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g^H Semi-Riemann Metric on the Tangent Bundle and its Index

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Abstract

In this paper, it is examined relations between the causal character of lifted tangent vector on the tangent bundle with g^H semi-Riemann metric of a semi-Riemann manifold and the causal character of the tangent vector on the semi-Riemann manifold with g semi-Riemann metric. Moreover, it is proved that g^H, which is obtained in term of the horizontal lift of a semi-Riemann metric with ν-index on a differentiable manifold, is a semi-Riemannian metric with n-index

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1. Introduction

The study about metrics on tangent bundle of a Riemannn manifold was introduced at the end of the 1950's. [Yano and Ishihara, 1970], The metrics on M were lifted TM by vertical, complete and horizontal lifts. [Yano and Ishihara, 1970], [Oproiu and Papaghiuc, 1998] examined the differential geometry of TM by this metric. Moreover, they defined that obtained metric on TM by the horizontal lift of Riemann or semi-Riemann metric on M is semi-Riemann metric without proof**.** In addition, They also defined that lifted metric on TM by horizontal lift is a semi-Riemann metric with n positive and n negative signs without proof.

In this paper, it is proved that (TM, g^H) is a semi-Riemann manifold. Then it is showed that the tangent vector in (M,g) can respectively spacelike, null and timelike if the tangent vector in (TM, g^H), which has been lifted a tangent vector in (M,g), is spacelike, null and timelike.
Finally, it is obtained that obtained metric on TM by the horizontal lift of a Riemann or a semi-Riemann metric on M is semi-Riemann metric with n positive and n negative signs.

2. A differentiable manifold and its tangent bundle

Let M be a differentiable n-dimensional manifold and TM be its tangent bundle. Suppose that $(x)=\{x^1,...,x^n\}$ is a system of local coordinates defined in the neighborhood $p\in U\subset M$. Since the canonical projection $\pi: TM\to M$ obtains the equality $\pi(Z_p)=p$, $\pi^{-1}(U)=U^/$ is open neighborhood of the point $\pi^{-1}(\{p\})$ in TM.

$$(x,y)(Z_p)=(x^1(p),...,x^n(p),Z_p[x^1],...,Z_p[x^n]) \qquad (1)$$

Therefore, the map (x,y) which is defined with the equality (1) is a local map in $U^/\subset TM$ and the system of $(x^i,y^i;1\le i\le n)$ is induced local coordinate system in TM.
TTM, the tangent bundle of TM, has subvector bundle $VTM = Çek((\tau_M)_*)$ which is called vertical distribution on TM and HTM which is called horizontal distribution on TM. In addition, it can be expressed TTM as direct sum of subvector bundles VTM and HTM

$$TTM = VTM \oplus HTM \qquad (2)$$

$\{\delta_i,\partial_i;1\le i\le n\}$ is adapted local frame in TM where δ_i is local frame in HTM

$$\delta_i=\left(\frac{\partial}{\partial x^i}\right)^H=\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^{\ j}\frac{\partial}{\partial y^j}, \qquad N_i^{\ j}=y^k\Gamma_{ki}^{\ \ j} \qquad (3)$$

and ∂_i is local frame in VTM

$$\partial_i=\left(\frac{\partial}{\partial x^i}\right)^V=\frac{\partial}{\partial y^i}. \qquad (4)$$

Furthermore, $\{\delta y^i,dx^i;1\le i\le n\}$ is adapted local dual frame in TM where

$$\delta y^i=dy^i+N^i{}_j dx^j, \qquad N^i{}_j=y^k\Gamma_{kj}^{\ \ i} \qquad (5)$$

(Oproiu ve Papaghiuc, 1998).
g^H is defined vector fields on TM as bellow

$$\begin{aligned} &g^H(X^V,Y^H)=g^H(X^H,Y^V)=(g(X,Y))^V,\\ &g^H(X^H,Y^H)=(g(X,Y))^H=0,\\ &g^H(X^V,Y^V)=0 \end{aligned} \qquad (6)$$

for any $X,Y\in\chi(M)$ and $X^V,X^H,Y^V,Y^H\in\chi(TM)$.

g^H, the horizontal lift of the metric tensor g with components g_{ij} on M, has components

$$g^H = g_{ij}dx^i \delta y^j + g_{ij}\delta y^i dx^j \tag{7}$$

with respect to adapted local frame in TM or

$$g^H = (g_{kj}\Gamma_{ji}^k y^i + g_{ik}\Gamma_{ij}^k y^j)dx^i dx^j + g_{ij}dx^i dy^j + g_{ji}dy^i dx^j \tag{8}$$

with respect to induced local coordinates in TM.
Moreover, the matrix representation of g^H is

$$g^H = \begin{bmatrix} g_{kj}\Gamma_{ji}^k y^i + g_{ik}\Gamma_{ij}^k y^j & g_{ij} \\ g_{ji} & 0 \end{bmatrix} \tag{9}$$

(Yano and Ishihara, 1970).

3. Semi-Riemann manifold

Semi-Riemann geometry involves a particular kind of (0,2) tensor on tangent spaces. To study these in general, let V be a finite dimensional real vector space. A bilinear form on V is an R-bilinear function $b: V \times V \to R$, and let b be a symmetric.

Definition 3.1 A symmetric bilinear form b on V is called positive [negative] definite provided $v \neq 0$ implies $b(v,v) > 0 [< 0]$ and is called nondegenerate provided $b(v,w) = 0$ for all $w \in V$ implies $v = 0$.
If b is a symmetric bilinear form on V then for any subspace W of V the restriction $b\big|_{(W \times W)}$, denoted merely by $b\big|_W$, is again symmetric and bilinear.

Definition 3.2 The index ν of a symmetric bilinear form b on V is the largest integer that is the dimension of a subspace $W \subset V$ on which $b\big|_W$ is negative definite.

Definition 3.3 A scalar product g on a vector space V is nondegenerate symmetric bilinear form on V.

Definition 3.4 If g smoothly assigns to each point p of M a scalar product g on the tangent space $T_p M$, and the index of g is the same for all p, a smooth manifold M furnished with a metric tensor g is called a semi-Riemann manifold. A semi-Riemann manifold is denoted an ordered pair (M, g).

Definition 3.5 A tangent vector v in a semi-Riemann manifold (M, g) is
i) spacelike vector if $g(v,v) > 0$ or $v = 0$,
ii) null vector if $g(v,v) = 0$ and $v \neq 0$,
iii) timelike vector if $g(v,v) < 0$.

The category into which a given tangent vector falls is called its causal character.

To prevention clumsy and to make computation easier, we will use normal coordinate system while computing the index of a semi-Riemann manifold.

Theorem 3.1 Let (M, g) be a semi-Riemann manifold with index ν. If $(x^1,...,x^n)$ is a normal coordinate system at $p \in M$, it is

i) $g_{ij}(p) = \delta_{ij}\varepsilon_j\,, \quad \varepsilon_j = \begin{cases} -1 & 1 \le j \le \nu \\ 1 & \nu+1 \le j \le n \end{cases},$ (10)

ii) $\Gamma_{ij}{}^{k}(p) = 0$ for $(1 \le i, j, k \le n)$ (11)

(O'Neill 1983).

4. The metric g^H in TM

Theorem 4.1 Let M be a differentiable manifold and g be a Riemann or semi-Riemann metric on M. If $\chi(TM)$ is a set of vector fields in TM and $C^\infty(TM, R)$ is a ring of differentiable function whose range set is real number, g^H is semi-Riemann metric in TM where g^H is

$$\begin{array}{cccc} g^H: & \chi(TM)\times\chi(TM) & \to & C^\infty(TM,R) \\ & (\widetilde{X},\widetilde{Y}) & \to & g^H(\widetilde{X},\widetilde{Y}) \end{array}. \tag{12}$$

Proof: Let X be a vector field in M and $\widetilde{X}$ be a vector field in TM. All $\widetilde{X}$ vector fields in TM are expressed with direct sum of vector fields X^V and X^H due to property (2) as below

$$\widetilde{X} = X^V + X^H. \tag{13}$$

We get

$$g^H(\alpha\widetilde{X} + \beta\widetilde{Y}, \widetilde{Z}) = g^H\big((\alpha X + \beta Y)^V + (\alpha X + \beta Y)^H, Z^V + Z^H\big)$$
$$= \alpha g^H(\widetilde{X}, \widetilde{Z}) + \beta g^H(\widetilde{Y}, \widetilde{Z})$$

and

$$g^H(\widetilde{X}, \alpha\widetilde{Y} + \beta\widetilde{Z}) = g^H\big((X^V + X^H, (\alpha Y + \beta Z)^V + (\alpha Y + \beta Z)^H\big)$$
$$= \alpha g^H(\widetilde{X}, \widetilde{Y}) + \beta g^H(\widetilde{X}, \widetilde{Z})$$

for any $X, Y, Z \in \chi(M)$, $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \chi(TM)$ and $\alpha, \beta \in R$.

Thus, we obtain that g^H is bilinear. By the equality (13), we get

$$g^H(\widetilde{X}, \widetilde{Y}) = g^H(\widetilde{Y}, \widetilde{X}).$$

Thus, we obtain that g^H is symmetric. By the definite nondegeneracy of a metric, we get

$$g^H(\widetilde{X}, Y^V) = g^H(X^V + X^H, Y^V) = g^H(X^H, Y^V) = \big(g(X,Y)\big)^V = 0$$

for $\forall\widetilde{X} \in \chi(TM)$ and $\widetilde{\mathrm{Y}} = \mathrm{Y}^{\mathrm{V}}$. We find that $Y^V = 0$ and by similary operation, we get

$$g^H(\widetilde{X}, Y^H) = \big(g(X,Y)\big)^V = 0.$$

We also find that $Y^H = 0$. Thus, we get

$$g^H(\widetilde{X},\widetilde{Y}) = 0 \Leftrightarrow \widetilde{Y} = 0 \quad \text{for } \forall \widetilde{X} \in \chi(TM).$$

Namely, g^H is nondegenerate. In conclusion, since it is provided non degenerate, symmetric and bilinear properties of g^H, g^H is a semi-Riemann metric in TM.

Theorem 4.2 Let (M,g) be a semi-Riemann manifold and g^H semi-Riemann metric in TM. Let T_pM be the tangent space at a point p in M and T_ZTM be the tangent space at a point Z in TM which provide equality $\pi(Z) = p$. If X, Z are spacelike or timelike vectors in T_pM, both $(X^V)_Z$ and $(X^H)_Z$ are null vectors in T_ZTM .

Proof: By the equation (6) and definition 3.5, it can be seen proof of the claim straightforward.

Theorem 4.3 Any vector which is defined in a semi-Riemann manifold (TM, g^H) is null vector if and only if

i) the vector which is defined on TM lies vertical or horizontal vector subspace in tangent space of TM or

ii) projected vector in T_pM of the vector which is defined in T_ZTM is null vector.

Proof:

i) By the equation (6), it is clear.

ii) By the equality (13), it is writen

$$\widetilde{X}_Z = (X^V)_Z + (X^H)_Z$$

where $\widetilde{X}_Z \in T_ZTM$ and $X, Z \in T_pM$. We get

$$g^H(\widetilde{X}_Z, \widetilde{X}_Z) = 2g(X_p, X_p) = 0.$$

In addition, we get

$$\widetilde{X}_Z \neq 0 \Rightarrow X_p \neq 0.$$

Thus, X_p is also null vector.

Theorem 4.4 If any vector which is defined on semi-Riemann manifold (TM, g^H) is spacelike vector, the vector which is projected on semi-Riemann manifold (M, g) is spacelike vector.

Proof: By the part (ii) of preceding theorem, it is writen

$$\widetilde{X}_Z = (X^V)_Z + (X^H)_Z$$

where $\widetilde{X}_Z \in T_ZTM$ and $X, Z \in T_pM$. By the definition 3.5, we get

$$g^H(\widetilde{X}_Z, \widetilde{X}_Z) = 2g(X_p, X_p) > 0$$

or

$$\tilde{X}_Z = 0 \Rightarrow X_p = 0.$$

Thus, X_p is spacelike vector.

Theorem 4.5 If any vector which is defined on semi-Riemann manifold (TM, g^H) is timelike vector, the vector which is projected on semi-Riemann manifold (M, g) is timelike vector.

Proof: By the part (ii) of theorem 4.3, it is writen

$$\tilde{X}_Z = \left(X^V\right)_Z + \left(X^H\right)_Z$$

where $\tilde{X}_Z \in T_Z TM$ and $X, Z \in T_p M$. By the definition 3.5, we get

$$g^H(\tilde{X}_Z, \tilde{X}_Z) = 2g(X_p, X_p) < 0.$$

Thus, if $\tilde{X}_Z$ is timelike vector, X_p is timelike vector.

5. The index of the metric g^H

Theorem 5.1 If (M, g) is a Riemann manifold, (TM, g^H) is a semi-Riemann manifold with n-index.

Proof: Let g be a Riemann metric on M. We take a normal coordinate system in M. In terms of the equalities (10), (11), it is obtained that

$$g_{ij}(p) = \delta_{ij} = \begin{Bmatrix} 1 & i = j \\ 0 & i \neq j \end{Bmatrix}, \quad \Gamma_{ij}^k(p) = 0 \tag{14}$$

for $\nu = 0$

The equalities in (14) taking into account (9), the matrix representation of g^H is

$$g^H = \begin{bmatrix} 0 & I_{nxn} \\ I_{nxn} & 0 \end{bmatrix}.$$

The eigenvalue of this obtained metric g^H can be seen by the equality

$$\left|\lambda I_{2nx2n} - g^H\right| = \begin{vmatrix} \lambda I_{nxn} & -I_{nxn} \\ -I_{nxn} & \lambda I_{nxn} \end{vmatrix} = (\lambda^2 - 1)^n. \tag{15}$$

The equality (15) can be proved by the methot of induction as below.
It is true that

$$\left|\lambda I_{2x2} - g^H\right| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda^2 - 1)^1 \quad \text{for } n = 1$$

We suppose that following equality is true $n = k$

$$\left|\lambda I_{2kx2k} - g^H\right| = \begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix} = (\lambda^2 - 1)^k.$$

The equality can be proved as follow for $n = k+1$

$$\left|\lambda I_{2(k+1)x2(k+1)} - g^H\right| = \det\begin{bmatrix} \lambda & \dots & 0 & -1 & \dots & 0 \\ \dots & \lambda I_{kxk} & & \dots & -I_{kxk} & \\ 0 & & & 0 & & \\ -1 & \dots & 0 & \lambda & \dots & 0 \\ \dots & -I_{kxk} & & \dots & \lambda I_{kxk} & \\ 0 & & & 0 & & \end{bmatrix}_{2(k+1)x2(k+1)}$$

$$= \lambda \begin{vmatrix} \lambda I_{kxk} & & 0 & -I_{kxk} & \\ & & \dots & & \\ & & 0 & & \\ 0 \;\dots\; 0 & & \lambda & \dots & 0 \\ -I_{kxk} & & \dots & \lambda I_{kxk} & \\ & & 0 & & \end{vmatrix} + (-1)^{k+4} \begin{vmatrix} 0 & \lambda I_{kxk} & & -I_{kxk} & \\ \dots & & & & \\ 0 & & & & \\ -1 & \dots & 0 \;\; 0 & \dots & 0 \\ \dots & -I_{kxk} & & \lambda I_{kxk} & \\ 0 & & & & \end{vmatrix}$$

$$= \lambda^2 (-1)^{2(k+1)} \begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix} + (-1)^{k+4}(-1)^{k+3} \begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix}$$

$$= \lambda^2 \begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix} + (-1)^{2k+7} \begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix} = \lambda^2 (\lambda^2 - 1)^k - (\lambda^2 - 1)^k$$

$$= (\lambda^2 - 1)^{k+1}.$$

Since the index of g^H is independent chose of map on TM [see the definition 3.4], g^H has n positive and n negative eigen value whose equal to +1 and -1. The eigen vectors which is corresponded the eigen value +1 are

$$X_1 = \begin{bmatrix} 1 \\ \dots \\ 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}, \dots, X_n = \begin{bmatrix} 0 \\ \dots \\ 1 \\ 0 \\ \dots \\ 1 \end{bmatrix},$$

and the eigen vectors which is corresponded the eigen value -1 are

$$X_{n+1} = \begin{bmatrix} 1 \\ \dots \\ 0 \\ -1 \\ \dots \\ 0 \end{bmatrix}, \dots, X_{2n} = \begin{bmatrix} 0 \\ \dots \\ 1 \\ 0 \\ \dots \\ -1 \end{bmatrix}.$$

The matrix which are obtained in terms of eigenvectors $X_1, \dots, X_{2n}$ and its inverse are

$$P = \begin{bmatrix} I_{nxn} & I_{nxn} \\ I_{nxn} & -I_{nxn} \end{bmatrix}$$

and

$$P^{-1} = \frac{1}{2}\begin{bmatrix} I_{nxn} & I_{nxn} \\ I_{nxn} & -I_{nxn} \end{bmatrix}.$$

Thus, $\tilde{g}^H = P^{-1} g^H P$, the diagonal matrix of g^H , is expressed as follow

$$\tilde{g}^H = \begin{bmatrix} I_{nxn} & 0 \\ 0 & -I_{nxn} \end{bmatrix}.$$

Finally, in terms of the matrix representation of $\tilde{g}^H$, we can see that g^H is a semi-Riemann metric with n positif and n negatif sign, straightforward.

Theorem 5.2 If (M, g) is a Riemann manifold with ν - index, (TM, g^H) is a semi-Riemann manifold with n-index.

Proof: Let M be a semi-Riemann manifold with ν - index. If we take normal coordinate system in M, it is

$$g_{ij}(p) = \left\{ \begin{matrix} -1 & ; & 1 \le j = i \le \nu \\ 1 & ; & \nu + 1 \le j = i \le n \\ 0 & & j \ne i \end{matrix} \right\} = I_n^\nu$$

where I_n^ν is diagonal square matrix. The first ν elements which are on the diagonal of the this matrix are -1 and rest $n - \nu$ elements are +1. Moreover, it is $\Gamma_{ij}^k(p) = 0$. Thus, the representation of g^H is that

$$g^H = \begin{bmatrix} 0 & I_n^\nu \\ I_n^\nu & 0 \end{bmatrix}.$$

The eigenvalue of this obtained metric g^H can be seen by the equality

$$\left| \lambda I_{2nx2n} - g^H \right| = \begin{vmatrix} \lambda I_{nxn} & -I_n^\nu \\ -I_n^\nu & \lambda I_{nxn} \end{vmatrix} = (\lambda^2 - 1)^n. \tag{16}$$

The equality (16) can be proved that it is cosidered two cases as follow.

<u>**Case 1:**</u> Suppose that n-changeable and ν – non changeable
we will the methot of induction so that we prove case 1. It is true that

$$\left| \lambda I_{4x4} - g^H \right| = \begin{vmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & -1 \\ 1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} = (\lambda^2 - 1)^2 \text{ for } n = 2$$

We suppose that following equality is true for $n = k$

$$\left|\lambda I_{2kx2k} - g^H\right| = \begin{vmatrix} \lambda I_{kxk} & -I_k^1 \\ -I_k^1 & \lambda I_{kxk} \end{vmatrix} = (\lambda^2 - 1)^k .$$

We can prove the true of the equality of (16) for $n = k+1$ as below

$$\left|\lambda I_{2(k+1)x2(k+1)} - g^H\right| = \det\begin{bmatrix} \lambda & \dots & 0 & 1 & \dots & 0 \\ \dots & \lambda I_{kxk} & & \dots & -I_{kxk} & \\ 0 & & & 0 & & \\ 1 & \dots & 0 & \lambda & \dots & 0 \\ \dots & -I_{kxk} & & \dots & \lambda I_{kxk} & \\ 0 & & & 0 & & \end{bmatrix}_{2(k+1)x2(k+1)}$$

$$= \lambda^2(-1)^{2(k+1)}\begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix} + (-1)^{k+3}(-1)^{k+2}\begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix}$$

$$\left|\lambda I_{2(k+1)x2(k+1)} - g^H\right| = \lambda^2\begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix} + (-1)^{2k+5}\begin{vmatrix} \lambda I_{kxk} & -I_{kxk} \\ -I_{kxk} & \lambda I_{kxk} \end{vmatrix}$$

$$= \lambda^2(\lambda^2 - 1)^k - (\lambda^2 - 1)^k$$

$$= (\lambda^2 - 1)^{k+1}.$$

<u>Case 2:</u> it is n-non changeable and ν changeable

It is true that

$$\left|\lambda I_{6x6} - g^H\right| = \begin{vmatrix} \lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 & -1 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & -1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & 0 & \lambda \end{vmatrix} = (\lambda^2 - 1)^3 \text{ for } n = 3 \text{ and } \nu = 1 .$$

and

$$\left|\lambda I_{6x6} - g^H\right| = \begin{vmatrix} \lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & 0 & \lambda \end{vmatrix} = (\lambda^2 - 1)^3 \text{ for } n = 3 \text{ and } \nu = 2 .$$

We suppose that following equality is true for $n = k$ and $\nu = \eta$

$$\left|\lambda I_{2kx2k} - g^H\right| = \begin{vmatrix} \lambda I_{kxk} & -I_n^\eta \\ -I_n^\eta & \lambda I_{kxk} \end{vmatrix} = (\lambda^2 - 1)^n .$$

We can prove that the true of the equality (20) for $n = k$ and $\nu = \eta + 1$ as follow

$$\left|\lambda I_{2kx2k}-g^H\right| = \begin{vmatrix} \lambda & & \ldots & 0 & 1 & & \ldots & 0 \\ \ldots & \lambda I_{k-1xk-1} & & \ldots & & -I_{k-1}^{\eta-1} & & \\ 0 & & & & 0 & & & \\ 1 & & \ldots & 0 & \lambda & & \ldots & 0 \\ \ldots & & -I_{k-1}^{\eta-1} & & \ldots & \lambda I_{k-1xk-1} & & \\ 0 & & & & 0 & & & \end{vmatrix}$$

$$= \lambda^2(-1)^{2k}\begin{vmatrix} \lambda I_{k-1xk-1} & -I_{k-1}^{\eta-1} \\ -I_{k-1}^{\eta-1} & \lambda I_{k-1xk-1}\end{vmatrix} + (-1)^{k+2}(-1)^{k+1}\begin{vmatrix} \lambda I_{k-1xk-1} & -I_{k-1}^{\eta-1} \\ -I_{k-1}^{\eta-1} & \lambda I_{k-1xk-1}\end{vmatrix}$$

$$= \lambda^2(\lambda^2-1)^{k-1} - (\lambda^2-1)^{k-1}$$

$$= (\lambda^2-1)^k.$$

Thus, the value of the deteminant $\left|\lambda I_{2nx2n}-g^H\right|$ is independent the choose of both n and ν. Furthermore, g^H has n positive and n negative eigenvalue which equal to +1 and -1.

The eigen vector which is corresponded the eigen value +1

$$X_1=\begin{bmatrix}1\\ \ldots\\ 0\\ 0\\ \ldots\\ 0\\ 1\\ \ldots\\ 0\\ 0\\ \ldots\\ 0\end{bmatrix},\ldots,X_\nu=\begin{bmatrix}0\\ \ldots\\ 1\\ 0\\ \ldots\\ 0\\ 0\\ \ldots\\ 1\\ 0\\ \ldots\\ 0\end{bmatrix},X_{\nu+1}=\begin{bmatrix}0\\ \ldots\\ 0\\ 1\\ \ldots\\ 0\\ 0\\ \ldots\\ 0\\ -1\\ \ldots\\ 0\end{bmatrix},\ldots,X_n=\begin{bmatrix}0\\ \ldots\\ 0\\ 0\\ \ldots\\ 1\\ 0\\ \ldots\\ 0\\ 0\\ \ldots\\ -1\end{bmatrix}$$

and the eigenvector which is corresponded the eigenvalue -1

$$X_{n+1}=\begin{bmatrix}1\\ \ldots\\ 0\\ 0\\ \ldots\\ 0\\ -1\\ \ldots\\ 0\\ 0\\ \ldots\\ 0\end{bmatrix},\ldots,X_{n+\nu}=\begin{bmatrix}0\\ \ldots\\ 1\\ 0\\ \ldots\\ 0\\ 0\\ \ldots\\ -1\\ 0\\ \ldots\\ 0\end{bmatrix},X_{n+\nu+1}=\begin{bmatrix}0\\ \ldots\\ 0\\ 1\\ \ldots\\ 0\\ 0\\ \ldots\\ 0\\ 1\\ \ldots\\ 0\end{bmatrix},\ldots,X_{2n}=\begin{bmatrix}0\\ \ldots\\ 0\\ 0\\ \ldots\\ 1\\ 0\\ \ldots\\ 0\\ 0\\ \ldots\\ 1\end{bmatrix}.$$

The matrix which are obtained in terms of eigenvectors $X_1,...,X_{2n}$ and its inverse are

$$P=\begin{bmatrix} I_{\nu\times\nu} & 0_{\nu\times\nu} & I_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & -I_{n-\nu\times n-\nu} \\ I_{\nu\times\nu} & 0_{\nu\times\nu} & -I_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} \end{bmatrix}$$

and

$$P^{-1}=\frac{1}{2}\begin{bmatrix} I_{\nu\times\nu} & 0_{\nu\times\nu} & I_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} \\ I_{\nu\times\nu} & 0_{\nu\times\nu} & -I_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{n-\nu\times n-\nu} & -I_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} \end{bmatrix}.$$

Thus, $\tilde{g}^H = P^{-1} g^H P$, the diagonal matrix of g^H , is expressed as follow

$$\tilde{g}^H=\begin{bmatrix} -I_{\nu\times\nu} & 0_{\nu\times\nu} & 0_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{n-\nu\times n-\nu} & I_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} \\ 0_{\nu\times\nu} & 0_{\nu\times\nu} & I_{\nu\times\nu} & 0_{\nu\times\nu} \\ 0_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & 0_{n-\nu\times n-\nu} & -I_{n-\nu\times n-\nu} \end{bmatrix}$$

Finally, in terms of the matrix representation of $\tilde{g}^H$, we can see that g^H is a semi-Riemann metric with n positive and n negative sign, straightforward

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