ORIGINAL PAPER

ANALYTICAL SOLUTIONS OF TIME FRACTIONAL NEGATIVE ORDER KdV-CALOGERO-BOGOYAVLENSKII-SCHIFF EQUATION

ALI KURT[1], ABDULSAMET BEKTAS[1]

Manuscript received: 27.03.2023; Accepted paper: 25.07.2023;
Published online: 30.09.2023.

Abstract. *In this work, we obtain the exact solutions of negative-order forms for KDV-Calogero-Bogoyavlenskii-Schiff equation (nKdV-nCBS) with conformable fractional derivative by using* $\exp(-\phi(\xi))$ *method. Chain rule and wave transform are used for transforming nonlinear fractional partial differential equation into nonlinear integer order ordinary differential equation. By this way, we don't need any normalization or reduction formulas which have complex calculation procedures. Moroever 3D graphical simulations are presented to show the geometrical behaviour of derived solutions.*

Keywords: $\exp(-\phi(\xi))$ *method; conformable fractional derivative; analytical solutions.*

1. INTRODUCTION

The story of the fractional order differential equations began on 30 September 1695, when L'Hospital wrote to Leibnitz questioning whether derivatives of the ½ nd order could be taken. To this question, Leibnitz replied, "Although this may seem like a paradox at the moment, it will one day have very useful consequences."

Mathematical modeling is needed to investigate the solutions of problems encountered in fields such as physics, chemistry, biology, and engineering and to explain the behavior of these solutions. In many mathematical models integer order derivatives and integrals have been used. As a result of the studies carried out, it has been understood that it is more advantageous to use fractional derivatives and integrals instead of full order derivatives or integrals in these models. These advantages can be explained as follows:

1. Fractional calculus can easily express the historical dependence of the evolution of system analysis by taking the global correlation into consideration, but integer calculus is not convenient to represent this process for its locality characteristics;
2. Fractional calculus overcomes the critical defect of integer calculus that the theoretical model results often fail to coincide with the experimental results. On the contrary, it could get a good coincidence by using a few parameters;
3. When describing complicated physical mechanics problems, fractional calculus has a clearer physical significance and a simpler expression compared with the nonlinear model.

Today, models obtained by using fractional order differentials in science, engineering and many other fields are attracting a lot of attention, and researches and studies on this subject are increasing. These studies are carried out in many different areas. Some of them are Chai and Chan [1] and Oldham [2] in the field of electrochemistry, Engheta [3] in the

[1] Pamukkale University, Faculty of Science, Department of Mathematics, 20160, Denizli, Turkey.
E-mail: akurt@pau.edu.tr; s.bektas33@hotmail.com.

electromagnetic field, Adolfsson and Olsson [4] and Alotta et al. [5], in the field of viscoelasticity, Magin [6] in bioengineering, in the field of circuit analysis Gomez-Aguilar et al. [7], Shiri and Baleanu [8] in the medical field.

There are many definitions of fractional derivatives in the literature. The most well-known and popular of these are the Riemann-Liouville and Caputo fractional derivatives. But these fractional derivatives have some drawbacks. For instance

- In the Riemann–Liouville fractional derivative approximation, the derivative of the constant is not zero. Let c be a constant Riemann-Liouville derivative do not satisfy

$$D_a^{\alpha}\left(c\right)=0$$

- The Riemann–Liouville and Caputo fractional derivative approximations do not provide the formula known as the derivative of the multiplication of two functions.

$$D_a^{\alpha}\left(f(t)g(t)\right)=f(t)D_a^{\alpha}\left(g(t)\right)+g(t)D_a^{\alpha}\left(f(t)\right)$$

- The Riemann–Liouville and Caputo fractional derivative approximations do not provide the formula known as the derivative of the quotient of two functions.

$$D_a^{\alpha}\left(\frac{f(t)}{g(t)}\right)=\frac{f(t)D_a^{\alpha}(g(t))-g(t)D_a^{\alpha}(f(t))}{[g(t)]^2}$$

- The Riemann-Liouville and Caputo fractional derivatives approximations do not satisfy the chain rule.

$$D_a^{\alpha}\left((fog)(t)\right)=f^{\alpha}\left(g\left(t\right)\right)g^{\alpha}\left(t\right)$$

- The Riemann–Liouville and Caputo fractional derivatives approaches do not satisfy the following equality.

$$D^{\alpha}D^{\beta}f(t)=D^{\alpha+\beta}f(t)$$

- The Caputo fractional derivative approach assumes that the function f is differentiable.

The Riemann–Liouville and Caputo fractional derivatives approaches have disadvantages because they do not provide the general properties of the derivative in classical calculus. Let's explain the derivative approach, which overcomes these disadvantages and provides the expressed properties, called the conformable derivative approach, with the definitions made below.

Definition 1. [9-12] Let $f:\left[0,\infty\right)\to\mathbb{R}$, $t>0$ and $\alpha\in\left(0,1\right)$. $\alpha-th$ order conformable fractional derivative of function f is given as

$$T_{\alpha}\left(f(t)\right)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f\left(t\right)}{\varepsilon}.$$

Let f be $\alpha-th$ order conformable differentiable function on the interval $(0,a)$ where $a>0$ and if the following limit exists

$$\lim_{t\to 0^+} f^{(\alpha)}(t)$$

then

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t).$$

Following theorem gives some basic properties of conformable fractional derivative.

Theorem 2. Let $\alpha\in(0,1]$ and the functions f and g be conformable differentiable. So the following properties are satisfied.

1. $T_\alpha\left(a\,f(t)+b\,g(t)\right)=aT_\alpha\left(f(t)\right)+bT_\alpha\left(g(t)\right),\ \forall a,b\in\mathbb{R}$.
2. $T_\alpha(t^p)=pt^{p-a},\ \forall p\in\mathbb{R}$.
3. Let $f(t)=c$ be a constant function then, $T_\alpha(c)=0$.
4. $T_\alpha\left(f(t)g(t)\right)=f(t)T_\alpha\left(g(t)\right)+g(t)T_\alpha\left(f(t)\right)$.
5. $T_\alpha\left(\dfrac{f(t)}{g(t)}\right)=\dfrac{g(t)T_\alpha\left(f(t)\right)-f(t)T_\alpha\left(g(t)\right)}{\left[g(t)\right]^2}$.
6. $T_\alpha\left(f(t)\right)=t^{1-\alpha}\dfrac{df(t)}{dt}$.

As can be seen from the above theorems, the conformable fractional derivative definition provides many of the properties provided by the integer order derivative used in classical calculus. These advantages of the conformable fractional derivative attracted a lot of attention and used in many scientific studies. In his published study, Abdeljawad [10] expressed the new properties of the conformable fractional derivative, as well as conformable fractional versions of basic concepts such as chain rule, exponential function, partial integration, Taylor series expansion, Laplace expansion. In another study published in 2015, Atangana et al. [11] expressed the concept of conformable partial derivative, conformable divergens theorem, conformable Green's theorem and gave some new properties of conformable fractional derivative. Zhao and Luo [13] defined the generalized concept of conformable fractional derivative and gave a physical explanation of this definition. Khalil et al. [14] provided a geometric interpretation of the conformable fractional derivative. Since the geometric interpretation of other fractional derivative definitions does not yet exist in the literature, it is a first in terms of making a geometric interpretation of a fractional derivative.

In this study $\exp(-\phi(\xi))$ method is used to get the exact solutions of negative-order KdV-CBS equation (nKdV-nCBS). First of all we will give a brief description of the considered method.

2. $\exp(-\phi(\xi))$ METHOD

Assume the nonlinear conformable fractional partial differential equations as

$$P(u, D_t^a u, D_x u, D_x D_t^a u, D_t^{(2a)} u, D_{xx} u, ...) = 0 \tag{1}$$

where $u = u(x,t)$ and $D_t^{(2a)}$ means applying the conformable fractional operator to function u two times consequently. The method can be given step by step as follows.

Step 1. Applying equation (1) the wave transforms

$$u(x,t) = u(\xi),\ \xi = cx - k\frac{t^\alpha}{\alpha} \tag{2}$$

where c is wave amplitude and k is the wave velocity and chain rule Eq. (1) turns into

$$Q(u, u', u'', ...) = 0 \tag{3}$$

nonlinear integer order ordinary differential equation.

Step 2. Assume the solutions of Eq. (3) will be given as follows.

$$u(\xi) = \sum_{n=0}^{m} a_n (\exp(-\phi(\xi))^n \tag{4}$$

that the function $\phi(\xi)$ is the solution of following differential equation.

$$\phi'(\xi) = \exp(-\phi(\xi)) + \mu \exp(\phi(\xi)) + \lambda \tag{5}$$

The solutions of the Eq. (5) will be given as

Solution Set 1. $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$

$$\phi_1(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c)\right) - \lambda}{2\mu}\right)$$

Solution Set 2. $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$

$$\phi_2(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c)\right) - \lambda}{2\mu}\right)$$

Solution Set 3. $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$

$$\phi_3(\xi) = -\ln\left(\frac{\lambda}{\cosh(\lambda(\xi+c)) + \sinh(\lambda(\xi+c)) - 1}\right)$$

Solution Set 4. $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda \neq 0$

$$\phi_4(\xi) = \ln\left(-\frac{2(\lambda(\xi+c)+2)}{\lambda^2(\xi+c)}\right)$$

Solution Set 5. $\lambda^2 - 4\mu = 0$, $\mu \neq 0$ and $\lambda \neq 0$

$$\phi_5(\xi) = \ln(\xi + c)$$

c is the arbitrary integration constant. In addition, the number n in Eq. (4) is found using the principle of homogeneous equilibrium between the highest-order nonlinear term in Eq. (3) and the highest-order linear term.

Step 3. Substituting Eq. (4) into Eq. (3) and gathering together all the same powers of $\exp(-\phi(\xi))$ together, we obtain a polynomial with respect to $\exp(-\phi(\xi))$. Equating all the coefficients of this polynomial to zero we obtain an algebraic equation system due to λ, k, c, μ, k. By solving this system, the solution of the Eq. (1) will be obtained.

3. EXACT SOLUTION OF (NKDV-NCBS) EQUATION

Consider the nKdV-nCBS equation as

$$D_t^\alpha D_x u + \theta D_x^2 u + \kappa D_x D_y u + D_x^3 D_y u + 2D_y u D_x^2 u + 4D_x u D_x D_y u + \psi D_x D_z u = 0 \tag{6}$$

Applying the wave transform $\xi = x + y + z + \dfrac{dt^\alpha}{\alpha}$ and the chain rule and integrating once, we get the following nonlinear integer order ordinary differential equation.

$$u''' + u'(d + \theta + \kappa + \psi) + 3(u')^2 = 0 \tag{7}$$

Using the homogenous balance method in the Eq. (7), solution of Eq. (7) will be assumed as

$$u(\xi) = a_0 + a_1 \exp(-\phi(\xi)). \tag{8}$$

If Eq. (8) and Eq. (5) are substituted in Eq. (6), the resulting equation is arranged according to the powers of the function $\exp(-\phi(\xi))$, and if each coefficient is equated to zero, the following system of equations is obtained.

$$3a_1^2\mu^2 - a_1 d\mu - a_1\theta\mu - a_1\kappa\mu - a_1\lambda^2\mu - 2a_1\mu^2 - a_1\mu\psi = 0,$$
$$6a_1^2\lambda - 12a_1\lambda = 0,$$

$$3a_1^2 - 6a_1 = 0,$$
$$3a_1^2\lambda^2 + 6a_1^2\mu - a_1 d - a_1\theta - a_1\kappa - 7a_1\lambda^2 - 8a_1\mu - a_1\psi = 0,$$
$$6a_1^2\lambda\mu - a_1 d\lambda - a_1\theta\lambda - a_1\kappa\lambda - a_1\lambda^3 - 8a_1\lambda\mu - a_1\lambda\psi = 0.$$

Solving the system by computer software MATHEMATICA led to

$$\kappa = -d - \theta - \lambda^2 + 4\mu - \psi, a_1 = 2.$$

By using this solution set, Eq. (8), solution sets of Eq. (5) and the wave transform $\xi = x + y + z + \frac{dt^\alpha}{\alpha}$ the analytical solutions of Eq. (6) can be obtained as follows.

- For $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$

$$u_1(x,y,z,t) = a_0 + \frac{4\mu}{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(C + \frac{dt^\alpha}{\alpha} + x + y + z\right)\right) - \lambda}.$$

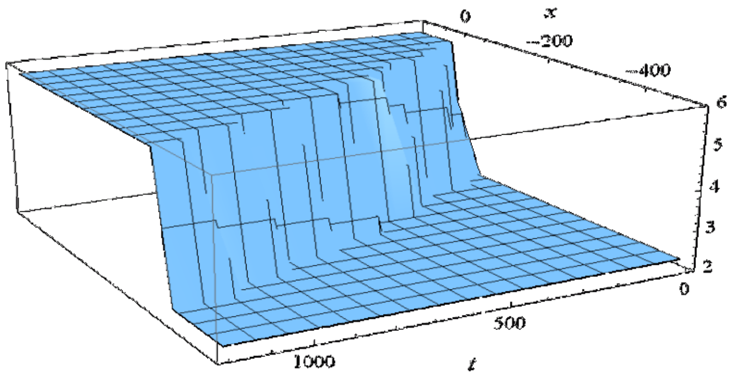

Figure 1. 3D graphical representation of $u_1(x, y, z, t)$ for $\alpha = 0.8, a_0 = 1, d = 1, y = 1, z = 1, \lambda = -3, \mu = 1.5, c = 1$.

- For $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$

$$u_2(x,y,z,t) = a_0 + \frac{4\mu}{\sqrt{4\mu - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(C + \frac{dt^\alpha}{\alpha} + x + y + z\right)\right) - \lambda}.$$

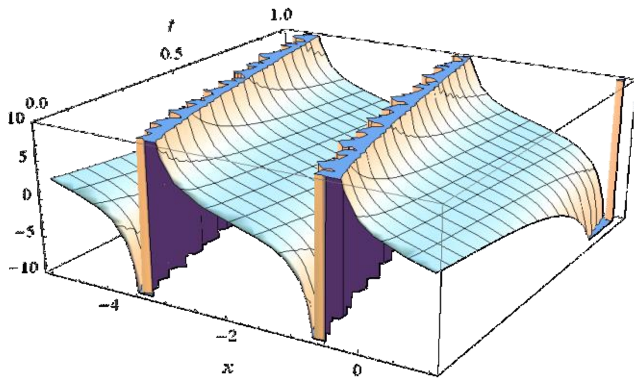

Figure 2. 3D graphical representation of $u_2\,(x, y, z, t)$ for $\alpha = 0.8$, $a_0 = 1$, $d = 1$, $y = 1$, $z = 1$, $\lambda = -1$, $\mu = 1.5$, $c = 1$.

For $\lambda^2 - 4\mu > 0$, $\lambda \neq 0$ and $\mu = 0$

$$u_3(x,y,z,t) = a_0 + \frac{2\lambda}{\sinh\left(\lambda\left(C + \dfrac{dt^\alpha}{\alpha} + x + y + z\right)\right) + \cosh\left(\lambda\left(C + \dfrac{dt^\alpha}{\alpha} + x + y + z\right)\right) - 1}.$$

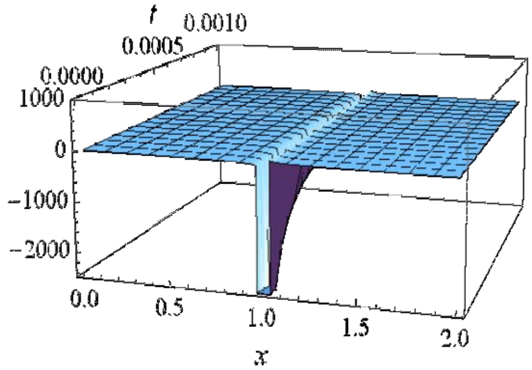

Figure 3. 3D graphical representation of $u_3\,(x, y, z, t)$ for $\alpha = 0.8$, $a_0 = 1$, $d = -1$, $y = -1$, $z = -1$, $\lambda = -1$, $\mu = 1.5$, $c = 1$.

4. CONCLUSION

In this article, fractional (3+1) dimensional nKdV-nCBS equation is considered. The exact solutions of this equation are obtained by using $\exp(-\phi(\xi))$ method. The solutions show that applied method is effective and applicable for the fractional partial differential equations involving conformable type derivative. Also, 3D graphical illustrations are given to show the physical behaviour of the obtained solutions.

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