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SOME PROPERTIES OF LIGTHLIKE HYPERSURFACES IN SEMI-REMANNIAN MANIFOLD

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Abstract: In this paper, by using the method of Duggal and Bejancu, we study the geometry of a lightlike hypersurfaces of a semi-Riemannian manifold. First of all, we presented the theorems giving the conditions in which lightlike hypersurfaces are totally umbilical and flat. Moreover, we gave the theorems about symmetric case of Ricci tensor of reduced connection. Finally we obtain the corresponding theorems of the well-known theorems about hypersurfaces, like Euler and Meusnier Theorem, for lightlike hypersurfaces.

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1. Introduction

The geometry of lightlike hypersurfaces of a semi-Riemannian manifold is one of the interesting topics of differential geometry. It is well known that lightlike hypersurfaces are metrics having vanishing determinants whose degeneracy leads several difficulty: The contravariant metric cannot immediately be defined, as

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a result, the connection cannot be specified uniquely in the normal way. Moreover, the normal is a lightlike vector lying in the tangent plane, which makes it necessary to look for some other vector to rig the hypersurface, and makes it impossible to normalise the normal in the usual way.

Duggal and Bejancu have been intensively studied of lightlike hypersurfaces of a semi-Riemannian manifolds endowed with some geometrical structures, see [1], [2], [4]. In these studies, Duggal and Bejancu have contructed a transversal vector bundle of a lightlike hypersurface, see [4]. In [7] D.N. Kupeli, using the canonical projection, has investigated the properties of the hypersurface. On the other hand, Bejancu, Ferrandez and Lucas have shown that the geometry of a lightlike (null, degenerate) hypersurface M in a semi-Euclidean space $\mathbb{R}^{n+2}_{\nu}$ can be investigated by using the geometry of M as a Riemannian hypersurface in a Euclidean space $\mathbb{R}^{n+1}_{0}$, see [3].

In this paper, we obtained the conditions under which lightlike hypersurfaces are totally umbilical and flat. In addition, we gave the theorems about symmetric case of Ricci tensor of reduced connection. Finally we obtained correponding theorems of the well-known theorems about hypersurface, as Euler and Meusnier Theorems, for lightlike hypersurface.

2. Preliminaries

Let M be a hypersurface of a $(n+1)-$ dimensional, $n>1$, semi-Riemannian manifold $\widetilde{M}$ with semi-Riemannian metric $\widetilde{g}$ of index $1\leq\nu\leq n$. We consider

$$T_xM^{\perp}=\left\{Y_x\in T_x\widetilde{M}\mid \widetilde{g}_x\left(Y_x,X_x\right)=0,\forall X_x\in T_xM\right\}$$

for any $x\in M$. We say that M is a *lightlike (null, degenerate) hypersurface* of $\widetilde{M}$ if $T_xM\cap T_xM^{\perp}\neq\{0\}$ at any $x\in M$.

An orthogonal complementary vector bundle of $TM^{\perp}$ in TM is non-degenerate subbundle of TM called the *screen distribution* on M and denoted $S\left(TM\right)$. We have the following splitting into orthogonal direct sum:

$$TM=S\left(TM\right)\perp TM^{\perp}. \tag{2.1}$$

The subbundle $S\left(TM\right)$ is non-degenerate, so is $S\left(TM\right)^{\perp}$, and the following holds:

$$T\widetilde{M}=S\left(TM\right)\perp S\left(TM\right)^{\perp}, \tag{2.2}$$

where $S\left(TM\right)^{\perp}$ is the orthogonal complementary vector bundle to $S\left(TM\right)$ in $T\widetilde{M}\Big|_M$.

Let $tr(TM)$ denote the complementary vector bundle of $TM^{\perp}$ in $S\,(TM)^{\perp}$. Then we have

$$S\,(TM)^{\perp} = TM^{\perp} \oplus tr(TM). \tag{2.3}$$

From (2.1), (2.2) and (2.3), we see that

$$T\widetilde{M}_{|M} = S\,(TM)\perp(TM^{\perp} \oplus tr(TM)). \tag{2.4}$$

Let $\widetilde{\nabla}$ and ∇ be Levi-civita and lineer connections on $\widetilde{M}$ and M, respectively. Then, by using the second form of the decomposition in (2.4), we get

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y$, $A_V X$ belong to $\Gamma(TM)$ while $h(X,Y)$ and $\nabla^{\perp}_X V$ belong to $\Gamma(tr(TM))$.

Let $\{\xi, N\}$ be a pair of rections on $\mathcal{U} \subset M$. Then bilineer form B and a $1-$ form τ on $\mathcal{U}$ are defined by

$$B(X,Y) = \widetilde{g}(h(X,Y),\xi) \tag{2.7}$$

and

$$\tau(X) = \widetilde{g}(\nabla^{\perp}_X V,\xi) \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$. Substituting (2.7) and (2.8) in (2.5) and (2.6) we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{2.9}$$

and

$$\widetilde{\nabla}_X V = -A_X V + \tau(X)N. \tag{2.10}$$

The following two theorems are due to Duggal, see [4].

Theorem 2.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $U \subset M$, there exists a unique section M of $tr(TM\ )$ on U satisfying:*

$$\tilde{g}(N,\xi) = 1\,, \quad g\,(N,N) = g\,(N,W) = 0\,.$$

Theorem 2.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then:*

(i) The lineer connection $\overset{*}{\nabla}$ is a metric connection on $S(TM)$.

(ii) The induced connection ∇ *on* M *satisfies*

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y)\,,\ \forall X,Y,Z \in \Gamma(TM)$$

for $X, Y \in \Gamma(TM)$, *where* $\eta(Z) = \widetilde{g}(Z,N)$.

Definition 2.1. Let M be $(n+1)$ -dimensional lightlike hypersurface and $S(TM)$ be a screen distributionon M. The lightlike mean curvature of M is defined by

$$H = -\sum_{a=1}^{n} B(E_a, E_a)\,,$$

where $\{E_1, \ldots, E_n\}$ is an orthonormal basis of $S(TM)$, see [4].

Definition 2.2. Let M be a lightlike hypersurface of a semi-Riemannain manifold of $\widetilde{M}$. Then M is called totally umbilical if there exists a smooth function F such that

$$B(X,Y) = Fg(X,Y)$$

for any $X, Y \in \Gamma(TM)$, see [4].

3. Some Properties of a Lightlike Hypersurfaces

It is well known that lightlike hypersurface have ineresting properties. For instance, the Ricci tensor of a lightlike hypersurface is not symmetric. As this is not desirable both for geometry and physics, we find geometric conditions so that the Ricci tensor is symmetric. In addition, after giving an example of a null cone in $\mathbb{R}^3_1$, we have found the conditions under which a lightlike hypersurface is total geodesic and flat.

Example 3.1. In $\mathbb{R}^3_1$ null cone is defined by

$$\Lambda = \{x \in \mathbb{R}^3_1 - \{0\} \mid g(x,x) = 0\}$$

or

$$\Lambda = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid f : \mathbb{R}^3_1 \to \mathbb{R}\}\,,$$
$$(x_1, x_2, x_3) \to f(x_1, x_2, x_3) = -x_1^2 + x_2^2 + x_3^2 = 0\,.$$

Thus, the gradient of f is given by

$$\operatorname{grad} f = (2x_1, 2x_2, 2x_3)$$

and $\operatorname{grad} f|_p \neq 0$ for any $p \in \Lambda$. On the other hand, as

$$g(\operatorname{grad} f, \operatorname{grad} f) = 0,$$

$\operatorname{grad} f$ is a lightlike vector field of null cone. Therefore, taking into account of the equation given by

$$T\widetilde{M} = S(TM) \perp RadTM \oplus tr(TM)$$

we obtain

$$N = \frac{(-x_1, x_2, x_3)}{2(x_1^2 + x_2^2 + x_3^2)},\ W_1 = \frac{(0, x_3, -x_2)}{\sqrt{x_3^2 + x_2^2}},\ W_2 = \frac{(0, -x_3, x_2)}{\sqrt{x_3^2 + x_2^2}},$$

where

$$\widetilde{g}(N, W_i) = \widetilde{g}(\operatorname{grad} f, W_i) = 0,\ i \in \{1, 2\}, \quad \widetilde{g}(N, \operatorname{grad} f) = 1$$

for any $N \in \Gamma(TM)$ and $W_i \in \Gamma(S(TM))$. Also, considering by Weingerten formulae for $X = (\partial_1, \partial_2, \partial_3) \in T_xM$, we have

$$A_N = \frac{1}{2(x_1^2 + x_2^2 + x_3^2)} I\,.$$

Proposition 3.1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$. Then if M is Einstein, the Ricci tensor of M is simetric.*

Proof. We suppose that M is a Einstein. Then taking into account Ricci tensor of a lightlike hypersurface M and using the Gauss formulae we have

$$Ric(X, Y) - Ric(Y, X) = -2d\tau(X, Y)$$

for any $X, Y \in \Gamma(TM)$ and $\{E_1, \ldots, E_n\}$ orthonormal basis of screen distribution $S(TM)$. Also using by definition of Einstein we get

$$kg(X, Y) - kg(Y, X) = -2d\tau(X, Y).$$

From above equation we have

$$d\tau(X, Y) = 0$$

which proves the assertion of proposition. □

Proposition 3.2. *Let M be a lightlike hypersurface of an $(n+1)$-dimensional, $n > 1$, semi-Riemannian manifold $\widetilde{M}$. Then lineer connection ∇^t on transversal vector bundle is flat if and only if lightlike transversal vector bundle N is parallel and M is totally geodesic.*

Proof. We assume that M is a lightlike hypersurface of a semi-Riemannian manifold $\widetilde{M}$. For $\widetilde{\nabla}$ and ∇^t on $T\widetilde{M}$ and $tr(TM)$, respectively, we have

$$\begin{aligned}\widetilde{R}(X, Y)N = &-\nabla_X(A_NY) - h(X, A_NY) - A_{\tau(Y)N}X + \nabla^t_X\nabla^t_Y N + \nabla_Y(A_NX)\\ &+ h(Y, A_NX) + A_{\tau(X)N}Y - \nabla^t_Y\nabla^t_X N - A_N[X, Y] + \nabla^t_{[X,Y]}N.\end{aligned}$$

Thus curvature tensor of a lightlike transversal vector bundle is taken by

$$\begin{aligned}\widetilde{R}^t(X, Y)N &= \nabla^t_X\nabla^t_Y N + \nabla^t_Y\nabla^t_X N + \nabla^t_{[X,Y]}N\\ &+ h(Y, A_NX) - h(X, A_NY)\end{aligned}$$

which show that ∇^t is flat if and only if N is parallel and M is totally geodesic.

Proposition 3.3. *Let M be a lightlike hypersurface of an $(n+1)$-dimensional, $n > 1$, semi-Riemannian space form $\widetilde{M}(c)$. Then the connection ∇^t on $S(TM)$ is flat if and only if M is a totally geodesic.*

Proof. We assume that $\widetilde{R}$ is curvature tensor of $\widetilde{M}(c)$. Then we get

$$\widetilde{R}(X,Y)N = \nabla_Y(A_N X) - \nabla_X(A_N Y) + h(Y, A_N X) - h(X, A_N Y) + \tau(X)A_N Y - \tau(Y)A_N X - A_N[X,Y] + \widetilde{R}^t(X,Y)N.$$

Considering definition of $\widetilde{M}(c)$ and the scaler product of the last equation with ξ, we obtain

$$g(\widetilde{R}^t(X,Y)N, \xi) = B(X, A_N Y) - B(Y, A_N X)$$

which proves proposition. □

Example 3.2. Suppose M is a hypersurface of R^4_2 given by the equation

$$x^3 - \frac{1}{2}(\cos x^1 + \sin x^1)^2 = x^0\,.$$

Then, we have

$$RadTM = Sp\{\frac{\partial}{\partial x^0} + (\cos x^1 + \sin x^1)\frac{\partial}{\partial x^1} - (\cos x^1 + \sin x^1)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\},$$

$$tr(TM) = -\frac{1}{2[1 + (\cos x^1 + \sin x^1)^2]}\{\frac{\partial}{\partial x^0} + (\cos x^1 + \sin x^1)\frac{\partial}{\partial x^1} + (\cos x^1 + \sin x^1)\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\}\,,$$

and

$$S(TM) = \{W_1 = \frac{\partial}{\partial x^1} - (\cos x^1 + \sin x^1)\frac{\partial}{\partial x^0} + W_2 = \frac{\partial}{\partial x^2} + (\cos x^1 + \sin x^1)\frac{\partial}{\partial x^1}\},$$

which satisfies the following equations

$$g(\xi,\xi) = g(\xi, W) = g(N, W) = 0$$

and

$$g(\xi, N) = 1.$$

Proposition 3.4. *Let M be a lightlike hypersurface of an semi-Riemannian manifold $\widetilde{M}$ with flat connection. Then if M and $S(TM)$ are totally umbilical, λ satisfies the partial differential equation*

$$\xi(\lambda) - \lambda(\tau(\xi) + c) = 0.$$

Proof. Let that $\widetilde{\nabla}$ be a Levi-civita connection of $\widetilde{M}$. Then we get

$$\widetilde{g}(\widetilde{R}(X,Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ).$$

Thus, knowing that $\widetilde{\nabla}$ is flat and $S(TM)$ is totally umbilical, we obtain

$$\begin{aligned}0 = X(\lambda)g(Y,PZ) + \{B(X,Y)\eta(PZ) + B(X,PZ)\eta(Y)\}\\ - Y(\lambda)g(X,PZ) + \{B(Y,X)\eta(PZ) + B(Y,PZ)\eta(Y)\}\\ + \tau(Y)g(X,PZ) - \tau(X)g(Y,PZ)\,.\end{aligned}$$

Taking $X = \xi$ and $B = cg$ in above equation, we have

$$\xi(\lambda) - \lambda(\tau(\xi) + c) = 0. \qquad \square$$

Proposition 3.5. *Let M be a totally umbilical lightlike hypersurface of an $(n+1)$-dimensional, $n > 1$, semi-Riemannian space form $\widetilde{M}(c)$. Then if screen distribution $S(TM)$ is totally umbilical, M has constant curvature.*

Proof. We assume that H is a mean curvature of M. Then taking into account that M and $S(TM)$ are totally umbilical, we have

$$\widetilde{g}(\overset{*}{A}_\xi X, Y) = B(X,PY) = \widetilde{g}(h(X,PY),\xi) = Hg(X,PY)$$

and

$$C(X,PW) = kg(X,PW).$$

Thus, from definition of $\widetilde{M}(c)$, we obtain

$$\begin{aligned} R(X,Y,Z,PW) &= c\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\}\\ &+ Hg(X,Z)kg(Y,PW) - Hg(Y,Z)kg(X,PW)\\ &= (c+Hk)\{g(X,Z)g(Y,PW) - g(Y,Z)g(X,PW)\}\end{aligned}$$

for any $X,Y,Z \in \Gamma(TM)$ and $PW \in \Gamma(S(TM))$.

4. The Geometry of a Lightlike Hypersurface in $\mathbb{R}^{n+2}_\nu$ with Respect to its Geometry in $\mathbb{R}^{n+2}_0$

In this section we show that the canonical lightlike transversal vector bundle of a lightlike hypersurface M in $\mathbb{R}^{n+2}_\nu$ is just the usual normal bundle of M as Riemannian hypersurface of the Euclidean space $\mathbb{R}^{n+2}_0$. As a consequence, we obtain corresponding theorems of the well known theorems about hypersurface, like Euler and Meusnier Theorems, for lightlike hypersurface.

Let $\mathbb{R}^{n+2}_\nu = (\mathbb{R}^{n+2}, \widetilde{g}_{SE})$ and $\mathbb{R}^{n+2}_0 = (\mathbb{R}^{n+2}, \widetilde{g}_E)$ be the $(n+2)$-dimensional semi-Euclidean space and Euclidean space, where $\widetilde{g}_{SE}$ and $\widetilde{g}_E$ stand for the semi-Euclidean metric of index $1 \leq \nu \leq n+1$ and the Euclidean metric given by

$$\widetilde{g}_{SE}(x,y) = -\sum_{i=0}^{\nu} x^i y^i + \sum_{a=\nu+1}^{n+1} x^a y^a \tag{4.1}$$

and

$$\widetilde{g}_{E}(x,y) = \sum_{A=0}^{n+1} x^A y^A \,. \tag{4.2}$$

Consider a hypersurface M of $\mathbb{R}^{n+2}$ given by the equation

$$F(x^o, \dots, x^{n+1}) = 0\,,$$

where F is smooth on an open set $U \subset M$ and $rank[F^{'}_{x^0}, \dots, F^{'}_{x^{n+1}}] = 1$ on M. Then the normal bundle of M with respect to $\widetilde{g}_{SE}$ is spanned by

$$\xi = \operatorname{grad} F_{SE} = -\sum_{i=0}^{\nu} F^{'}_{x^i} \frac{\partial}{\partial x^i} + \sum_{a=\nu+1}^{n+1} F^{'}_{x^a} \frac{\partial}{\partial x^a} \,. \tag{4.3}$$

Thus, M is a lightlike hypersurface if and only if F is a solition of the partial differential equation

$$\sum_{i=0}^{\nu} (F^{'}_{x^i})^2 = \sum_{a=\nu+1}^{n+1} (F^{'}_{x^a})^2 \,. \tag{4.4}$$

In order to get transversal vector bundle and screen distribution, considering a vector field which is nowhere tanjant to M, and defined by

$$V = \sum_{i=0}^{\nu} F^{'}_{x^i} \frac{\partial}{\partial x^i}, \tag{4.5}$$

we have following equation

$$\widetilde{g}_{SE}(V, \xi) = \sum_{i=0}^{\nu} (F^{'}_{x^i})^2 \neq 0.$$

Thus, it is obtained that the canonical lightlike transversal vector bundle $tr(TM)$ is spanned by

$$N = \frac{1}{2} \Big(\sum_{i=0}^{\nu} (F^{'}_{x^i})^2\Big)^{-1} \sum_{A=0}^{n+1} F^{'}_{x^A} \frac{\partial}{\partial x^A} \,. \tag{4.6}$$

From now on we denote the unit normal vector field on M with respect to the Euclidean metric $\widetilde{g}_E$, by $N_{0.}$ Then from (4.5) and (4.6) we have

$$N = \alpha(x) N_0, \qquad \alpha(x) = \frac{1}{\sqrt{2}} \Big(\sum_{i=0}^{\nu} (F^{'}_{x^i})^2\Big)^{-\frac{1}{2}}, \qquad \text{see [3]}. \tag{4.7}$$

Let $\widetilde{\nabla}$ be the Levi-civita connection on $\mathbb{R}^{n+2}$ with respect to both metric $\widetilde{g}_{SE}$ and $\widetilde{g}_{E.}$ Then taking into account of (4.7), the Gauss and Weingarten formulaes are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N = \nabla_X Y + \alpha(x)B(X,Y)N_0 \tag{4.8}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N = -A_N X + \alpha(x)\tau(X)N_0\,. \tag{4.9}$$

From (4.7), (4.8) and (4.9) we have

$$B_0(X,Y) = \alpha(x)B(X,Y) \tag{4.10}$$

and

$$A_N X = \alpha(x)A_{N_0}X \tag{4.11}$$

for any $X, Y \in \Gamma(TM)$, see [3].

Example 4.1. Taking into account of (2.5), the gradient and unit normal vector field with respect to Euclidean space are given, respectively, by

$$\text{grad}\, f_E = (-2x_1, 2x_2, 2x_3)$$

and

$$N_0 = \frac{\text{grad}\, f_E}{\|\ \text{grad}\, f_E \ \|} = \frac{(-x_1, x_2, x_3)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,.$$

Thus we obtain the transversal vector field and shape operator of null cone of $\mathbb{R}^3_1$ given by

$$N = \frac{1}{2\sqrt{x_1^2 + x_2^2 + x_3^2}} N_0$$

and

$$A_N = \frac{1}{2\sqrt{x_1^2 + x_2^2 + x_3^2}} A_{N_0}.$$

Proposition 4.1. *The lightlike hypersurface M of semi-Euclidean space $\mathbb{R}^{n+2}_\nu$ for F function on smooth open set $U \subset R^{n+2}$ is given by $F(x_0, \ldots, x_{n+1}) = 0$ where*

$$Rank[\frac{\partial F}{\partial x^1}, \ldots, \frac{\partial F}{\partial x^n}] = 1.$$

Then vector field $\xi = \nabla F$ is the eigenvector corresponding a zero eigenvalue of $A_{N.}$

Proof. Suppose that M a lightlike hypersurface of semi-Euclidean space $\mathbb{R}^{n+2}_\nu$. Then the Gauss equation is

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N$$

for any $X, Y \in \Gamma(TM)$. Using the Gauss equation for $\xi \in \Gamma(TM)$, we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + B(X, \xi)N.$$

Thus, scalar product of both side, we obtain

$$B(X, \xi) = \widetilde{g}_{SE}(\widetilde{\nabla}_X \xi, \xi).$$

On the other hand, as $\widetilde{g}_{SE}(\xi, \xi) = 0$, $B(X, \xi) = 0$ is obtained. Considering (4.10), we get

$$\widetilde{g}_{SE}(A_{N_0}\xi, X) = 0$$

for any $X \in \Gamma(TM)$. From (4.11), we see that A $_N\xi = 0$. This shows that vector field ξ is a eigenvector corresponding a zero eigenvalue of $A_{N.}$ □

Conclusion 4.1. *Let $\{\xi, X_1, \ldots, X_n\}$ be eigenvectors corresponding to different eigenvalues of A_N shape operator. Then the matris of A_N with respect to eigenvalues $\{0, k_1, \ldots, k_n\}$ is given by*

$$A_N = \alpha(x) \begin{pmatrix} k_1 & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & k_n & \\ 0 & & & & & 0 \end{pmatrix}.$$

Definition 4.1. Let M be a lightlike hypersurface of sem-Euclidean space $\mathbb{R}_\nu^{n+2.}$ When the normal curve of M in $\mathbb{R}_0^{n+2}$ direction of X_p is K_p,

$$k_p = \alpha(x)\mathcal{K}_p$$

is called normal curvature of M in $\mathbb{R}_\nu^{n+2}$ according to lightlike transversal vector field.

Proposition 4.2. *Let M be a lightlike hypersurface of semi-Euclidean space. Where shape operator of A_{N_0} of M in $\mathbb{R}^{n+2}$ and set of eigenvectors that is equal to different eigenvalues $\{\xi, X_1, \ldots, X_n\}$ if the Euclidean angle of $X_p \in \Gamma(TM)$ with $X_1, \ldots, X_n$ are in turn $\theta_1, \ldots, \theta_n$,is obtained as*

$$k_p = \alpha(x) \sum_{i=1}^{n} \mathcal{K}_i \cos^2 \theta_i \,.$$

Proof. Let M be a lightlike hypersurface and k_p be normal curvatur for lightlike tranversal vector field N. Then, when $\mathcal{K}_p$ is normal curvature of M, it is known that $\mathcal{K}_p$ is

$$k_p = \sum_{i=1}^{n} \mathcal{K}_i \cos^2 \theta_i \,.$$

Thus, from (4.9) we obtain

$$k_p = \alpha(x) \sum_{i=1}^{n} \mathcal{K}_i \cos^2 \theta_i \,. \qquad \square$$

Proposition 4.3. *Let p be a point on the curve β. Assume that the curve of intersection of M and a plane passing through the tanget line of β at p, making the angle $\theta_i (0 < \theta_i < \frac{\pi}{2})$ with the unit normal of M at p is β_i. The k normal curve of M at the point $p \in \beta \cap \beta_i$ for the values K and K_{0_i} in the direction of β and β_i normal curves for M, with respect to N transversal vector field is*

$$k = \alpha^{-1}(x) \mathcal{K}_{0_i} \cos \theta_i \,.$$

Proof. Let M be a lightlike hypersurface of semi-Euclidean space $\mathbb{R}^{n+2}_{\nu}$ where M is the hypersurface of $\mathbb{R}^{n+2}_0$ if $\mathcal{K}$ normal curve of β curve at the direction of X_p is taken, it is clearly as follow $k = \mathcal{K}_{0_i} \cos \theta_i$. From (4.10), the curve according to N lightlike transversal vector field is

$$k = \alpha^{-1}(x) \mathcal{K}_{0_i} \cos \theta_i \,. \qquad \square$$

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