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Two-parameter deformed quantum mechanics based on Fibonacci calculus and Debye crystal model of two-parameter deformed quantum statistics

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Abstract Starting on the basis of Fibonacci calculus and Fibonacci oscillator algebra, we introduce the main properties to develop a new formalism for the two-parameter (q_1, q_2)-deformed quantum mechanics, where q_1 and q_2 are real positive independent deformation parameters. As applications of such a two-parameter deformed formalism, we investigate the behavior of a quantum particle in some different physical phenomena covering the free particle and the inverse-harmonic potential case. The effect of two deformation parameters on the wave functions for these applications is studied. Another application is carried out onto the quantum statistics of lattice oscillations through a model of the (q_1, q_2)-deformed phonon gas, and it is shown that the high- and low-temperature behavior of the model specific heat differs notably from the classical theories for the interval $0 < (q_1, q_2) < \infty$. We also construct a two-parameter deformed non-extensive entropy based on some elements of the Fibonacci calculus and discuss its possible connection with the Tsallis entropy in non-extensive statistical mechanics. Finally, other possible application areas of the present two-parameter (q_1, q_2)-deformed construction on quantum mechanics are discussed.

1 Introduction

As is known that physics of the microscopic world is governed by the quantum mechanics. Nevertheless, quantum systems in nature have inherently complex, many-body interactions in that understanding the physical mechanism behind a nonlinear behavior of them often requires novel theories (or models) beyond the formalism of standard quantum mechanics. In addition, such theories should properly describe some modified (or deformed) versions of the well-known quantum mechanical postulates so that under some limiting case(s), they should reduce to the standard ones. In microscopic level, such models can have their own deformed many-particle systems such that they formed by indistinguishable particles satisfying some special deformed commutation or deformed anti-commutation relations. These relations accordingly form representations of deformed particle algebras with one (or more) deformation parameters. Most notably, one of the most effective approaches to deal with some nonlinearity (or non-ideality) in a complex system, where the standard theories do not work, is to use representations of quantum groups [1–4] and their associated quantum algebras [5–7]. Such an approach can entail studies on oscillator constructions of deformed particle algebras, whose properties can be linked to a quantum group representation structure via the use of Jordan–Schwinger mapping [8–14]. However, these q-deformed objects should not be confused with anyons [15, 16], since the q-deformed bosonic and fermionic oscillators, where q is a real deformation parameter, can exist in any dimensions while anyons can only live in strictly two-dimensional space. In addition, they are originally different from the quon operators [17], which exhibits another example of fractional statistics particles [18, 19].

Besides, it was argued in the literature that the one- and two-parameter deformed bosonic and fermionic oscillator algebras can be used to describe continuous interpolation between the Bose–Einstein (BE) and Fermi–Dirac (FD) statistics [20–23]. These investigations also led to a suitable framework for discussing an intermediate-statistics behavior in a system such as in [24, 25].

Furthermore, after different constructions of deformed oscillator systems were carried out, possible consequences of the use of their underlying quantum algebras (either bosonic or fermionic) in several application areas of research have been extensively discussed in the literature. Some of them can be summarized as in probing higher-order effects in the many-body interactions in nuclei [14, 26], addressing the quantum statistics of charged, extremal black holes via the one-parameter deformed statistics [27, 28], analyzing a generalized uncertainty relation which was thought that it results from the quantum gravity effect [29], constructing a consistent formulation of the q-deformed version of the Bardeen–Cooper–Schrieffer (BCS) theory for an interacting fermionic

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system such as a superconductor [30] and discussing possible role of the deformation parameter q on the thermostatistical properties of a q-deformed relativistic ideal Fermi gas [31].

On the other hand, another interesting application was observed in the field of condensed matter physics such as in viewing quantum groups as hidden symmetries of quantum impurities [32], where it was asserted that the deformation parameter q contains all the information about many-particle interactions in the system. Also, an analytical approach to the Bose polaron problem was carried out by using a q-deformed Lie algebra [33], whose oscillator construction was made by the Arik–Coon (AC)-type q-bosonic oscillator algebra in Eqs. (1) and (2). Recently, thermal and electrical properties of a solid have been tackled through the properties of bosonic Fibonacci oscillator algebra [34] and it was shown that the change in thermal conductivity of a given material can be described by two deformation parameters (q_1, q_2) of the Fibonacci oscillator algebra as one inserts impurities [35]. They also discussed the possibility of adjusting the Fibonacci oscillators acting as defects or impurities in a crystal lattice [34]. In addition, thermoelectric properties of BiSbTe alloy nanofilms produced by DC sputtering [36] have been examined by contrasting with the results of a deformed model based on the Biedenharn–Macfarlane (BM)-type q-bosonic oscillator algebra in Eqs. (4) and (5). In the work of [36], a possible relation between the deformation parameter q and the annealing temperature was also discussed.

Besides, the high-temperature thermostatistical characteristics of a gas of the three-parameter (q_1, q_2, μ)-deformed oscillators have been discussed in [37], where the hybrid deformed statistics was also introduced and the three-parameter (q_1, q_2, μ)-deformed oscillator algebra is formed by an association of the two-parameter (q_1, q_2)-deformed algebra of Fibonacci oscillators with the μ-deformed oscillator algebra [38–42].

Moreover, it is known that the thermodynamic curvature serves as a measure of statistical interaction and a sign for the phase transition in a system under consideration, the thermodynamic geometry of q-deformed bosons and fermions has been analyzed in [43], where the BM-type q-oscillator algebra given in Eqs. (4) and (5) was considered by assigning q-bosons with $k = 1$ and q-fermions with $k = -1$. Thereafter, this approach was extended to a two-parameter (p, q)-deformed system via the use of one-dimensional Chakrabarti–Jagannathan (CJ)-type qp-deformed quantum algebra [44], where condensation characteristics of a qp-deformed statistic gas were tackled with the behavior of thermodynamic curvature of the system.

Recently, thermodynamical properties of the μ-deformed analog of the Bose gas model based on the μ-deformed oscillator algebra have been proposed for effective modeling of main physical properties of dark matter constituents in [45], where the related μ-calculus was also used for the analysis of thermodynamic geometry of the μ-Bose gas. In the meantime, the q-deformed statistics based on the BM-type q-oscillator algebra has been used for discussing the properties of dark matter in [46], where the q-deformed condensate and some of its thermodynamical virtues were examined in terms of different regimes for values of the deformation parameter q.

In [47], it was proposed that the q-deformed bosonic oscillators can be used to treat many-body problems of composite particles. As a two-parameter extension of this study, the $(\widetilde{\mu}, q)$-deformed Bose gas model based on both the Jannussis μ-oscillator [38] and the AC-type q-oscillator [5] algebras has recently been constructed and it was used to effectively account for both compositeness of particles and their interactions in a system under consideration [48–50]. In [39, 40, 49], it was also shown that the two-parameter $(\widetilde{\mu}, q)$-deformed Bose gas model is very efficient and effective to describe the observed non-Bose like behavior of two-pion correlation intercepts observed in the experiment on relativistic heavy-ion collisions. In [51], a special correspondence between the second virial coefficients in the equations of state of the ideal q- and μ-deformed Bose gases has been discussed.

As is proved in the works of [41, 52], composite bosons (or quasi-bosons) can be realizable by deformed oscillators, and therefore, it is also natural to address their possible consequences in the research field of quantum information theory. In this context, among many others, some of the applications in this direction can be mentioned as follows: It was shown in [53] that a direct connection between the degree of the entanglement within a composite boson and the deformation parameter q of the AC-type q-boson algebra has been established. In another study, a spin-deformed boson model was used as a mean for controlling the decoherence process in quantum computation [54], where the Hamiltonian of the model was constructed by the generators of a deformed oscillator algebra.

On the other hand, the thermal radiation laws of a one-parameter q-deformed boson system have been examined in the works of [55, 56]. In these studies, the authors have put forward an interesting hypothesis such that the cosmic microwave background radiation (CMBR) can be regarded as the radiation of q-deformed bosonic oscillators and in this way, the deviations from the Planck radiation law in the CMBR may be explained to some extent. These studies have also been asserted that the CMBR should be composed by the radiation of dressed photons, which may accordingly be formed by the interactions among bare photons [55, 56].

With the above considerations in mind, one can observe that the following one-parameter q-deformed oscillator algebras have been mostly studied in the literature: The AC-type [5] q-deformed bosonic oscillator algebra was introduced by

$$aa^* - qa^*a = 1, \tag{1}$$

where the relation between the number operator and step operators is $a^*a = [\widehat{N}]_q$, whose spectrum is given by the following q-basic number definition:

$$[n] \equiv [n]_q = \frac{q^n - 1}{q - 1}. \tag{2}$$

The basic number definitions of deformed oscillator algebras are intrinsically connected with the Jackson's q-calculus (also called quantum calculus) via the Jackson derivative (JD) operator [57, 58]. For instance, the q-basic number in Eq. (2) is intimately connected to the JD operator defined as

$$\hat{\partial}_x^{(q)} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{3}$$

which reduces to the ordinary derivative in the limit $q \to 1$. Similar connection was also constructed by the BM-type [6, 7] q-oscillator algebra defined as

$$aa^* - qa^*a = q^{-\hat{N}}, \tag{4}$$

whose number operator spectrum was defined by the following $q \leftrightarrow q^{-1}$ symmetric basic number:

$$[n] \equiv [n]_{q\leftrightarrow q^{-1}} = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{5}$$

Equations (4) and (5) imply the JD operator for an arbitrary real function $f(x)$ as

$$\hat{\partial}_x^{(q)} f(x) = \frac{f(qx) - f\left(q^{-1}x\right)}{\left(q - q^{-1}\right)x}. \tag{6}$$

We should emphasize that the JD operator plays a crucial role in the framework of generalized statistical mechanics such that the entire thermostatistics of the one-parameter q-deformed bosons and fermions can be established on the formalism of Jackson's q-calculus [23, 59, 60]. But the situation is different for another generalized formalism called the non-extensive thermostatistics, where the Tsallis entropy function has a central role [61, 62] and the Tsallis $\widetilde{q}$-exponential function was defined [62–64] as

$$e_{\tilde{q}}(x) = \left[1 + (1 - \tilde{q})x\right]^{\frac{1}{1-\tilde{q}}}, \tag{7}$$

where $\widetilde{q}$ is the entropic index in the Tsallis formalism and $(x, \widetilde{q}) \in R$, $\widetilde{q} \neq 1$. Note that its related $\widetilde{q}$-derivative was also defined [65–67] as

$$\hat{\partial}_x^{(\tilde{q})} f(x) = (1 + (1 - \tilde{q})x)\frac{df}{dx}. \tag{8}$$

The main difference between the q-deformed quantum group field algebraic structures and the non-extensive $\widetilde{q}$-statistics comes originally from the fact that the Tsallis entropy is essentially based on a deformation of the logarithmic function of the Boltzmann-Gibbs entropy function $S = k_B \log W$ with the usual statistical weight W, whereas as we shall show below, the two-parameter (q_1, q_2)-deformed entropy function $S^{(q_1,q_2)}$ will be based on a two-parameter modification of the statistical weight $W \equiv W^{(q_1,q_2)}$. In this context, it would be important to recall that possible connections between the one-parameter deformed quantum group field algebras and the non-extensive Tsallis thermo statistics have been investigated by several authors [68–72]. At this point, we should also mention that one other formalism called the κ-deformed statistical mechanics has been proposed by the works of [73–75], where the κ-deformed exponential function was given by

$$e_\kappa(x) = \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{\frac{1}{\kappa}}, \tag{9}$$

where $0 < \kappa < 1$. This κ-deformed exponential induced the κ-deformed derivative as

$$\hat{\partial}_x^{(\kappa)} f(x) = \sqrt{1 + \kappa^2 x^2}\frac{\mathrm{d}f}{\mathrm{d}x}. \tag{10}$$

Furthermore, to the best of our knowledge from the literature that the four different approaches have been intensively addressed to construct some version of a deformed quantum mechanical formalism. The first one was proposed by studies on quantum mechanics in the q-deformed calculus [76–80] based on the JD operator in Eqs. (3) or (6), and the second one was introduced by studies on the $\widetilde{q}$-deformed quantum mechanics [81–83] based on the $\widetilde{q}$-deformed derivative emerging in the non-extensive Tsallis statistics as given in Eq. (8). Besides, after realizing the first deformation on the Heisenberg algebra by Wigner [84] and thereafter Yang [85] as in the form

$$[\widehat{x}, \widehat{p}] = i\hbar\left(1 + 2\nu\widehat{\mathcal{P}}\right), \tag{11}$$

where $\widehat{\mathcal{P}}$ is the reflection operator obeying $\widehat{\mathcal{P}} f(x) = f(-x)$ and ν is a real parameter, the third method of introducing a deformed quantum mechanical formalism was appeared by employing the Dunkl derivative [86, 87]. It was defined [86] as

$$\widehat{\partial}_x^{(\mathrm{Dunkl})} = \widehat{\partial}_x + \frac{\nu}{x}\left(1 - \widehat{\mathcal{P}}\right), \tag{12}$$

which has been used to study the properties of Wigner–Dunkl quantum mechanics in one dimension along with its implications on the spin-less particle confined to the one-dimensional infinite potential and the harmonic oscillator problem [88].

On the other hand, the fourth method of introducing a deformed quantum mechanical formalism is to consider the κ-deformed statistics [73, 74, 89]. This led to studies on the construction of deformed quantum theory with κ-translation symmetry [75], where the κ-deformed derivative from the pseudo derivative was proposed. Note also that there has recently been appeared a new map leading to a different deformed statistical theory in [90], where the α-deformed entropy based on the α-deformed addition was proposed. After the introduction of the α-deformed thermodynamics by [90], the α-boson gas condensation and its thermostatistical aspects were also discussed in [91]. Here, it was interestingly shown that the critical temperature at which the α-boson condensation would occur depends on the shape of the container [91]. But, as far as we know from the literature that the deformed quantum theory with such an α-additive property has not yet been studied. In addition, it is worth recalling that based on the Dunkl derivative in Eq. (12), the ideal Bose gas and the blackbody radiation in the Dunkl formalism have been discussed in [92]. It was then shown that the Dunkl formalism modifies total radiated energy in the Dunkl-ideal Bose gas [92].

From the above analysis, we should stress that all the above methods reviewed have been accompanied by discussions on different constructions of the one-parameter deformed Heisenberg algebra as well as the one-parameter deformed time-dependent Schrödinger equation. Here, we should remark that both the feasibility and the effectiveness of these one-parameter deformed formalisms to describe a nonlinearity in a real phenomenon depend crucially on the physical problem under investigation, in which the closest one to the experimental data may be chosen to model such a nonlinearity.

Although some versions of the one-parameter deformed quantum mechanics have been constructed by several researchers as mentioned above, a complete and fully consistent deformed theory beyond the standard quantum mechanical formalism is still under active consideration. Nevertheless, an attempt to address such an aim is a remarkable issue, since we believe that collective excitations in many-body interacting quantum systems can be approximated by some effective models through deformed quantum mechanical theories. Here, it is relevant to emphasize that most of the studies in the literature as reviewed above were carried out using the one-parameter q-deformed derivative operator, which led to have a framework for constructing the one-parameter deformed quantum mechanics. But, the two-parameter ones in the same lines of research have relatively been less studied and less applied to different physical applications. What is more interesting here that as far as we know from the literature, the two-parameter (q_1, q_2)-deformed Fibonacci oscillator algebra along with its related Fibonacci difference operator has not yet been used for establishing the properties of a two-parameter deformed quantum mechanics.

Motivated by the above points, in this work, our first aim is to fill this gap in the literature, and our second aim is to develop a formalism for the two-parameter (q_1, q_2)-deformed quantum mechanics. Our third objective is to apply such a two-parameter deformed formalism into some physical phenomena covering the free particle and the inverse-harmonic potential cases. For each application, the corresponding wave functions will be derived via the two-parameter (q_1, q_2)-deformed Schrödinger equation, and the effect of two deformation parameters on the wave functions will be analyzed. Moreover, another application onto lattice oscillations through a model of the (q_1, q_2)-deformed phonon gas will be worked out and its quantum statistics will be discussed in detail. Our last objective in this work is to study further the two-parameter deformed non-extensive entropy function $S^{(q_1,q_2)}$, and is to discuss its possible connection with the Tsallis entropy function $S^{(\widetilde{q})}$ in the framework of non-extensive statistical mechanics.

The plan of this study is organized as follows: In Sect. 2, we present the bosonic and fermionic Fibonacci oscillator algebras together with their algebraic and representative properties. In Sect. 3, we introduce various elements of the Fibonacci calculus, which allow us to have a suitable framework to develop a two-parameter deformed formalism. In Sect. 4, we construct the two-parameter (q_1, q_2)-deformed quantum mechanics in the one-dimensional space and formulate its deformed postulates. In Sect. 5, we apply our two-parameter deformed formalism into some physical phenomena covering the free particle and the inverse-harmonic potential cases. This will be followed by a derivation of the energy eigenfunctions in terms of some functions of the parameters q_1 and q_2. Another application will be studied by a Debye solid covering (q_1, q_2)-deformed phonons and its deformed quantum statistics will be discussed in detail, where the focus will be on the behavior of the specific heat of such a two-parameter deformed phonon gas in the thermodynamical limit. In Sect. 6, we introduce a two-parameter deformed non-extensive entropy obtained by some elements of the Fibonacci calculus, and discuss its possible connection with the Tsallis entropy. Section 7 is reserved to give a concise discussion and concluding remarks about our results along with other potential application areas of the present construction on two-parameter deformed quantum mechanical formalism.

2 The bosonic and fermionic Fibonacci oscillator algebras

In this section, we shall present the main algebraic and representative properties of bosonic and fermionic Fibonacci oscillator algebras. First, it is worth recalling that the multi-dimensional two-parameter deformed bosonic Fibonacci oscillators are of two kinds [93]: commuting and covariant. The two types are related by a transformation and the diagonal commutation relations for these two kinds of oscillators are the same. The algebra generated by the commuting Fibonacci oscillators a_i and their corresponding creation operators a_i^* is defined by the following relations [93]:

$$\left[a_i, a_j^*\right] = 0, \ \ i \neq j,$$

$$\left[a_i, a_j\right] = 0, \ \ i, j = 1, 2, \cdots, d,$$

$$a_i a_i^* - q_1^2 a_i^* a_i = q_2^{2\hat{N}_i},$$
$$a_i a_i^* - q_2^2 a_i^* a_i = q_1^{2\hat{N}_i}, \tag{13}$$
$$a_i^* a_i = \left[\hat{N}_i\right],\ a_i a_i^* = \left[\hat{N}_i + 1\right],$$

where the spectrum of the deformed boson number operators $\left[\widehat{N}_i\right]$ is given by the generalized Fibonacci basic integers as

$$[n] \equiv [n]_{q_1,q_2} = \frac{q_1^{2n} - q_2^{2n}}{q_1^2 - q_2^2}, \tag{14}$$

where $q_1 \neq q_2$, $(q_1, q_2) \in \mathbb{R}^+$. On the other hand, the covariant Fibonacci oscillator algebra is defined by the following relations [93]:

$$b_i b_j = q_1 q_2^{-1} b_j b_i,\ i < j,$$
$$b_i b_j^* = q_1 q_2 b_j^* b_i,\ i \neq j,$$
$$b_1 b_1^* - q_1^2 b_1^* b_1 = q_2^{2\hat{N}}, \tag{15}$$
$$b_j b_j^* - q_1^2 b_j^* b_j = b_{j-1} b_{j-1}^* - q_2^2 b_{j-1}^* b_{j-1},\ \ j = 2, \cdots, d,$$
$$q_1^{2\hat{N}} = b_{\mathrm{d}} b_{\mathrm{d}}^* - q_2^2 b_{\mathrm{d}}^* b_{\mathrm{d}},$$

where the total deformed boson number operator for this system is

$$\sum\nolimits_{i=1}^{d} b_i^* b_i = \left[\hat{N}_1 + \hat{N}_2 + \cdots + \hat{N}_{\mathrm{D}}\right] = \left[\hat{N}\right], \tag{16}$$

whose spectrum is given by Eq. (14). The generalized Fibonacci basic integer $[n]_{q_1,q_2}$ in Eq. (14) has some important limiting cases as follows: When we take the limit $q_1 = \sqrt{q}$ and $q_2 = 1$, we obtain the AC-type bosonic q-oscillators' basic number as in Eq. (2). In addition, the limit $q_1 = \sqrt{q}$ and $q_2 = (1/\sqrt{q})$ gives the BM-type bosonic q-oscillators' basic number as in Eq. (5). Considering the limit $q_1 = q_2 = \sqrt{q}$ in Eqs. (13) and (14) reveals the multi-dimensional q-deformed bosonic Newton oscillator algebra [94] with $SU(d)$-symmetry defined as

$$a_i a_j^* - q a_j^* a_i = q^{\hat{N}} \delta_{ij}, \quad i, j = 1, 2, \ldots, d, \tag{17}$$

where the spectrum of the total deformed boson number operator was given by the relation $[n]_q = nq^{(n-1)}$ with $n = n_1 + n_2 + \cdots + n_{\mathrm{d}}$. Also, in the limit $q_1 = q_2 = 1$, the bosonic Fibonacci oscillator algebra together with the generalized Fibonacci basic integer $[n]_{q_1,q_2}$ in Eqs. (13) and (14) reduces to the usual (undeformed) boson algebra with the usual spectrum of eigenvalues n. Note also that from Eq. (14), one can observe the first few generalized Fibonacci basic integers as

$$\begin{aligned}
&[0] = 0,\\
&[1] = 1,\\
&[2] = q_1^2 + q_2^2,\\
&[3] = q_1^4 + q_1^2 q_2^2 + q_2^4,\\
&[4] = q_1^6 + q_1^4 q_2^2 + q_1^2 q_2^4 + q_2^6,\\
&[5] = q_1^8 + q_1^6 q_2^2 + q_1^4 q_2^4 + q_1^2 q_2^6 + q_2^8,\\
&\ldots\ldots
\end{aligned} \tag{18}$$

It is worth emphasizing that the q-deformed commutation relations corresponding to the AC- and BM-type bosonic algebras in Eqs. (1) and (4) are special cases of the following general linear second-order homogenous difference equation [93]:

$$[N + 2] = \alpha[N + 1] + \beta[N], \tag{19}$$

where $\alpha = q_1^2 + q_2^2$ and $\beta = -q_1^2 q_2^2$. Thus, the number operator spectrum of this relation yields the generalized Fibonacci basic integer $[n]_{q_1,q_2}$ with the initial conditions $[0] = 0$ and $[1] = 1$ as in Eq. (14). From Eq. (19), we have

$$[0] = 0,$$
$$[1] = 1,$$
$$[2] = \alpha,$$
$$[3] = \alpha^2 + \beta, \tag{20}$$
$$[4] = \alpha^3 + 2\alpha\beta,$$

$$[5] = \alpha^4 + 3\alpha^2\beta + \beta^2,$$
$$[6] = \alpha^5 + 4\alpha^3\beta + 3\alpha\beta^2,$$
$$[7] = \alpha^6 + 5\alpha^4\beta + 6\alpha^2\beta^2 + \beta^3,$$
$$\ldots\ldots$$

This form leads to the usual Fibonacci sequence as $\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$ in the limit $\alpha = \beta = 1$. The Fibonacci oscillator algebra in Eqs. (13)–(16) has symmetry under the interchange of the deformation parameters q_1 and q_2. Also, Eqs. (15) and (16) show the most general quantum group covariant two-parameter deformed bosonic oscillator algebra with the quantum group $SU_r(d)$-symmetry, where $r = q_1/q_2$. We should remark that the commuting Fibonacci oscillators in Eqs. (13) and (14) are relevant for constructing both the coherent states and the unitary quantum Lie algebras, whereas the covariant Fibonacci oscillators in Eqs. (15) and (16) are needed for quantum group covariance and a bilinear Hamiltonian with a degenerate spectrum [93]. Note also that one immediate application of the commuting Fibonacci oscillators in Eqs. (13) and (14) has been recently carried out by the work of [95], where the (q_1, q_2)-deformed Hermite polynomials have been computed by replacing the quantum harmonic oscillator problem to the bosonic Fibonacci oscillators.

Moreover, for the bosonic Fibonacci oscillator algebra, the (q_1, q_2)-Fock space spanned by the orthonormalized eigenstates $|n_1, n_2, \ldots, n_\mathrm{d} >$ can be constructed according to

$$|n_1, n_2, \ldots, n_\mathrm{d} > = \frac{1}{\sqrt{[n_1]_{q_1,q_2}!\,[n_2]_{q_1,q_2}!\ldots[n_\mathrm{d}]_{q_1,q_2}!}}\left(b_\mathrm{d}^*\right)^{n_\mathrm{d}}\ldots\left(b_2^*\right)^{n_2}\left(b_1^*\right)^{n_1}|0, 0, \ldots, 0 >, \tag{21}$$
$$b_i|0, 0, \ldots, 0 >= 0, \; i = 1, 2, \ldots, d, \tag{22}$$

where the (q_1, q_2)-deformed basic factorial is defined as

$$[n]_{q_1,q_2}! = [1]_{q_1,q_2}[2]_{q_1,q_2}[3]_{q_1,q_2}\cdots[n-1]_{q_1,q_2}[n]_{q_1,q_2}. \tag{23}$$

where the generalized Fibonacci basic integer $[n]_{q_1,q_2}$ is defined in Eq. (14). For instance, we can construct the two-particle representations for the above algebra by considering the actions of the operators b_i, b_i^* on the Fock state $|n_1, n_2 >$ as follows:

$$\begin{aligned}
b_1|n_1, n_2 >&= q_2^{n_2}\sqrt{[n_1]}|n_1 - 1, n_2 >,\\
b_1^*|n_1, n_2 >&= q_2^{n_2}\sqrt{[n_1+1]}|n_1 + 1, n_2 >,\\
b_2|n_1, n_2 >&= q_1^{n_1}\sqrt{[n_2]}|n_1, n_2 - 1 >,\\
b_2^*|n_1, n_2 >&= q_1^{n_1}\sqrt{[n_2+1]}|n_1, n_2 + 1 >,\\
b_1^*b_1|n_1, n_2 > &= q_2^{2n_2}[n_1]|n_1, n_2 >,\\
b_2^*b_2|n_1, n_2 > &= q_1^{2n_1}[n_2]|n_1, n_2 >,
\end{aligned} \tag{24}$$

where $n_1, n_2 = 0, 1, 2, \ldots$, and in turn, the last two relations in Eq. (24) read

$$[n_1 + n_2] = [n_1]q_2^{2n_2} + q_1^{2n_1}[n_2], \tag{25}$$

where the generalized Fibonacci basic integer $[n] \equiv [n]_{q_1,q_2}$ is given by Eq. (14).

On the other hand, the fermionic Fibonacci oscillator algebra is defined by the following deformed anti-commutation relations [96]:

$$\begin{aligned}
c_i c_j &= -q_1 q_2^{-1} c_j c_i, \quad i < j,\\
c_i c_j^* &= -q_1 q_2 c_j^* c_i, \quad i \neq j,\\
c_i^2 &= 0, \quad c_1 c_1^* + q_2^2 c_1^* c_1 = q_2^{2\hat{N}},\\
c_i c_i^* + q_1^2 c_i^* c_i &= c_{i+1}c_{i+1}^* + q_2^2 c_{i+1}^* c_{i+1}, \quad i = 1, 2, \cdots, d-1,\\
q_1^{2\hat{N}} &= c_\mathrm{d} c_\mathrm{d}^* + q_1^2 c_\mathrm{d}^* c_\mathrm{d}
\end{aligned} \tag{26}$$

where the total deformed fermion number operator is

$$c_1^* c_1 + c_2^* c_2 + \ldots + c_\mathrm{d}^* c_\mathrm{d} = \left[\hat{N}_1 + \hat{N}_2 + \ldots + \hat{N}_\mathrm{d}\right] = [\hat{N}], \tag{27}$$

whose spectrum is the same as in Eq. (14). But, for the fermionic Fibonacci oscillator algebra, each $\widehat{N}_i$ in Eq. (27) can only have eigenvalues 0 and 1. Notice that in Eqs. (26) and (27), the usual (undeformed) fermionic oscillator algebra can be recovered in the limit $q_1 = q_2 = 1$. This two-parameter deformed fermionic oscillator algebra has symmetry under the interchange of the deformation parameters q_1 and q_2, and that is also covariant under the quantum group $SU_r(d)$ with $r = q_2/q_1$. Here, it is worth emphasizing that as proved in [96], as far as the quantum group covariance is concerned, the maximum number of parameters should be just two. We should also mention that when we take the limit $q_1 = 1$ and $q_2 = q$ in Eqs. (26) and (27), the one-parameter deformed $SU_q(d)$-covariant fermion algebra can be obtained [97]. When we apply for the limit $q_1 = q_2 = \sqrt{q}$ in Eqs. (26) and (27), we get the multi-dimensional q-deformed fermionic Newton oscillator algebra invariant under the undeformed Lie group $SU(d)$ [98].

Working in the context of the above properties on Fibonacci oscillators, in the next section, we shall study the main elements of Fibonacci calculus.

3 The Fibonacci calculus

Let us consider the main features of Fibonacci calculus and some basic two-parameter deformed elementary functions, which will be useful in the next sections. For the commuting Fibonacci oscillator algebra in Eqs. (13) and (14), the transformation from Fock space of observables to the configuration space may be formally accomplished by the following map (for the sake of simplicity, from now on, we shall omit the particle index i):

$$a \Rightarrow \hat{D}_x^{(q_1,q_2)}, \quad a^* \Rightarrow x, \tag{28}$$

where $\widehat{D}_x^{(q_1,q_2)}$ is the modified Fibonacci difference operator [99] defined as

$$\hat{D}_x^{(q_1,q_2)} f(x) = \left\{ \frac{(q_1^2 - q_2^2)}{\left[\ln\left(q_1^2/q_2^2\right)\right]} \right\} \hat{\partial}_x^{(q_1,q_2)} f(x), \tag{29}$$

where $\widehat{\partial}_x^{(q_1,q_2)} f(x)$ is the Fibonacci difference operator as

$$\hat{\partial}_x^{(q_1,q_2)} f(x) = \frac{f\left(q_1^2 x\right) - f\left(q_2^2 x\right)}{(q_1^2 - q_2^2)x}, \tag{30}$$

for an analytic function $f(x)$. We also have $\left(\widehat{D}_x^{(q_1,q_2)}\right)^2 = \widehat{D}_x^{(q_1,q_2)}(\widehat{D}_x^{(q_1,q_2)})$, which reads

$$\left(\hat{D}_x^{(q_1,q_2)}\right)^2 f(x) = \frac{\left[q_1^2 f\left(q_2^4 x\right) - (q_1^2 + q_2^2) f\left(q_1^2 q_2^2 x\right) + q_2^2 f\left(q_1^4 x\right)\right]}{q_1^2 q_2^2 x^2 \left[ln\left(q_1^2/q_2^2\right)\right]^2}. \tag{31}$$

The modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29) can be regarded as a two-parameter extension of the usual JD operator given in Eq. (3). Note that it can be deduced from Eq. (29) up to some multiplicative factor $[(q-1)/lnq]$ by applying the limit $q_1 = \sqrt{q}$ and $q_2 = 1$. We should also note that the operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29) is different from the $\widetilde{q}$-derivative in Eq. (8), which was appeared by the formalism of Tsallis thermostatistics [62, 67].

Besides, some further elements of the Fibonacci calculus based on the above construction can be studied as follows: The action of the operator $\widehat{D}_x^{(q_1,q_2)}$ on a monomial $f(x) = x^n$, where $n \geq 0$, is derived by

$$\hat{D}_x^{(q_1,q_2)}\left(x^n\right) = \left\{ \frac{(q_1^2 - q_2^2)}{\left[\ln\left(q_1^2/q_2^2\right)\right]} \right\} [n]_{q_1,q_2} x^{n-1}, \tag{32}$$

where the Fibonacci basic number $[n]_{q_1,q_2}$ is defined in Eq. (14). Moreover, one can prove the following (q_1, q_2)-deformed version of the Leibniz rule:

$$\begin{aligned} \hat{D}_x^{(q_1,q_2)}(f(x)g(x)) &= f\left(q_1^2 x\right)\left[\hat{D}_x^{(q_1,q_2)} g(x)\right] + g\left(q_2^2 x\right)\left[\hat{D}_x^{(q_1,q_2)} f(x)\right] \\ &= g\left(q_1^2 x\right)\left[\hat{D}_x^{(q_1,q_2)} f(x)\right] + f\left(q_2^2 x\right)\left[\hat{D}_x^{(q_1,q_2)} g(x)\right] \end{aligned} \tag{33}$$

In addition to this, we have the relation

$$\hat{D}_{ax}^{(q_1,q_2)} f(x) = \frac{1}{a} \hat{D}_x^{(q_1,q_2)} f(x), \tag{34}$$

where a is some real constant different from zero. Let us introduce the following (q_1, q_2)-deformed exponential function as

$$e_{q_1,q_2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_{q_1,q_2}!} = 1 + x + \frac{x^2}{[2]_{q_1,q_2!}} + \frac{x^3}{[3]_{q_1,q_2!}} + \dots, \tag{35}$$

where the (q_1, q_2)-factorial $[k]_{q_1,q_2}!$ is described in Eq. (23). Remarkably, this form is also consistent with the following representation of the (q_1, q_2)-deformed analogue of the Taylor expansion:

$$f(x) = f(a) + (x-a)\left[\hat{D}_x^{(q_1,q_2)} f(x)\right]\Big|_{x=a} + \frac{(x-a)^2}{[2]_{q_1,q_2}!}\left[\left(\hat{D}_x^{(q_1,q_2)}\right)^2 f(x)\right]\Big|_{x=a} + \ldots, \tag{36}$$

where $\left[(\widehat{D}_x^{(q_1,q_2)})^2 f(x)\right]$ is given in Eq. (31) and $[k]_{q_1,q_2}!$ is also introduced in Eq. (23). It is also worth to add that the (q_1, q_2)-deformed exponential function satisfies the following relation:

$$\hat{D}_x^{(q_1,q_2)} e_{q_1,q_2}(ax) = a e_{q_1,q_2}(ax), \tag{37}$$

and its dual relation is given by

$$\int_0^x e_{q_1,q_2}(ay) d_{q_1,q_2} y = \frac{1}{a}\left[e_{q_1,q_2}(ax) - 1\right]. \tag{38}$$

Besides, it is important to point out the following relation for the (q_1, q_2)-deformed addition law:

$$[x+y]_{q_1,q_2} = [y]_{q_1,q_2}[x+1]_{q_1,q_2} + [x]_{q_1,q_2}[y+1]_{q_1,q_2} - \left(q_1^2 + q_2^2\right)[x]_{q_1,q_2}[y]_{q_1,q_2}, \tag{39}$$

where the Fibonacci basic number $[n]_{q_1,q_2}$ is given in Eq. (14). The expression in Eq. (39) was first introduced by Arik et al. [93] during a study on the construction of the most general quantum group covariant bosonic oscillator algebra.

On the other hand, among many properties, it would be important to introduce the two-parameter (q_1, q_2)-deformed trigonometric functions as

$$e_{q_1,q_2}(ix) = C_{q_1,q_2}(x) + i S_{q_1,q_2}(x), \tag{40}$$

which constitutes a two-parameter (q_1, q_2)-deformed analogue of the Euler's identity. In Eq. (40), the (q_1, q_2)-cosine and (q_1, q_2)-sine functions are defined by

$$S_{q_1,q_2}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{[2n+1]_{q_1,q_2}!}, \tag{41}$$

$$C_{q_1,q_2}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{[2n]_{q_1,q_2}!}, \tag{42}$$

where the (q_1, q_2)-factorial $[x]_{q_1,q_2}!$ is given in Eq. (23). The above functions reduce to their corresponding ordinary functions in the limit $q_1 = q_2 = 1$ and also, we should note that the one-parameter deformed analogues of them can be obtained in the special limiting cases $q_1 = \sqrt{q}$, $q_2 = 1$ [79] and $q_1 = \sqrt{q}$, $q_2 = 1/\sqrt{q}$[80], respectively.

Hence, we expect below that the above properties would have some crucial roles in the following introduction to a consistent formulation of the two-parameter (q_1, q_2)-deformed quantum mechanics.

4 The two-parameter (q_1, q_2)-deformed quantum mechanics

Starting from the properties studied in the previous sections, we are now able to develop a formalism for the two-parameter (q_1, q_2)-deformed quantum dynamics in the framework of Fibonacci calculus. In this regard, we shall adopt the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29), which will have a central role to formulate the (q_1, q_2)-deformed quantum mechanics. For the construction of the two-parameter (q_1, q_2)-deformed quantum mechanics through some elements of Fibonacci calculus studied above and for the sake of compatibility, from now on, we shall consider the one-dimensional case for a construction of the representations of (q_1, q_2)-deformed quantum mechanics. For such an aim, one may introduce the following coordinate realizations for the two-parameter deformed momentum and position operators:

$$\hat{p} = \frac{\hbar}{i}\hat{D}_x^{(q_1,q_2)}, \ \hat{x} = x, \tag{43}$$

where the (q_1, q_2)-derivative operator $\widehat{D}_x^{(q_1,q_2)}$ is defined as in Eq. (29). Thus, we obtain the two-parameter (q_1, q_2)-deformed Heisenberg relation as

$$\left[\hat{x}, \hat{p}\right] = i\hbar\left\{\frac{(q_1^2 - q_2^2)}{\left[\ln\left(q_1^2/q_2^2\right)\right]}\right\}. \tag{44}$$

Accordingly, this leads to the (q_1, q_2)-deformed uncertainty relation as

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2}\left\{\frac{\left(q_1^2 - q_2^2\right)}{\left[\ln\left(q_1^2/q_2^2\right)\right]}\right\}. \tag{45}$$

From Eq. (43) and considering a physical state represented by a wave function $\Psi(x, t)$, we can write the time-dependent (q_1, q_2)-deformed Schrödinger equation as

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat{H}_{q_1,q_2}\Psi(x,t), \tag{46}$$

where the (q_1, q_2)-deformed Hamiltonian is

$$\hat{H}_{q_1,q_2} = \frac{\hat{p}^2}{2m} + U\left(\hat{x}\right) = -\frac{\hbar^2}{2m}\left(\hat{D}_x^{(q_1,q_2)}\right)^2 + U(x) \tag{47}$$

with a particle mass m and the potential energy $U(x)$. In Eq. (47), the square of the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ is given as in Eq. (31). Equation (46) can be factorized by setting $\Psi(x, t) = e^{-iEt/\hbar}\varphi(x)$, where the wave function $\varphi(x)$ is the solution of time-independent (q_1, q_2)-deformed Schrödinger equation as

$$\hat{H}_{q_1,q_2}\varphi(x) = \left[-\frac{\hbar^2}{2m}\left(\hat{D}_x^{(q_1,q_2)}\right)^2 + U(x)\right]\varphi(x) = E\varphi(x). \tag{48}$$

Moreover, in the Hilbert space related to the (q_1, q_2)-deformed quantum mechanics, the inner (scalar) product is defined as

$$\langle\varphi|\psi\rangle \equiv \langle\varphi|\psi\rangle_{q_1,q_2} = \int_{-\infty}^{+\infty}\varphi^*(x)\psi(x)d_{q_1,q_2}x, \tag{49}$$

and the expectation value of a physical operator $\widehat{A}$ with respect to the state $\Psi(x, t)$ is defined as follows:

$$\langle\hat{A}\rangle = \langle\Psi|\hat{A}\Psi\rangle = \int_{-\infty}^{+\infty}\Psi^*(x,t)\hat{A}\Psi(x,t)d_{q_1,q_2}x, \tag{50}$$

where $\widehat{A} \equiv \widehat{A}_{q_1,q_2}$ will be a (q_1, q_2)-Hermitian operator if it obeys

$$\langle\hat{A}\Psi|\Psi\rangle = \langle\Psi|\hat{A}\Psi\rangle. \tag{51}$$

According to the above prescriptions, it follows that the (q_1, q_2)-deformed Hamiltonian in Eq. (47) is a (q_1, q_2)-Hermitian operator, and the time development of a system represented by the wave function $\Psi(x, t)$ is given in Eq. (46). In fact, it also implies a consistent conservation in time for the probability density $\rho(x, t) = \Psi^*(x, t)\Psi(x, t)$, which can be deduced from the relation

$$i\hbar\frac{\partial}{\partial t}\left[\int\limits_{-\infty}^{+\infty}\Psi^*\Psi d_{q_1,q_2}x\right] = \int\limits_{-\infty}^{+\infty}\left[\Psi^*\hat{H}_{q_1,q_2}\Psi - \left(\hat{H}_{q_1,q_2}\Psi^*\right)\Psi\right]d_{q_1,q_2}x = 0. \tag{52}$$

The above properties introduced provide a self-consistent theoretical background for the two-parameter (q_1, q_2)-deformed quantum mechanics in the framework of Fibonacci calculus. It is worth pointing out that the present analysis serves as not only a two-parameter extension for studies on quantum mechanics covering both the Jackson's q-calculus [79] and the Tsallis $\widetilde{q}$-calculus [83], but also it refers to a new perspective to deal with nonlinear quantum behavior for interacting bosonic and fermionic particle systems by enabling appropriate postulates on the two-parameter deformed formalism based on Fibonacci calculus.

In the next section, based on the above properties, we shall investigate possible consequences of introducing two deformation parameters on some physical applications covering different quantum dynamical problems.

5 Some physical applications

In this section, using both the mathematical background developed by some elements of Fibonacci calculus and the properties related to the two-parameter (q_1, q_2)-deformed quantum mechanics introduced in Sects. 3 and 4, we deal with some physical applications covering the free particle and the inverse-harmonic potential cases. In this context, another application containing a model of the (q_1, q_2)-deformed Debye solid is studied along with its deformed quantum statistics. Such an analysis will also help us to work out possible effects of two deformation parameters on the physical behavior of these systems.

5.1 The free particle

As a first application, let us consider the case of a free particle ($U(x) = 0$ everywhere) described by the wave function $\varphi^{\text{free}}_{q_1,q_2}(x)$. Then, Eq. (48) becomes

$$\left(\hat{D}_x^{(q_1,q_2)}\right)^2 \varphi^{\text{free}}_{q_1,q_2}(x) + k^2 \varphi^{\text{free}}_{q_1,q_2}(x) = 0, \tag{53}$$

where $k = \sqrt{2mE}/\hbar$. The solution to this equation can be deduced in terms of the (q_1, q_2)-deformed exponential function as

$$\varphi^{\text{free}}_{q_1,q_2}(x) = N e_{q_1,q_2}(ikx), \tag{54}$$

where N is some normalization constant. It is noteworthy that Eq. (54) exhibits a two-parameter extension of the plane wave function in the framework of Fibonacci calculus. In addition, one can also search for the momentum eigenfunction satisfying

$$\hat{p}\varphi_p(x) = p\varphi_p(x). \tag{55}$$

which gives rise to the following relation via Eq. (43):

$$\hat{D}_x^{(q_1,q_2)} \varphi_p(x) = \frac{ip}{\hbar}\varphi_p(x). \tag{56}$$

From Eqs. (35), (37) and (56), we have the plane wave solution with momentum p as follows:

$$\varphi_p(x) = N\left[e_{q_1,q_2}(x)\right]^{(ip/\hbar)}. \tag{57}$$

In this regard, the time-dependent plane wave can also be given by the relation

$$\Psi_p(x,t) = N\left[e_{q_1,q_2}(x)\right]^{(ip/\hbar)} e^{-i\omega t}. \tag{58}$$

5.2 The inverse-harmonic potential

To observe possible effects of two deformation parameters in a physical model, we now consider an application to the case of inverse-harmonic potential described by the following relation:

$$U(x) = \frac{\lambda(q_1,q_2)}{x^2}, \tag{59}$$

where $\lambda(q_1, q_2) = 1$ in the limit $q_1 = q_2 = 1$. Hence, by means of the two-parameter deformed time-independent Schrödinger equation in Eq. (48), one can calculate the energy eigenfunctions for this potential in the position representation. This consideration yields

$$\left[-\frac{\hbar^2}{2m}\left(\hat{D}_x^{(q_1,q_2)}\right)^2 + \frac{\lambda(q_1,q_2)}{x^2}\right]\varphi_N(x) = -E_N\varphi_N(x),\ N = 1,2,3,\ldots \tag{60}$$

This can also be written as

$$\left(\hat{D}_x^{(q_1,q_2)}\right)^2 \varphi_N(x) = \left(\frac{\alpha}{x^2} + \epsilon_N\right)\varphi_N(x), \tag{61}$$

where

$$\epsilon_N = \frac{2mE_N}{\hbar^2},\ \alpha = \frac{2m\lambda}{\hbar^2}, \tag{62}$$

Thus, using Eqs. (31) and (61), we have

$$q_1^2\varphi_N\left(q_1^4x\right) - \left(q_1^2 + q_2^2\right)\varphi_N\left(q_1^2q_2^2x\right) + q_2^2\varphi_N\left(q_2^4x\right) = \left(\beta + \mu_N x^2\right)\varphi_N(x), \tag{63}$$

where

$$\mu_N = q_1^2q_2^2\left[\ln\left(q_1^2/q_2^2\right)\right]^2\epsilon_N,\ \beta = q_1^2q_2^2\left[\ln\left(q_1^2/q_2^2\right)\right]^2\alpha. \tag{64}$$

Using Eq. (63), we can study the scattering solution for the inverse-harmonic potential in Eq. (59) by considering the cases $x > x_0 > 0$, $\lambda > 0$. For the sake of simplicity, from now on, we will omit the index N. Under such circumstances, we set the solution in the following form:

$$\varphi(x) = \sqrt{x}\phi(x). \tag{65}$$

Hence, by inserting Eq. (65) into Eq. (63), we get

$$q_1^4\phi\left(q_1^4x\right) - q_1q_2\left(q_1^2 + q_2^2\right)\phi\left(q_1^2q_2^2x\right) + q_2^4\phi\left(q_2^4x\right) = \left(\beta + \mu x^2\right)\phi(x), \tag{66}$$

which leads to a solution in the form of (q_1, q_2)-deformed Bessel function. Now, let us set

$$\sqrt{\mu}x = \xi. \tag{67}$$

Inserting Eq. (67) into Eq. (66), we obtain

$$q_1^4\phi\left(q_1^4\frac{\xi}{\sqrt{\mu}}\right) - q_1q_2(q_1^2+q_2^2)\phi\left(q_1^2q_2^2\frac{\xi}{\sqrt{\mu}}\right) + q_2^4\phi\left(q_2^4\frac{\xi}{\sqrt{\mu}}\right) = (\beta+\xi^2)\phi\left(\frac{\xi}{\sqrt{\mu}}\right). \tag{68}$$

Here, we should also remark that the ordinary Bessel differential equation can be recovered in the limit $q_1 = q_2 = 1$. So now, we search for the solution of Eq. (68) by setting

$$\phi(X) = X^a y(X) = X^a \sum_{n=0}^{\infty} a_n X^n, \tag{69}$$

which leads to

$$q_1^{4a+4}y\left(q_1^4\frac{\xi}{\sqrt{\mu}}\right) - q_1^{2a+1}q_2^{2a+1}(q_1^2+q_2^2)y\left(q_1^2q_2^2\frac{\xi}{\sqrt{\mu}}\right) + q_2^{4a+4}y\left(q_2^4\frac{\xi}{\sqrt{\mu}}\right) = (\beta+\xi^2)y\left(\frac{\xi}{\sqrt{\mu}}\right), \tag{70}$$

and then we have

$$\begin{aligned} &q_1^{4a+4}\sum_{n=0}^{\infty} a_n\left(\frac{q_1^4}{\sqrt{\mu}}\right)^{2n}\xi^{2n} - q_1^{2a+1}q_2^{2a+1}(q_1^2+q_2^2)\sum_{n=0}^{\infty} a_n\left(\frac{q_1^2q_2^2}{\sqrt{\mu}}\right)^{2n}\xi^{2n} + q_2^{4a+4}\sum_{n=0}^{\infty} a_n\left(\frac{q_1^4}{\sqrt{\mu}}\right)^{2n}\xi^{2n} \\ &= \beta\sum_{n=0}^{\infty} a_n\left(\frac{1}{\sqrt{\mu}}\right)^{2n}\xi^{2n} + \sum_{n=0}^{\infty} a_n\left(\frac{1}{\sqrt{\mu}}\right)^{2n}\xi^{2n+2}. \end{aligned} \tag{71}$$

This relation gives

$$\left[q_1^{4a+4}\left(q_1^4\right)^{2n} - q_1^{2a+1}q_2^{2a+1}\left(q_1^2+q_2^2\right)\left(q_1^2q_2^2\right)^{2n} + q_2^{4a+4}\left(q_2^4\right)^{2n} - \beta\right]a_n = a_{n-1}, \tag{72}$$

which can be written as

$$-\beta\left[1 - A\left(q_1^4\right)^{2n} + B\left(q_1^4q_2^4\right)^n - C\left(q_2^4\right)^{2n}\right]a_n = a_{n-1}, \tag{73}$$

where

$$A = \frac{1}{\beta}q_1^{4a+4}, \tag{74}$$

$$B = \frac{1}{\beta}q_1^{2a+1}q_2^{2a+1}\left(q_1^2+q_2^2\right), \tag{75}$$

$$C = \frac{1}{\beta}q_2^{4a+4}. \tag{76}$$

If we set

$$(A, a; q_1, q_2)_{n-1} = \prod_{k=0}^{n-1}\left[1 - A\left(q_1^{2a+1}q_1^{4k} - q_2^{2a+1}q_2^{4k}\right)\left(q_1^{2a+3}q_1^{4k} - q_2^{2a+3}q_2^{4k}\right)\right], \tag{77}$$

then we get

$$y(\xi) = \sum_{n=1}^{\infty}\frac{\left(-\frac{1}{\beta}\right)^{n-1}}{\left(\frac{1}{\beta}, a; q_1, q_2\right)_{n-1}}\xi^{2(n-1)}. \tag{78}$$

Thus, from Eqs. (65), (67), (69), (78), the complete solution for Eq. (63) can implicitly be expressed as

$$\varphi(x) = \mu^{(a/2)}x^{(a+1/2)}y\left(\sqrt{\mu}x\right). \tag{79}$$

As a result, the two-parameter (q_1, q_2)-deformed time-independent Schrödinger equation in Eq. (60) for the inverse-harmonic potential case leads to the wave function as in Eq. (79). Moreover, Eq. (68) can also be considered as the (q_1, q_2)-deformed Bessel differential equation, whose solution having the undeformed limit $q_1 = q_2 = 1$ leads to the ordinary Bessel function with the form $J_{\sqrt{\alpha+\frac{1}{4}}}(\xi)$, where the term α is defined in Eq. (62). But Eq. (63) does not have a closed solution for ϵ_N, since we do not have the chain rule for the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29). This point is beyond the scope of the present work, which remains an open problem to be pursued in a separate study in the future.

5.3 The ($\boldsymbol{q}_1$, $\boldsymbol{q}_2$)-deformed Debye solid

We now make another application of our construction into the case of a two-parameter deformed Bose gas model as a statistical mechanical application. As examples of such kind of bosonic gases, one can think about phonons in crystalline solids or excitons in the high-density limit such as in a nanomaterial. Not only in such examples but also in interacting theories of bosons, it is interesting to observe that two deformation parameters could play some roles for understanding the details about the nature of interactions, which can cover applications onto either controlling the statistics of deformed (quasi)particles in the system or fitting the standard theory-based results with experimental data. Thus, we expect that two deformation parameters in a real phenomenological application can give some extra advantageous rather than the one-parameter ones, since our construction along with wide spectrum of the model parameters' values as ($0 < q_1 < \infty$), ($0 < q_2 < \infty$) gives a chance to increase the adjustment range, and thereby reveals more flexibility to calibrate the theoretical results (namely expectation values) with experimental data.

To find possible effects of two deformation parameters on the thermostatistical properties of a Debye crystalline solid, we have a (q_1, q_2)-deformed boson gas model, whose algebraic structure is based on some elements of the Fibonacci calculus studied in Sect. 3. By viewing lattice phonons as the quasi-particles obeying the commuting Fibonacci oscillators algebra in Eqs. (13) and (14), a (q_1, q_2)-deformed boson gas model can be constructed via the bosonic Hamiltonian

$$\hat{\mathcal{H}} \equiv \hat{\mathcal{H}}_{q_1,q_2} = \sum_i (\varepsilon_i - \mu) N_i, \tag{80}$$

where μ is the chemical potential and ε_i is the kinetic energy of a particle in the state i associated with the number operator N_i. In fact, the model Hamiltonian in Eq. (80) is essentially a (q_1, q_2)-deformed Hamiltonian, which implicitly depends on the two deformation parameters via Eq. (16). Hence, our (q_1, q_2)-deformed boson gas serves as a model to develop a Debye solid for studying its thermostatistics when the phonons in a crystal lattice are regarded as the commuting set of Fibonacci oscillators. Before entering the details of such a Debye solid, we start with the calculation of the mean occupation number n_i in the two-parameter deformed statistics. The thermal average of an operator $\widehat{O}$ can be calculated [100–102] by

$$\langle \hat{O} \rangle = Tr\Big(\rho \hat{O}\Big),\ \rho = \frac{\exp\Big(-\beta \hat{\mathcal{H}}\Big)}{Z}\ ,\ Z = Tr\Big(\exp\Big(-\beta \hat{\mathcal{H}}\Big)\Big), \tag{81}$$

where $\beta = 1/k_{\mathrm{B}}T$ and ρ is the density operator and Z is the grand canonical partition function. The mean value of the (q_1, q_2)-deformed occupation number $n_i \equiv n_{i,q_1,q_2}$ can be obtained by using

$$[n_{i,q_1,q_2}] \equiv \langle [N_i] \rangle = \frac{Tr\Big(\exp\Big(-\beta \hat{\mathcal{H}}\Big) a_i^* a_i\Big)}{Z}. \tag{82}$$

From the commuting Fibonacci oscillators algebra in Eqs. (13) and (14) and applying the cyclic property of the trace [18, 23, 103–111], one can deduce

$$[n_{i,q_1,q_2}] = \exp(-\beta(\varepsilon_i - \mu))[1 + n_{i,q_1,q_2}]. \tag{83}$$

Using Eqs. (13), (14), (80)–(83), we obtain

$$n_{i,q_1,q_2} = \frac{1}{\left[\ln\left(q_1^2/q_2^2\right)\right]} \ln\left[\frac{z^{-1}\exp(\beta\varepsilon_i) - q_2^2}{z^{-1}\exp(\beta\varepsilon_i) - q_1^2}\right], \tag{84}$$

where $q_1 \neq q_2$ and $z = \exp(\beta\mu)$ is the fugacity of the system. It may be called as a two-parameter deformed BE statistical distribution function of the present (q_1, q_2)-deformed boson gas, and it also describes a model system covering intermediate-statistics particles for the interval $0 < (q_1, q_2) < \infty$.

On the other hand, the logarithm of the grand partition function can be expressed as

$$\ln Z = -\sum_i \ln\big[1 - z\exp(-\beta\varepsilon_i)\big], \tag{85}$$

which leads to the usual (undeformed) form for the total number of particles $N^{(1,1)}$ as

$$N^{(1,1)} = z\left(\frac{\partial}{\partial z}\right)\ln Z = \sum_i n_{i,1,1}, \tag{86}$$

where $n_{i,1,1}$ corresponds to the usual (undeformed) BE distribution function defined by $n_{i,1,1} = [1/\big(z^{-1}\exp(\beta\varepsilon_i) - 1\big)]$. However, the total number of (q_1, q_2)-deformed bosons in our model cannot be obtained by using ordinary thermodynamic derivatives. Instead of this, we can obtain it via the relation

$$N^{(q_1,q_2)} = z\hat{D}_z^{(q_1,q_2)} \ln Z = \sum_i n_{i,q_1,q_2}, \tag{87}$$

where n_{i,q_1,q_2} is given in Eq. (84) and the modified Fibonacci difference operator $\widehat{D}_z^{(q_1,q_2)}$ is defined as in Eq. (29) with the use of z instead of x. Considering the high-temperature limit, namely $z \ll 1$, we can also derive a relationship between $N^{(q_1,q_2)}$ and $N^{(1,1)}$. By means of a power series expansion of Eq. (84) in such a limiting case, we infer the following relation:

$$N^{(q_1,q_2)} = \left(\frac{q_1^2 - q_2^2}{\left[\ln\left(q_1^2/q_2^2\right)\right]}\right) N^{(1,1)}, \tag{88}$$

where $q_1 \neq q_2$. Interestingly enough, as a consequence of the commuting Fibonacci oscillators algebra in Eqs. (13) and (14), Eqs. (84)–(88) exhibit a key role played by the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29).

Furthermore, as in the usual (undeformed) Debye solid [112], we have the following total number of normal modes of vibration:

$$\int_0^{\omega_\mathrm{D}} D(\omega)\mathrm{d}\omega = 3N, \tag{89}$$

where $D(\omega)$ is the normal mode density with the volume V and the particle number N through the expression

$$D(\omega) = \frac{3V\omega^2}{2\pi^2\upsilon^3}, \tag{90}$$

where υ is the propagation velocity of the wave in a solid and $\omega = 2\pi\nu$. From Eq. (89), we find the Debye frequency as

$$\omega_\mathrm{D} = \left(6\pi^2 n\right)^{1/3}\upsilon, \tag{91}$$

where $n = N/V$ is the particle density. If we set $\varepsilon_i = \hbar\omega_i = h\nu_i$ with the frequency ν_i of a (q_1, q_2)-phonon along with the use of Eq. (84), we have the following expected (q_1, q_2)-energy value for the mode i:

$$\langle E_i\rangle_{q_1,q_2} = \frac{h\nu}{\left[\ln\left(q_1^2/q_2^2\right)\right]} \ln\left[\frac{\exp(h\nu/k_\mathrm{B}T) - q_2^2}{\exp(h\nu/k_\mathrm{B}T) - q_1^2}\right], \tag{92}$$

where the case $z = 1$ is considered for the fugacity of the (q_1, q_2)-deformed phonon gas and we have also assumed that all the (q_1, q_2)-phonons have the same frequency given by ν. Thus, the total energy $U^{(q_1,q_2)}$ for our model containing the (q_1, q_2)-phonons can be derived as

$$U^{(q_1,q_2)} = \int_0^{\omega_\mathrm{D}} \varepsilon(\omega)D(\omega)n(\omega, q_1, q_2)\mathrm{d}\omega, \tag{93}$$

which leads to the following relation via the use of Eqs. (84) and (90):

$$U^{(q_1,q_2)} = \int_0^{\omega_\mathrm{D}} \frac{3V\omega^2}{2\pi^2\upsilon^3}\left[\frac{\hbar\omega}{\left(\ln\left(q_1^2/q_2^2\right)\right)} \ln\left(\frac{\exp(\hbar\omega/k_\mathrm{B}T) - q_2^2}{\exp(\hbar\omega/k_\mathrm{B}T) - q_1^2}\right)\right]\mathrm{d}\omega. \tag{94}$$

For the present (q_1, q_2)-deformed Debye solid, we can obtain its specific heat through the relation $c_V^{(q_1,q_2)}(T) = (\partial U^{(q_1,q_2)}/\partial T)_{V,N}$. It reads

$$c_V^{(q_1,q_2)}(T) = 9Nk_\mathrm{B}\left(\frac{q_1^2 - q_2^2}{\left(\ln\left(q_1^2/q_2^2\right)\right)}\right)\left(\frac{1}{x_\mathrm{D}^3}\right)\int_0^{x_\mathrm{D}} \frac{x^4 e^x \mathrm{d}x}{\left[\left(\exp(x) - q_1^2\right)\left(\exp(x) - q_2^2\right)\right]}, \tag{95}$$

where $x = (\hbar\omega/k_\mathrm{B}T)$ and $x_\mathrm{D} = (\theta_\mathrm{D}/T)$ with the Debye temperature $\theta_\mathrm{D} = (\hbar\omega_\mathrm{D}/k_\mathrm{B})$. If we define a (q_1, q_2)-deformed Debye function $D(x_\mathrm{D}; q_1, q_2)$ as

$$D(x_\mathrm{D}; q_1, q_2) = \frac{3}{x_\mathrm{D}^3}\left(\frac{q_1^2 - q_2^2}{\left(\ln\left(q_1^2/q_2^2\right)\right)}\right)\int_0^{x_\mathrm{D}} \frac{x^4 e^x \mathrm{d}x}{\left[\left(\exp(x) - q_1^2\right)\left(\exp(x) - q_2^2\right)\right]}, \tag{96}$$

then we get

$$c_V^{(q_1,q_2)}(T) = 3Nk_\mathrm{B}D(x_\mathrm{D}; q_1, q_2). \tag{97}$$

Within the framework of two-parameter deformed quantum statistics, Eq. (97) will reveal the effect of Fibonacci oscillators onto the specific heat of a Debye solid. Equation (96) can be integrated out to find

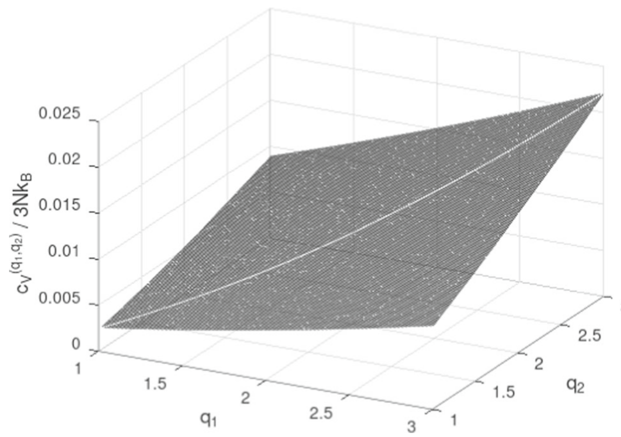

Fig. 1 The approximate (q_1, q_2)-deformed specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_\mathrm{B}$) of the ($q_1$, q_2)-phonon gas model as a function of the deformation parameters q_1 and q_2 for high temperatures for the case (q_1, q_2) > 1

$$\begin{aligned}
&D(x_\mathrm{D};q_1,q_2)\\
&= -\frac{3x_\mathrm{D}}{\left(\ln\left(q_1^2/q_2^2\right)\right)}\left[\ln\left(\frac{q_2^2q_1^2 - q_2^2\exp(x_\mathrm{D})}{q_1^2q_2^2 - q_1^2\exp(x_\mathrm{D})}\right)\right] - \frac{36}{x_\mathrm{D}^3(\ln(q_1/q_2))}\\
&\left[Li_5\left(q_2^{-2}\exp(x_\mathrm{D})\right)) - Li_5\left(q_1^{-2}\exp(x_\mathrm{D})\right))\right] \quad - \frac{36}{x_\mathrm{D}^2(\ln(q_1/q_2))}\left[Li_4\left(q_1^{-2}\exp(x_\mathrm{D})\right) - Li_4\left(q_2^{-2}\exp(x_\mathrm{D})\right))\right]\\
&- \frac{18}{x_\mathrm{D}(\ln(q_1/q_2))}\left[Li_3\left(q_2^{-2}\exp(x_\mathrm{D})\right)) - Li_3\left(q_1^{-2}\exp(x_\mathrm{D})\right))\right] - \frac{6}{(\ln(q_1/q_2))}\left[Li_2\left(q_1^{-2}\exp(x_\mathrm{D})\right)) - Li_2\left(q_2^{-2}exp(x_\mathrm{D})\right))\right]\\
&- \frac{36}{x_\mathrm{D}^3(\ln(q_1/q_2))}\left[Li_5\left(q_1^{-2}\right) - Li_5\left(q_2^{-2}\right)\right],
\end{aligned} \tag{98}$$

where the polylogarithm function $Li_n(r)$ is defined as

$$Li_n(r) = \sum_{k=1}^{\infty}\frac{r^k}{k^n}. \tag{99}$$

Hence, it turns out that the values of the specific heat $c_V^{(q_1,q_2)}(T)$ in Eq. (97) depend on both the temperature T and the deformation parameters (q_1, q_2). In this context, we can further study the high- and low-temperature behaviors of the above two-parameter deformed specific heat in the two limiting cases involving $x_\mathrm{D} \ll 1$ for high temperatures and $x_\mathrm{D} \gg 1$ for low temperatures, respectively. For $x_\mathrm{D} \ll 1$, namely $T \gg \theta_\mathrm{D}$, from Eqs. (97) and (98), the specific heat of the (q_1, q_2)-deformed Debye solid can be approximated as

$$c_V^{(q_1,q_2)}(T) \approx 36Nk_\mathrm{B}\left(\frac{q_1^2 - q_2^2}{q_1^2q_2^2\left(\ln\left(q_1^2/q_2^2\right)\right)}\right), \tag{100}$$

where $q_1 \neq q_2$. In effect, it differs from the behavior of the usual (undeformed) specific heat of a Debye solid in the high-temperature limit represented by the following Dulong–Petit law:

$$c_V^{(1,1)}(T) = 3Nk_\mathrm{B}. \tag{101}$$

In order to visualize the behavior of the specific heat $c_V^{(q_1,q_2)}(T)$ in Eq. (100), in Figs. 1 and 2, we show the plots of the specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_\mathrm{B}$) for the ($q_1$, q_2)-phonon gas model as a function of the deformation parameters q_1 and q_2 for high temperatures for the cases (q_1, q_2) > 1 and (q_1, q_2) < 1, respectively. When we compare the result of the classical theory in Eq. (101), it is interestingly found that the specific heat $c_V^{(q_1,q_2)}(T)$ can have lower values depending on the values of the model deformation parameters q_1 and q_2 as shown in Figs. 1 and 2. Such a result could have some implications especially in studies on anharmonic effects of interatomic interactions either in lattice or in molecular vibrations.

On the other hand, for the limiting case $x_\mathrm{D} \gg 1$, namely $T \ll \theta_\mathrm{D}$, from Eqs. (97) and (98), the specific heat of the (q_1, q_2)-deformed Debye solid can be approximated as

$$c_V^{(q_1,q_2)}(T) \approx 216Nk_\mathrm{B}\left(\frac{T}{\theta_\mathrm{D}}\right)^3\frac{1}{\left(\ln\left(q_1^2/q_2^2\right)\right)}\left[Li_5\left(q_1^2\right) - Li_5\left(q_2^2\right)\right], \tag{102}$$

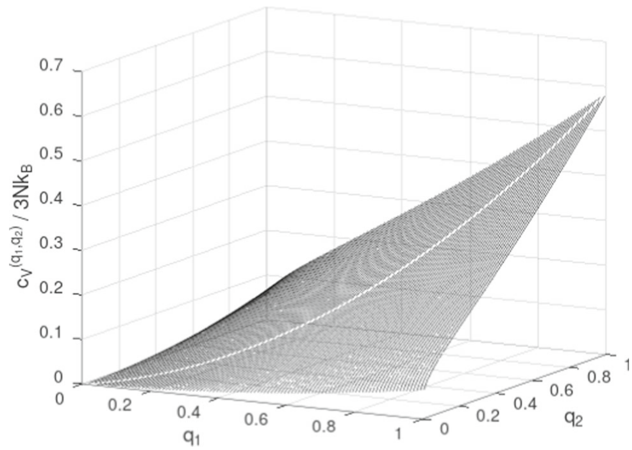

Fig. 2 The approximate (q_1, q_2)-deformed specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_{\mathrm{B}}$) of the ($q_1$, q_2)-phonon gas model as a function of the deformation parameters q_1 and q_2 for high temperatures for the case (q_1, q_2) < 1

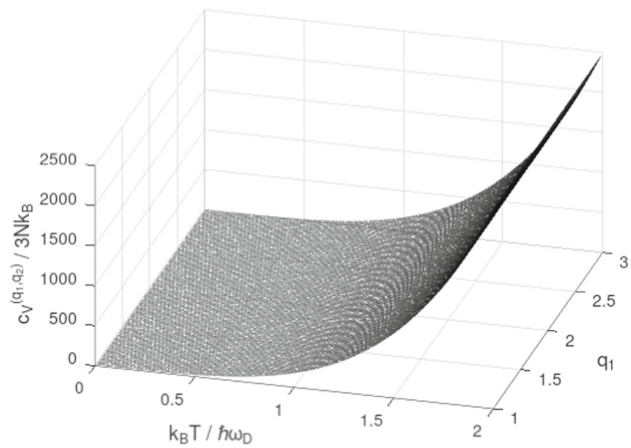

Fig. 3 The approximate (q_1, q_2)-deformed specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_{\mathrm{B}}$) of the ($q_1$, q_2)-phonon gas model as a function of ($k_{\mathrm{B}}T/\hbar\omega_{\mathrm{D}}$) for some value of the second deformation parameter $q_2 = 2$ for low temperatures for the case $q_1 > 1$

where $q_1 \neq q_2$ and the polylogarithm function $Li_n(r)$ is defined in Eq. (99). It turns out that the low-temperature specific heat of the (q_1, q_2)-phonon gas model is directly proportional to the term T^3, which is due to the phonon excitation. This agrees with experimental results [112]. Nonetheless, the result in Eq. (102) still differs from the usual (undeformed) Debye T^3-law in the sense that the values of the deformation parameters (q_1, q_2) can affect the behavior of the specific heat through the polylogarithm functions $Li_5(q_1^2)$ and $Li_5(q_2^2)$. We can further express the behavior of the low-temperature specific heat of the present (q_1, q_2)-phonon gas by considering small values of the terms ($q_1^2 - 1$) and ($q_2^2 - 1$) in Eq. (102). It results in

$$c_V^{(q_1,q_2)}(T) \approx \frac{12}{5}\pi^4 Nk_{\mathrm{B}}\left(\frac{T}{\theta_{\mathrm{D}}}\right)^3\left(\frac{q_1^2 - q_2^2}{\left(\ln\left(q_1^2/q_2^2\right)\right)}\right). \tag{103}$$

For comparison, in Figs. 3, 4 and 5, we show the plots of the specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_{\mathrm{B}}$) for the ($q_1$, q_2)-deformed phonon gas model and the specific heat ($c_V^{(1,1)}(T)/3Nk_{\mathrm{B}}$) of an undeformed phonon gas as a function of ($k_{\mathrm{B}}T/\hbar\omega_{\mathrm{D}}$) for some values of the deformation parameter q_2 for low temperatures for the cases $q_1 > 1$, $q_1 < 1$, respectively. When compared with the values of the specific heat of an undeformed phonon gas shown in Fig. 5, at the same value of ($k_{\mathrm{B}}T/\hbar\omega_{\mathrm{D}}$), the values of the specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_{\mathrm{B}}$) for the case ($q_1$, q_2) > 1 in Fig. 3 are larger than the results of an undeformed specific heat ($c_V^{(1,1)}(T)/3Nk_{\mathrm{B}}$), whereas it has lower values than those of the function ($c_V^{(1,1)}(T)/3Nk_{\mathrm{B}}$) for the case ($q_1$, q_2) < 1 as shown in Fig. 4. Our results may also be contrasted with a real data on the experimental specific heat capacity of a given material. For instance, the experimental specific heat value of Germanium at ($k_{\mathrm{B}}T/\hbar\omega_{\mathrm{D}}$) ≈ 0.54 was measured as about 20.93 J/mol K [112], which corresponds to the values of the model deformation parameters ($q_1 = 0.64$) and ($q_2 = 0.1$) for the range (q_1, q_2) < 1, whereas it corresponds to the values of the model deformation parameters ($q_1 = 1.65$) and ($q_2 = 2$) for the range (q_1, q_2) > 1. Another example of such a comparison can be given by the experimental specific heat value of Silicon at ($k_{\mathrm{B}}T/\hbar\omega_{\mathrm{D}}$) ≈ 0.46, which was measured as about 22.50 J/mol K [112]. This corresponds to the values of the model deformation parameters ($q_1 = 0.49$) and ($q_2 = 0.1$) for the range (q_1, q_2) < 1, whereas it corresponds to the values of the model deformation parameters ($q_1 = 1.32$) and ($q_2 = 2$) for the range (q_1, q_2) > 1.

We believe that the above analysis on the low- and high-temperature behavior of the specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_{\mathrm{B}}$) of the ($q_1$, q_2)-deformed phonon gas can reveal new insights for further studies on thermal and electrical properties of materials, where anharmonic crystal interactions and imperfections may play the role of quasi-particle excitations, which can be approximated by the

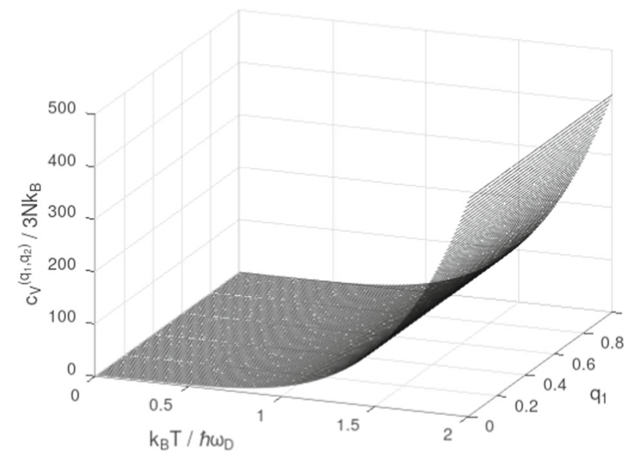

Fig. 4 The approximate (q_1, q_2)-deformed specific heat ($c_V^{(q_1,q_2)}(T)/3Nk_B$) of the (q_1, q_2)-phonon gas model as a function of ($k_BT/\hbar\omega_D$) for some value of the second deformation parameter $q_2 = 0.1$ for low temperatures for the case $q_1 < 1$

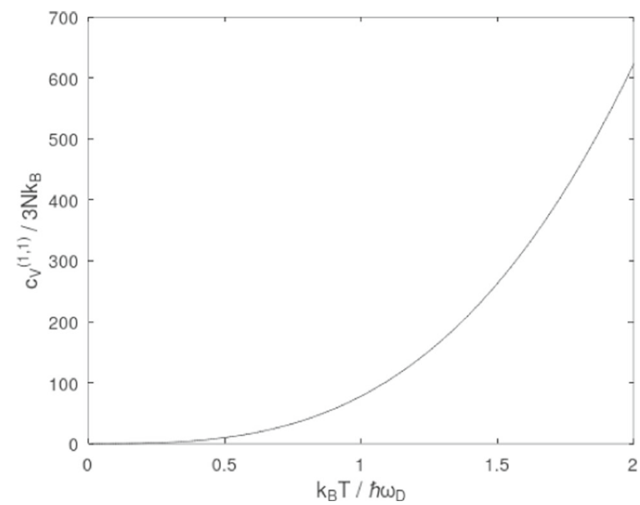

Fig. 5 The standard specific heat ($c_V^{(1,1)}(T)/3Nk_B$) for an undeformed phonon gas as a function of ($k_BT/\hbar\omega_D$) for low temperatures for the case $q_1 = q_2 = 1$

present model of Fibonacci oscillators. We further envisage that the two deformation parameters can be seen as fitting parameters between the theoretical results and the experimental data in a real phenomenological application. Here, we should stress that another remarkable effect of the model deformation parameters q_1 and q_2 can be seen as their roles for controlling the quantum statistics of a boson gas such as the one for a phonon gas developed in Eqs. (84)–(103).

Before closing this section, we should also mention that the above applicational results associated with the two-parameter (q_1, q_2)-deformed quantum statistical mechanics are different from the ones discussed in [23, 34, 35, 37, 44, 66, 67, 111, 113, 114], where the authors employed different algebraic structures on the one- and two-parameter deformed oscillator systems with or without quantum group symmetry as well as the non-extensive Tsallis thermostatistics.

6 The two-parameter (q_1, q_2)-deformed non-extensive entropy

Now we deal with a construction on the two-parameter (q_1, q_2)-deformed entropy through some elements of Fibonacci calculus. In this context, we should mention that although possible connections between the thermostatistical properties of a gas model of bosonic Fibonacci oscillators and the properties of the non-extensive Tsallis thermostatistics have been discussed to some extent in [115, 116], here we continue to further study on the two-parameter deformed entropy in order to deduce new implications based on the Fibonacci calculus. It is known that the Tsallis entropy $S_{\tilde{q}}^T$ was introduced by the following relation [61, 62]:

$$S_{\tilde{q}}^T = -k_B \sum_i \frac{P_i^{\tilde{q}} - P_i}{\tilde{q} - 1}, \tag{104}$$

where k_B is the Boltzmann constant. Recently, using the Jackson's q-calculus [57, 58], the Tsallis entropy in Eq. (104) has been rewritten by Abe [69] as

$$S_q^T = -k_B \frac{d'}{d(\alpha; q)} \sum_i (P_i)^\alpha \Big|_{\alpha=1}, \tag{105}$$

where in accordance with Eq. (104), the entropic index in the Tsallis formalism is assumed as $\tilde{q} \equiv q$, and $d'/d(\alpha; q)$ is the JD operator defined as

$$\frac{d' f(\alpha)}{d(\alpha; q)} = \frac{f(q\alpha) - f(\alpha)}{q\alpha - \alpha}. \tag{106}$$

It is worth to emphasize here that as is discussed in the first section, the JD operator in Eqs. (3) or (106) is connected with the basic number definition of the AC-type q-bosonic oscillator algebra in Eqs. (1) and (2).

Inspired by the work of Abe [69], we introduce a two-parameter deformed entropy function as

$$S^{(q_1,q_2)} = -k_B \hat{D}_\alpha^{(q_1,q_2)} \left\{ \sum_{i=1}^{W} (P_i)^\alpha \Big|_{\alpha=1} \right\}, \tag{107}$$

where the modified Fibonacci difference operator $\widehat{D}_\alpha^{(q_1,q_2)}$ is given in Eq. (29) with the variable α instead of x. Based on the non-additivity property given in Eq. (39), we know that the two-parameter deformed entropy function $S^{(q_1,q_2)}$ exhibits a non-extensive entropic structure except that the limit $q_1 = q_2 = 1$. Hence, from Eq. (107), the two-parameter deformed non-extensive entropy $S^{(q_1,q_2)}$ can be rewritten as

$$S^{(q_1,q_2)} = -k_B \left\{ \frac{\left(q_1^2 - q_2^2\right)}{\left[\ln\left(q_1^2/q_2^2\right)\right]} \right\} \sum_{i=1}^{W} \frac{(P_i)^{q_1^2} - (P_i)^{q_2^2}}{\left(q_1^2 - q_2^2\right)}, \tag{108}$$

which manifests the invariance under the operation$q_1 \leftrightarrow q_2$. Note also that considering the limit $q_1 = \sqrt{\tilde{q}}$, $q_2 = 1$, up to some multiplicative factor$[(\widetilde{q} - 1)/\ln\widetilde{q}]$, the two-parameter generalized entropy function in Eq. (108) reduces to the Tsallis entropy function as in Eq. (104).

We can further analyze the two-parameter deformed non-extensive entropy function in Eq. (108), such that it can be connected to the Tsallis entropy function in Eq. (104). Therefore, we deduce such a possible connection as follows:

$$S^{(q_1,q_2)} = \frac{\left[\left(q_1^2 - 1\right) S^T_{q_1^2} - \left(q_2^2 - 1\right) S^T_{q_2^2}\right]}{\left[\ln\left(q_1^2/q_2^2\right)\right]}. \tag{109}$$

In this regard, we should also remark that the two-parameter deformed non-extensive entropy function $S^{(q_1,q_2)}$ in Eq. (108) is different from the ones studied in [69, 117]. Since our analysis is based on both the Fibonacci oscillator algebra and some elements of Fibonacci calculus discussed in Sects. 2 and 3.

As a result, the above analysis allows one to conclude that both the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29) and the Fibonacci basic number $[n]_{q_1,q_2}$ in Eq. (14) can play some crucial roles to construct a self-consistent deformed formalism for developing the two-parameter generalized non-extensive thermostatistics. This notion also can pave the way for new insights for further studies on a possible construction of deformed quantum thermodynamics as well.

In the last section, our concluding remarks on the two-parameter model approach developed here as well as its other potential applications will be discussed.

7 Discussion and conclusions

As is extensively discussed in the first section, in the recent years, there have been growing interest in studies on different schemes of deformed formalism beyond the standard quantum mechanics, such as the $\widetilde{q}$-deformed quantum theory based on the non-extensive Tsallis statistics, the quantum mechanics with Dunkl derivative and the quantum theory with quantum group field algebras depending on the Jackson's q-calculus, since they can particularly provide much insight into the research fields of condensed matter physics, thermodynamics and cosmology. It is worth pointing out that all these deformed schemes on quantum mechanics are in essence established by using one deformation parameter. In this sense, the two-parameter ones, especially based on the quantum group field algebras, have relatively been less studied and less applied to pure quantum mechanical issues in the literature.

This work first aimed to fill this gap in the literature by constructing a new formalism for the two-parameter (q_1, q_2)-deformed quantum mechanics based on some elements of the Fibonacci calculus and its associated oscillator algebra. In this framework, we introduced the necessary postulates for the two-parameter (q_1, q_2)-deformed quantum mechanics, where the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29) has a central role. We then formulated the (q_1, q_2)-deformed Heisenberg algebra. Using this algebraic structure, we found the (q_1, q_2)-deformed time-dependent Schrödinger equation as in Eq. (46).

The second objective of the present work was to observe possible consequences of introducing two deformation parameters on some physical models. In this regard, we carried out three different applications of the current two-parameter approach covering the free particle, the inverse-harmonic potential case and thermostatistics of the (q_1, q_2)-deformed Debye solid. By means of the time-independent (q_1,q_2)-deformed Schrödinger equation given in Eq. (48), we derived the wave functions for both the free particle and the inverse-harmonic potential cases depending on some functions of the deformation parameters q_1 and q_2.

The third objective of this work was to further study on the two-parameter deformed non-extensive entropy. This was addressed in Sect. 6 by considering some elements of the Fibonacci calculus and expressing it in terms of the operator $\widehat{D}_x^{(q_1,q_2)}$ as in Eqs. (107) and (108). In addition to this, we revisited the main properties of the two-parameter deformed entropy function $S^{(q_1,q_2)}$ and discussed its possible connection with the Tsallis entropy function S_q^T via a relation given in Eq. (109).

Moreover, we applied our two-parameter model approach onto the case of inverse-harmonic potential as studied in Sect. 5.2. As far as we know from the literature that this is the first attempt to address such a potential field in connection with the (q_1,q_2)-deformed Schrödinger equation in Eq. (48) based on the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29). Accordingly, we introduced the (q_1,q_2)-deformed Bessel differential equation as in Eq. (68), which leads to a solution involving the (q_1,q_2)-deformed Bessel function. The insertion of the modified Fibonacci difference operator $\widehat{D}_x^{(q_1,q_2)}$ in Eq. (29) into the quantum harmonic oscillator problem was also discussed by [95], where the (q_1,q_2)-deformed Hermite polynomials through the use of bosonic Fibonacci oscillators in Eq. (13) and (14) were calculated. In addition to this, we know from the literature that the ground-state solution of the harmonic plus inverse-harmonic potential in the corresponding Schrödinger equation was introduced for modeling the fusion process of two identical nuclei in [118]. Hence, we conjecture that our results in Eqs. (63)–(79) may gain new insight for further studies on interacting theories of nuclear structures as well as in the inclusion of impurity factors in materials.

Besides, as is studied in Sect. 5.3, we applied our two-parameter deformed construction on some quantum statistical features of a deformed boson gas. In this context, we studied the thermostatistics of (q_1,q_2)-deformed Debye solid and we found many of the thermostatistical functions of the model in terms of the deformation parameters (q_1,q_2) and compared it with the results of usual (undeformed) Bose gas having the case $q_1 = q_2 = 1$. As shown in Figs. 1 and 2, the modification of the specific heat $c_V^{(q_1,q_2)}(T)$ increasing with the values of the deformation parameters q_1 and q_2 becomes more prominent in the (q_1,q_2)-phonon gas model. When we compare with the result of the classical theory in Eq. (101), it is interestingly found that the specific heat $c_V^{(q_1,q_2)}(T)$ can have lower values depending on the values of the model deformation parameters q_1 and q_2 as shown in Figs. 1 and 2. Such an observation may gain new insights for further studies on intermediate-statistics in thermoelectric properties of materials, which hopefully contain new or exotic quantum states in different physical regimes associated with the (q_1,q_2)-deformation in the system under consideration.

On the other hand, as is visualized in Figs. 3, 4 and 5, we analyzed the low-temperature behavior of the specific heat of the (q_1,q_2)-deformed Debye solid. We found that when compared with the values of the specific heat of an undeformed phonon gas shown in Fig. 5, at the same value of $(k_B T/\hbar\omega_D)$, the values of the specific heat $(c_V^{(q_1,q_2)}(T)/3Nk_B)$ for the case $(q_1, q_2) > 1$ in Fig. 3 are larger than the results of an undeformed specific heat $(c_V^{(1,1)}(T)/3Nk_B)$, whereas it has lower values than those of the function $(c_V^{(1,1)}(T)/3Nk_B)$ for the case $(q_1, q_2) < 1$ as shown in Fig. 4. Our results may also be contrasted with a real data on the experimental specific heat capacity of a given material. For instance, the experimental specific heat value of Germanium at $(k_B T/\hbar\omega_D) \approx 0.54$ was measured as about 20.93 J/mol K [112], which corresponds to the values of the model deformation parameters $(q_1 = 0.64)$ and $(q_2 = 0.1)$ for the range $(q_1, q_2) < 1$, whereas it corresponds to the values of the model deformation parameters $(q_1 = 1.65)$ and $(q_2 = 2)$ for the range $(q_1, q_2) > 1$. Another example of such a comparison can be given by the experimental specific heat value of Silicon at $(k_B T/\hbar\omega_D) \approx 0.46$, which was measured as about 22.50 J/mol K [112]. This corresponds to the values of the model deformation parameters $(q_1 = 0.49)$ and $(q_2 = 0.1)$ for the range $(q_1, q_2) < 1$, whereas it corresponds to the values of the model deformation parameters $(q_1 = 1.32)$ and $(q_2 = 2)$ for the range $(q_1, q_2) > 1$. In this context, we note that our findings are different from the results of the works of [44, 113], where different realizations of the q- and qp-deformed quantum algebras were employed. We should also notice that our results in Sect. 5.3 differ from the ones in [34, 35] such that they considered different definitions of both the model Hamiltonian and the Debye function. Hence, we should emphasize that one other distinct feature of our construction comes from the inclusion of a form of the (q_1,q_2)-deformed statistical distribution function as in Eq. (84). In this sense, our two-parameter model approach would be more effective to deal with nonlinearities in quantum systems, where collective excitations can induce some exotic quasi-particle states. This kind of states can even entail to exploit some form of the non-standard quantum statistics such as in Eq. (84).

From the analysis discussed above, we should remark that studying with two deformation parameters could give more flexibility when dealing with phenomenological applications for the analysis of complex systems. The present model approach also provides greater possibility to fit experimental data with theoretical ones. More specifically, our results support the idea that by inserting two deformation parameters q_1 and q_2 rather than of the one-parameter case increases the adjustment range on a physical quantity under consideration to calibrate some discrepancy between the theory-based results, namely expectations values, and experimental data on the same quantity.

Yet another potential application area of the present (q_1,q_2)-deformed quantum mechanical formalism, one can use it to study further the quantum statistical behavior of excitons inside a nanomaterial. Since it can be approximated as a deformed bosonic gas in the high-density limit. Here, it is also noteworthy to remark that even in the interacting scenario, our two-parameter deformed formalism can be used to deduce different characteristics for analyzing a nonlinearity in the system under consideration. Beside the application to an exciton gas, the present two-parameter deformed formalism can also be applied to the electronic structure calculations in solids as well as in metals and semiconductors. However, the energy gap between the valance and conduction bands in metallic substances is strongly sensitive to collective excitations. From the quantum mechanical perspective, the deformation parameters q_1 and q_2 may give some clues about the details of interaction dynamics in such many-body particle systems, which in turn may help to understand the nature of existence of quasi-particle bound states.

In addition, the Kondo effect and the Bose polaron model can be studied through the use of some properties of the two-parameter (q_1,q_2)-deformed quantum mechanics, in which quasi-particles occurred in such physical effects can be viewed as either (q_1,q_2)-deformed bosons or (q_1,q_2)-deformed fermions obeying the bosonic or fermionic Fibonacci oscillator algebras in Eqs. (13)–(16) or (26), (27). Such an approach could possibly provide not only in better understanding of the details of interaction mechanism between quasi-particles such as phonons and ordinary (undeformed) particles such as electrons, but also in simplifying algebraic and representative calculations in the system under consideration.

Last but not the least, when we consider the recent developments on some other applications of quantum deformed algebraic structures such as in analyzing some electronic properties of quasi-crystals modeled by the Fibonacci quasi-crystal in one dimension [119] and in discussing different symmetry properties occurred in topological insulators as well as in Bloch electron systems such as in [120], possible new applications of the present two-parameter deformed quantum mechanical formalism are expected to use in the same lines of research as pointed out above.

In conclusion, we hope that the present construction on the (q_1,q_2)-deformed quantum mechanics could have new implications into interacting theories of bosons and fermions, and it may be useful for further studies on condensed matter theory-based materials design.

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