f)\right)(\tilde{P}) = X^V({}^{C}f \circ \varphi)(\varphi(\tilde{P})) = X^V(f^C)(P) \qquad (37)$$
$$ = (Xf)^V(P) = \left((Xf)^V \circ \varphi^{-1}\right)(\tilde{P}) = {}^{V}(Xf)(\tilde{P})$$

Since the vector field ${}^{V}X$ is provide identities at (36) and (37), it is called the vertical lift of X to T^*M. As to components of the ${}^{V}X$, we can straightforward calculate as follows

$$ {}^{V}X = \varphi_*\left((X^i)^V . \frac{\partial}{\partial y^i}\right) = {}^{V}\left(X^i g_{ij}\right)\frac{\partial}{\partial p_j}.$$

Theorem 8 *If* X^C *is the complete lift of a vector field* X *on* M *to* TM *then* ${}^{C}X$ *, the push-forward of* X^C *by* φ *, is the complete lift of* X *to* T^*M *which has the local expression by*

$$ {}^{C}X = {}^{V}(X^i)\frac{\delta}{\delta q^i} + p_i . {}^{V}(\nabla_j X^i)\frac{\partial}{\partial p_j}$$

with respect to adapted frame on T^*M *, where* $\nabla_j X^i = \frac{\partial X^i}{\partial x^j} + X^k \Gamma^i_{jk}$.

Proof. For all $\tilde{P} \in \pi^{-1}(U) \subset T^*M$ and all $P \in \tau^{-1}(U) \cap \varphi^{-1}(\pi^{-1}(U)) \subset TM$, we get

$$ {}^{C}X(\tilde{P}) = \left(\varphi_* \circ X^C \circ \varphi^{-1}\right)(\tilde{P})$$
$$ = \varphi_*\left(X^C(P)\right).$$

Moreover, we have

$$ {}^{C}X({}^{V}f)(\tilde{P}) = \left(\varphi_* \circ X^C\right)({}^{V}f)(\tilde{P}) = X^C(P)({}^{V}f \circ \varphi)(P) \qquad (38)$$
$$ = (Xf)^V(P) = \left((Xf)^V \circ \varphi^{-1}\right)(\tilde{P}) = {}^{V}(Xf)(\tilde{P})$$

and for all points $\tilde{P}$ and P providing equality of $\varphi(P) = \tilde{P}$, we get

$$ {}^{C}X\left({}^{C}f\right)(\tilde{P}) = \left(\left(\varphi_* \circ X^C\right)({}^{C}f)\right)(\tilde{P}) = X^C({}^{C}f \circ \varphi)(\varphi(\tilde{P})) \qquad (39)$$
$$ = X^C(f^C)(P) = (\left((Xf)^C \circ \varphi^{-1}\right)(\tilde{P}) = {}^{C}(Xf)(\tilde{P}).$$

Since the vector field ${}^{C}X$ is provide identities at (38) and (39), it is called the complete lift of X to T^*M . By using (16), (18), (29), (30) and (33), we can straightforward calculate as follows

$$^{C}X(\tilde{P}) = \varphi_*\left(X^C(P)\right)$$

$$= \varphi_*\left\{ (X^i)^V(P)\frac{\delta}{\delta x^i}\bigg|_P + (\nabla_j X^i)^V(P).y^j(P).\frac{\partial}{\partial y^i}\bigg|_P \right\}$$

$$^{C}X(\tilde{P}) = \left({}^V(X^i)\frac{\delta}{\delta q^i} + p_i.^V\left(\nabla_j X^i\right)\frac{\partial}{\partial p_j} \right)(\tilde{P}).$$

Since this equality is true all P and $\tilde{P}$ providing equality of $\varphi(P) = \tilde{P}$, The components of ^{C}X is seen to be correct.

Definition 9 *Let* X *is a vector field on* M *with the components* $X = X^i \frac{\partial}{\partial x^i}$ *and the vector field* X^H *is the horizontal lift of* X *to* TM *then* ^{H}X *which is defined the push-forward* X^H *by* φ *is called the horizontal lift of* X *to* T^*M *such that*

$$^{H}X = {}^{C}X - \varphi_*\left(\gamma(\nabla X)\right),$$

where

$$\gamma \ : \Im_s^r(M) \to \Im_{s-1}^r(TM).$$

$\gamma(\nabla X)$ has the local expression by

$$\gamma(\nabla X) = (\nabla_j X^i)^V.y^j.\frac{\partial}{\partial y^i}$$

with respect to adapted frame on TM . The image of $\gamma(\nabla X)$ under the φ_* is

$$\varphi_*\left(\gamma(\nabla X)\right) = p_i.^V\left(\nabla_j X^i\right)\frac{\partial}{\partial p_j}. \qquad (40)$$

Thus, by helping the components of ^{C}X in theorem 8 and (40), the horizontal lift of X to T^*M ^{H}X has the local expression by

$$^{H}X = {}^{V}(X^i)\frac{\delta}{\delta q^i}$$

with respect to adapted frame on T^*M .

Theorem 10 *If* X^H *is the horizontal lift of a vector field* X *on* M *to* TM *then* ${}^H X$ *, the push-forward of* X^H *by* φ *, is the horizontal lift of* X *to* T^*M *which has the local expression by*

$$ {}^H X = {}^V (X^i) \frac{\delta}{\delta q^i} $$

with respect to adapted frame on T^*M *.*

Proof. For all $\tilde{P} \in \pi^{-1}(U) \subset T^*M$ and all $P \in \tau^{-1}(U) \cap \varphi^{-1}(\pi^{-1}(U)) \subset TM$, we get

$$ \begin{aligned} {}^H X(\tilde{P}) &= \varphi_*\left(X^H(P)\right) \\ &= \left({}^V (X^i) \frac{\delta}{\delta q^i} \right)(\tilde{P}). \end{aligned} $$

Since this equality is true all P and $\tilde{P}$ providing equality of $\varphi(P) = \tilde{P}$, the components of ${}^H X$ is seen to be correct.

3.3 $\mathcal{L}$ -dual lifted one forms between the tangent bundle and the cotangent bundle of the Lagrange manifold

Let θ be an one form on M then θ^V, θ^C, θ^H be one forms on TM .

Definition 11 *Let* $\theta^{\sim}$ *be a one form on* TM *. The pull-back* $\theta^{\sim}$ *by* φ^{-1} *is defined by*

$$ {}^{\sim}\theta = \left(\varphi^{-1}\right)^* \circ \theta^{\sim} \circ \varphi^{-1}, \tag{41} $$

and the pull-back ${}^{\sim}\theta$ *by* φ *is defined by*

$$ \theta^{\sim} = \varphi^* \circ {}^{\sim}\theta \circ \varphi. \tag{42} $$

The one forms ${}^{\sim}\theta$ *and* $\theta^{\sim}$ *are called*
$\mathcal{L}-$ *dual one forms (see [5]).*

Theorem 12 *If* θ^V *is the vertical lift to* TM *of a one form* θ *on* M *then* ${}^V\theta$ *, pull-back of* θ^V *by* φ^{-1} *is the vertical lift of* θ *to* T^*M *which has the local expression by*

$$^{V}\theta = ^{V}(\theta^{i})dq^{i}$$

with respect to dual adapted frame on $T^{*}M$.

Proof. Let $\tilde{P}$ be a point in open neighborhood $\pi^{-1}(U)$ of $T^{*}M$ with the induced coordinate (q^{i}, p_{i}) . There is a point P on TM providing equality of $\varphi(P) = \tilde{P}$. To obtain the component of $^{V}\theta$, if we put θ^{V} instead of $\theta^{\sim}$ at the composition operation in (41), we get

$$^{V}\theta(\tilde{P}) = \left((\varphi^{-1})^{*} \circ \theta^{V} \circ \varphi^{-1}\right)(\tilde{P}) = (\varphi^{-1})^{*}\left(\theta^{V}(P)\right).$$

Moreover, we have

$$^{V}\theta(^{V}X)(\tilde{P}) = \left((\varphi^{-1})^{*}(\theta^{V})\right)(^{V}X)(\tilde{P}) = \theta^{V}(\varphi_{*}^{-1}(^{V}X))(P) = \theta^{V}\left(X^{V}\right)(P) = 0 \quad (43)$$

and for all points $\tilde{P}$ and P providing equality of $\varphi(P) = \tilde{P}$, we get

$$\begin{aligned} ^{V}\theta\left(^{C}X\right)(\tilde{P}) &= \left((\varphi^{-1})^{*}(\theta^{V})(^{C}X)\right)(\tilde{P}) = \theta^{V}(\varphi_{*}^{-1}(^{C}X))(P) \\ &= \theta^{V}(X^{C})(P) = \left((\theta(X))^{V} \circ \varphi^{-1}\right)(\tilde{P}) = ^{V}(\theta(X))(\tilde{P}) \end{aligned} \quad (44)$$

Since the one form $^{V}\theta$ is provide identities at (43) and (44), it is called the vertical lift of θ to $T^{*}M$. As to components of the $^{V}\theta$, we can straightforward calculate as follows

$$^{V}\theta = (\varphi^{-1})^{*}\left((\theta_{i})^{V}.dx^{i}\right) = ^{V}(\theta_{i}).(\varphi^{-1})^{*}\left(dx^{i}\right) = ^{V}(\theta_{i})dq^{i}.$$

Theorem 13 *If* θ^{C} *is the complete lift of a one form* θ *on* M *to* TM *then* $^{C}\theta$ *, the pull-back of* θ^{C} *by* φ^{-1} *, is the complete lift of* θ *to* $T^{*}M$ *which has the local expression by*

$$^{C}\theta = ^{V}(g^{ji}\theta_{j})\delta p_{i} + ^{V}(\nabla_{j}\theta_{i})^{V}(g^{jk})p_{k}dq^{i}$$

with respect to dual adapted frame on $T^{*}M$ *, where* $\nabla_{j}\theta_{i} = \frac{\partial\theta_{i}}{\partial x^{j}} - \theta_{k}\Gamma_{ji}^{k}$.

Proof. For all $\tilde{P} \in \pi^{-1}(U) \subset T^{*}M$ and all $P \in \tau^{-1}(U) \cap \varphi^{-1}(\pi^{-1}(U)) \subset TM$, we get

$$^{C}\theta(\tilde{P}) = \left(\left(\varphi^{-1}\right)^{*} \circ \theta^{C} \circ \varphi^{-1}\right)(\tilde{P})$$
$$= \left(\varphi^{-1}\right)^{*}\left(\theta^{C}(P)\right).$$

Moreover, we have

$$\begin{aligned} ^{C}\theta(^{V}X)(\tilde{P}) &= \left(\left(\varphi^{-1}\right)^{*} \circ \theta^{C}\right)(^{V}X)(\tilde{P}) = \theta^{C}(\varphi_{*}^{-1}(^{V}X))(P) \\ &= \theta^{C}\left(X^{V}\right)(P) = \left((\theta(X))^{V} \circ \varphi^{-1}\right)(\tilde{P}) = {}^{V}(\theta(X))(\tilde{P}), \end{aligned} \tag{45}$$

and for all points $\tilde{P}$ and P providing equality of $\varphi(P) = \tilde{P}$, we get

$$\begin{aligned} ^{C}\theta\left(^{C}X\right)(\tilde{P}) &= \left(\left(\varphi^{-1}\right)^{*} \circ \theta^{C})(^{C}X)\right)(\tilde{P}) = \theta^{C}(\varphi_{*}^{-1}(^{C}X))(P) \\ &= \theta^{C}(X^{C})(P) = \left(\theta(X)\right)^{C} \circ \varphi^{-1}\right)(\tilde{P}) = {}^{C}(\theta(X))(\tilde{P}). \end{aligned} \tag{46}$$

Since the one form $^{C}\theta$ is provide identities at (45) and (46), it is called the complete lift of θ to $T^{*}M$. By using the theorem 2, we can straightforward calculate coordinate expression of $^{C}\theta$ as follows

$$\begin{aligned} ^{C}\theta(\tilde{P}) &= \left(\varphi^{-1}\right)^{*}\left(\theta^{C}(P)\right) \\ &= \left(\varphi^{-1}\right)^{*}\left\{(\theta_{i})^{V}(P)\delta y^{i}\big|_{P} + (\nabla_{j}\theta_{i})^{V}(P).y^{j}(P).dx^{i}\big|_{P}\right\} \\ &= {}^{V}(\theta_{i}).\left(\varphi^{-1}\right)^{*}\left(\delta y^{i}\big|_{P}\right) + {}^{V}(\nabla_{j}\theta_{i}).\frac{\partial H}{\partial p_{j}}.\left(\varphi^{-1}\right)^{*}\left(dx^{i}\big|_{P}\right) \\ &= \left({}^{V}(g^{ij}\theta_{j})\delta p_{i} + {}^{V}(\nabla_{j}\theta_{i})^{V}(g^{jk})p_{k}dq^{i}\right)(\tilde{P}). \end{aligned}$$

Since this equality is true all P and $\tilde{P}$ providing equality of $\varphi(P) = \tilde{P}$, the components of $^{C}\theta$ is seen to be correct .

Definition 14 *Let* θ *is a one form on* M *with the components* $\theta = \theta_{i}dx^{i}$ *and the one form* θ^{H} *is the horizontal lift of* θ *to* TM *then* $^{H}\theta$ *which is defined the pull-back* θ^{H} *by* φ^{-1} *is called the horizontal lift of* θ *to* $T^{*}M$ *such that*

$$^{H}\theta = {}^{C}\theta - \left(\varphi^{-1}\right)^{*}\left(\gamma(\nabla\theta)\right),$$

where

$$\gamma \ : \Im_s^r(M) \to \Im_{s-1}^r(TM).$$

$\gamma(\nabla\theta)$ has the local expression by

$$\gamma(\nabla\theta) = (\nabla_j\theta_i)^V.y^j.dx^i$$

with respect to adapted frame on TM. The image of $\gamma(\nabla\theta)$ under the $(\varphi^{-1})^*$ is

$$(\varphi^{-1})^*(\gamma(\nabla X)) = {}^V(\nabla_j\theta_i)^V(g^{jk})p_k dq^i. \quad (47)$$

Thus by helping the components of ${}^C\theta$ in theorem 13 and (47), the horizontal lift of θ to T^*M ${}^H\theta$ has the local expression by

$${}^H\theta = {}^V(g^{ij}\theta_j)\delta p_i$$

with respect to dual adapted frame on T^*M.

Theorem 15 *If θ^H is the horizontal lift of a one form θ on M to TM then ${}^H\theta$, the pull-back of θ^H by φ^{-1}, is the horizontal lift of θ to T^*M which has the local expression by*

$${}^H\theta = {}^V(g^{ij}\theta_j)\delta p_i$$

with respect to adapted frame on T^*M.

Proof. By using the theorem 2, we get

$$\begin{aligned}{}^H\theta(\tilde{P}) &= (\varphi^{-1})^*(\theta^H(P)) \\ &= (\varphi^{-1})^*\{(\theta_i)^V(P)\delta y^i|_P\} \\ &= {}^V(\theta_i)(\tilde{P}).(\varphi^{-1})^*(\delta y^i|_P) \\ &= ({}^V(g^{ij}\theta_j)\delta p_i)(\tilde{P}).\end{aligned}$$

Since this equality is true all P and $\tilde{P}$ providing equality of $\varphi(P) = \tilde{P}$, the components of ${}^H\theta$ is seen to be correct.

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