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Investigating the new perspectives of Caudrey–Dodd–Gibbon equation arising in quantum field theory

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Abstract
The main purpose of the paper is obtaining the analytical results for beta fractional Caudrey–Dodd–Gibbon equation which is used to resolve complex problems in fluid dynamics, chemical kinetics, plasma physics, quantum field theory, crystal dislocations, and nonlinear optics by using auxiliary method. Beta derivative is a useful fractional operator due to satisfying basic properties of integer order derivative and also, allows us using chain rule and wave transform to turn nonlinear fractional partial differential equations into integer order ordinary differential equations. By the way many analytical methods can be applied to these equations. In order to understand the physical features of the solutions, 3D and 2D graphical illustrations are given. Finally, authors expect that the obtained solutions may give a deep insight for the explanation of physical phenomena in the fluid dynamics, chemical kinetics, plasma physics, quantum field theory, crystal dislocations, and nonlinear optics.

Keywords Beta derivative · Auxiliary equation method · Analytical solution

1 Introduction

Over the past few decades, a growing number of researchers have been paying attention to obtain the exact solutions of fractional nonlinear partial differential equations (FNLPDEs). Because acquiring the exact solutions of FNLPDEs takes a significant role in the explanation and examination of physical phenomena. The FNLPDEs has applications in

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a wide range of disciplines such as physical sciences, fluid mechanics, plasma physics, oceanography, astrophysics, chemical physics, chemistry, ocean engineering, biomedical engineering and many other engineering areas.

In order to achieving the solutions of FNLPDEs, efficient and reliable methods have been emerged in recent years. For example Jacobi elliptic function expansion method (Tasbozan et al. 2016), Backlund transformation (Huang and Yang 2019), the simplest equation method (Chen and Jiang 2018), the modified extended tanh function method (Dubey and Chakraverty 2022), the G'/G-expansion method (Zheng 2012), the variational iteration method (Odibat and Momani 2009), the Adomian decomposition method (Duan et al. 2012; Daftardar-Gejji and Jafari 2005), the homotopy perturbation method (Wang 2007), the F-expansion method (Wang et al. 2022), the Cole–Hopf transformation (Li and Li 2020), the exponential rational function method (Ekici and Ünal 2020) and etc.

Using fractional derivatives gives us great achievements. For instance models expressed by using fractional derivatives coincides with the experimental results of considered model. The historical dependence of the evolution of system analysis can be expressed by fractional calculus easily. Some of the popular fractional derivative definitions can be stated as Khalil et al. (2014):

Definition 1.1 *(Riemann–Liouville Definition)* For $\alpha \in [n-1, n)$, the α derivative of function f is

$$D_a^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

Definition 1.2 *(Caputo Definition)* or $\alpha \in [n-1, n)$, the α derivative of function f is

$$D_a^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

Many applications for different types of fractional derivatives are made by various scientists (Özkan and Akar 2022; Wang et al. 2021). For instance Abdelwahed et al. (2023) applied the unified technique for nonlinear Schrödinger equations to extract new structures of waves where the derivatives are in conformable sense. The residual power series method for solving nonlinear Caputo time fractional reaction–diffusion equations is introduced by Tchier et al. (2016). In the study of Abdelwahed et al. (2022), they aimed to introduce new closed form of solutions to the fractional ion sound and Langmuir waves with conformable derivative. Fan et al. (2023) applied three distinct methods to obtain semi-analytical solutions for fractional Fitzhugh–Nagumo equation with Caputo fractional derivative.

Although many studies have been done on the fractional derivative, there are some flaws of commonly used fractional derivative definitions (Khalil et al. 2014). For example derivative of a constant is not zero with Riemann–Liouville definition. Also the initial/boundary conditions of considered mathematical models involving Riemann–Liouville derivative must be expressed with fractional order. Moreover, some basic properties(the derivative of quotient of two functions, the derivative of product of two functions, chain rule and etc.) of Newtonian concept derivative are not satisfied by these popular derivative definitions. In order to avoid these limitations, the definition of beta fractional derivative is expressed. This definition obeys the basic properties of conventional derivative definition. The definition of new fractional derivative and some basic properties are given below.

Definition 1.3 Let g be a function, such that $f:[a,\infty)\rightarrow\mathbb{R}$. The beta derivative of a function g is described as

$$D_x^{\beta}(g(x))=\lim_{\epsilon\rightarrow 0}\frac{g\left(x+\epsilon(x+1/\Gamma(\beta))^{1-\beta}\right)-g(x)}{\epsilon}$$

for all $x\geq a$, $\beta\in[0,1]$. When the above limit exists, it can be said that the functions g is beta differentiable(Atangana and Doungmo Goufo 2014; Atangana et al. 2016; Atangana 2015).

Following theorem gives some properties of beta derivative.

Theorem 1.1 *Let f, g are beta differentiable functions and $g\neq 0$ where $\beta\in(0,1]$. Then following properties are satisfied (Atangana and Doungmo Goufo 2014; Atangana et al. 2016; Atangana 2015).*

- $D_x^{\beta}(af(x)+bg(x))=aD_x^{\beta}(af(x))+bD_x^{\beta}(af(x))$ where $a,b\in\mathbb{R}$.
- $D_x^{\beta}(c)=0$ where c is constant.
- $D_x^{\beta}(f(x)g(x))=g(x)(D_x^{\beta}(f(x)))+f(x)D_x^{\beta}(g(x))$.
- $D_x^{\beta}(f(x)/g(x))=\frac{g(x)(D_x^{\beta}(f(x)))-f(x)D_x^{\beta}(g(x))}{g^2(x)}$.

Theorem 1.2 (Atangana and Doungmo Goufo 2014; Atangana et al. 2016; Atangana 2015) *$f,g:[a,\infty)\rightarrow\mathbb{R}$ are function and both of them are differentiable also f is beta differentiable. Then following rule is provided.*

$$D_x^{\beta}(gof(x))=\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}f'(x)g'(f(x)).$$

Definition 1.4 (Atangana and Doungmo Goufo 2014; Atangana et al. 2016; Atangana 2015) $g:[a,\infty)\rightarrow\mathbb{R}$ is a function, beta integral of f is as follows.

$$I_x^{\beta}(g(x))=\int_a^x\left(\tau+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}g(\tau)d\tau.$$

There are many studies made including mathematical models where the derivative or integral in beta sense (Akar and Özkan 2023; Ozkan et al. 2023; Özkan and Özkan 2021; Ozkan 2022; Elsherbeny et al. 2023; Zafar et al. 2023). The main motivation for these studies is that the fractional derivative of the beta derivative is more applicable than other fractional derivatives and provides the basic properties of the Newtonian concept derivative.

In these days, getting exact wave solutions of NLPDEs become more accessible by means of the swift development of mathematical techniques with computer software's such as Mathematica, Maple and Matlab which makes complicated algebraic calculations possible on computer.

Nonlinear optics let us to alter the color of a beam of light, differ its shape in space and time and generate the shortest events ever made by humans. Nonlinear optical phenomena corresponds to the basis of many components of optical communication systems, optical sensing, and materials research. Because of these importance we consider the time fractional Caudrey–Dodd–Gibbon equation (Singh et al. 2022) as

$$\frac{\partial^\beta u}{\partial t^\beta}+\frac{\partial^5 u}{\partial x^5}+30u\frac{\partial^3 u}{\partial x^3}+30\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+180u^2\frac{\partial u}{\partial x}=0 \tag{1}$$

which corresponds to model in fluid dynamics, chemical kinetics, plasma physics, quantum field theory, crystal dislocations, and nonlinear optics. Also, fractional model of regarded equations is studied as a mathematical model of the internal waves in a shallow density-stratified fluid and surface waves of small amplitude and long wave length on shallow water. It plays a very significant role in plasma physics and laser optics too.

Many authors studied to get its solitary solutions (Wazwaz 2006; Neamaty et al. 2016). Dark and soliton solutions of considered equations with truncated $\mathbb{M}$ fractional derivative are given in Ref. Majeed et al. (2022), it's semi-analytical solutions acquired by natural decomposition method with Atangana–Baleanu fractional operator (Fathima et al. 2023) and some other methods applied to solve this equation such as the q-homotopy analysis transform method (Veeresha and Prakasha 2020) and homotopy analysis Sumudu transform method (Singh et al. 2022) and etc.

In this paper the exact solutions of fractional Caudrey–Dodd–Gibbon equation is obtained by using auxiliary equation method as a tool where the fractional derivative is in beta sense. All the calculations are made by the help of computer software called Wolfram Mathematics. To the best of our knowledge acquired results including beta derivative are never seen in the literature before. These solutions may give a new perspective to the scientists studying on fluid dynamics, chemical kinetics, plasma physics, quantum field theory, crystal dislocations, and nonlinear optics and laser optics. In addition, scientists will be able to compare the concordance between experimental data and theoretical knowledge.

2 Basic idea of auxiliary equation method

Before explaining the application procedure of auxiliary equation method, the balancing procedure (Malfliet 1992) which is the basically used in analytical techniques can be explained as follows.

The balancing procedure describes finding the upper bound of the complete solution given in total. In a nonlinear ordinary differential equation, a constant number is obtained between the highest order linear term and the highest degree nonlinear term. Let the highest order linear term be $\frac{d^q u}{d\xi^q}$ and the highest order nonlinear term $u^p\left(\frac{d^r u}{d\xi^r}\right)^s$. If $u=\tau^N$ transform is made balancing relation is obtained as $N+q=Np+s(N+r)$, where p, q, r, s are positive integers and N is balancing number. From this equation, N positive balancing number is obtained (Tasbozan et al. 2016).

Auxiliary equation method is utilizable to achieve exact solutions for NLPDEs (Sirendaoreji and Jiong 2003; Abdou 2008; Akbulut and Kaplan 2018; Ma et al. 2009; Durur et al. 2020; Yılmaz and Tasbozan 2020; Göktaş et al. 2022). In order to implement auxiliary equation method to NPLDEs, equation must be formed by only even-order or only odd-order partial derivative terms.

Now let us give a brief description of auxiliary equation method. Assume that the a FNLPDE is given as

$$H\left(u,\frac{\partial u}{\partial x},\frac{\partial^\beta u}{\partial t^\beta},\frac{\partial^2 u}{\partial x^2},\frac{\partial^{2\beta} u}{\partial t^{2\beta}},\dots\right)=0 \tag{2}$$

where $\frac{\partial^{2\beta}u}{\partial t^{2\beta}}$ denotes two times sequential beta fractional derivative of unknown function $u(x,t)$. Through to chain rule and $\xi = mx + n\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}$ wave transformation, FNLPDE turns into integer order nonlinear differential equation as

$$G\left(U, U_{\xi}, U_{\xi\xi}, U_{\xi\xi\xi}, \dots\right) = 0. \tag{3}$$

On the other hand auxiliary equation method is based on the ensuing differential equation

$$\left(\frac{dz}{d\xi}\right)^{2} = az^{2}(\xi) + bz^{3}(\xi) + cz^{4}(\xi) \tag{4}$$

where a, b, c are real constants and in $u(\xi)$ can shown as

$$U(\xi) = \sum_{i=0}^{N} a_i z^i(\xi) \tag{5}$$

also, $z(\xi)$ is solution of nonlinear differential equation (4), a_i is real constant an N is a positive integer to be determined by the balancing procedure (Malfliet 1992). By the way, suitable $z(\xi)$ values can be selected from following forms with $\varepsilon = \pm 1$ and $\Delta = b^2 - 4ac$.

1) $\frac{-ab\,\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2-ac\left(1+\varepsilon\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)}, \quad a > 0$

2) $\frac{abc\,\mathrm{csch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2-ac\left(1+\varepsilon\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)}, \quad a > 0$

3) $\frac{2a\,\mathrm{sech}\left(\sqrt{a}\xi\right)}{\varepsilon\sqrt{\Delta}-b\,\mathrm{sech}\left(\sqrt{a}\xi\right)}, \quad a > 0, \Delta > 0$

4) $\frac{2a\sec\left(\sqrt{-a}\xi\right)}{\varepsilon\sqrt{\Delta}-b\sec\left(\sqrt{-a}\xi\right)}, \quad a < 0, \Delta > 0$

5) $\frac{2a\,\mathrm{csch}\left(\sqrt{a}\xi\right)}{\varepsilon\sqrt{-\Delta}-b\,\mathrm{csch}\left(\sqrt{a}\xi\right)}, \quad a > 0, \Delta < 0$

6) $\frac{2a\csc\left(\sqrt{-a}\xi\right)}{\varepsilon\sqrt{\Delta}-b\csc\left(\sqrt{-a}\xi\right)}, \quad a < 0, \Delta > 0$

7) $\frac{-a\,\mathrm{sech}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b+2\varepsilon\sqrt{ac}\tanh\left(\frac{\sqrt{a}}{2}\xi\right)}, \quad a > 0, c > 0$

8) $\frac{-a\sec^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b+2\varepsilon\sqrt{-ac}\tan\left(\frac{\sqrt{-a}}{2}\xi\right)}, \quad a < 0, c > 0$

9) $\frac{a\,\mathrm{csch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b+2\varepsilon\sqrt{ac}\coth\left(\frac{\sqrt{a}}{2}\xi\right)}, \quad a > 0, c > 0$

10) $\frac{-a\csc^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b+2\varepsilon\sqrt{-ac}\cot\left(\frac{\sqrt{-a}}{2}\xi\right)} \quad a < 0, c > 0$

11) $-\frac{a}{b}\left[1+\varepsilon\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right], \quad a > 0, \Delta = 0$

12) $-\frac{a}{b}\left[1+\varepsilon\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right], \quad a > 0, \Delta = 0$

13) $\frac{4ae^{\varepsilon\sqrt{a}\xi}}{\left(e^{\varepsilon\sqrt{a}\xi}-b\right)^2-4ac}, \quad a > 0$

14) $\frac{\pm 4ac\varepsilon e^{\varepsilon\sqrt{a}\xi}}{1-4ace^{2\varepsilon\sqrt{a}\xi}}, \quad a > 0, b = 0$

3 Application of the method

We consider the time-fractional Caudrey–Dodd–Gibbon equation with β-derivative

$$D_x^5 u + 30uD_x^3 u + 30D_x u D_x^2 u + D_t^{\beta} u + 180u^2 D_x u = 0 \tag{6}$$

where u is dependent, x and t are independent variables. By using the chain rule and $\xi = mx + n\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}$ wave transform with some basic operations we get

$$u(x,t) = U(\xi)$$

and integration once, the equation turns into integer order ordinary differential equation as follows

$$m^5U^{(4)} + 30m^3UU'' + nU + 60mU^3 = 0. \tag{7}$$

where U' denotes the derivative of function U with respect to new variable ξ. We acquire $N = 2$ in Eq. (7) by using balancing procedure. Hence, we get

$$U(\xi) = a_0 + a_1z(\xi) + a_2z^2(\xi). \tag{8}$$

Finally, putting equation (8) into Eq. (7) with Eq. (4), we obtain an algebraic equation witch respects to a_0, a_1, a_2, m, n and if we equate all the coefficient of same powers of $z(\xi)$ to 0 it gives an algebraic equation system. Solving the system and finding unknown values a_0, a_1, a_2, m, n, and putting them in (8) with suitable values of $z(\xi)$ following solutions sets arises,

$$u_{1,2} = \frac{\left(15-\sqrt{105}\right)a_0\operatorname{sech}^4\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{\left(1-\frac{\left(1\pm\tanh\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}{4}\right)^2}$$

$$a_0 + \frac{\left(\sqrt{105}-15\right)a_0\operatorname{sech}^2\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{2\left(1-\frac{\left(1\pm\tanh\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}{480a_0^5n^2}\right)}$$

$$u_{3,4} = \frac{\left(15-\sqrt{105}\right)a_0\operatorname{csch}^4\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{\left(1-\frac{\left(1\pm\coth\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}{4}\right)^2}$$

$$a_0 - \frac{\left(\sqrt{105}-15\right)a_0\operatorname{csch}^2\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{2\left(1-\frac{\left(1\pm\coth\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}{480a_0^5n^2}\right)}$$

$$u_{5,6}=\frac{\left(15-\sqrt{105}\right)a_0\operatorname{sech}^4\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{4\left(1\pm\tanh\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}$$

$$a_0+\frac{\left(\sqrt{105}-15\right)a_0\operatorname{sech}^2\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{2\left(1\pm\tanh\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)}$$

$$u_{7,8}=\frac{\left(15-\sqrt{105}\right)a_0\sec^4\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{4\left(1\pm i\tan\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}$$

$$a_0+\frac{\left(\sqrt{105}-15\right)a_0\sec^2\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{2\left(1\pm i\tan\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)}$$

$$u_{9,10}=\frac{\left(15-\sqrt{105}\right)a_0\operatorname{csch}^4\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{4\left(1\pm\coth\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}$$

$$a_0-\frac{\left(\sqrt{105}-15\right)a_0\operatorname{csch}^2\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{2\left(1\pm\coth\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)}$$

$$u_{11,12}=\frac{\left(15-\sqrt{105}\right)a_0\csc^4\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{4\left(1\pm i\cot\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}$$

$$a_0+\frac{\left(\sqrt{105}-15\right)a_0\csc^2\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)}{2\left(1\pm i\cot\left(30a_0^2\sqrt{a_0\left(15-\sqrt{105}\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)}$$

$$u_{13,14}=\frac{\left(15-\sqrt{105}\right)a_0\left(1\pm\tanh\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}{4}$$

$$a_0+\frac{\left(\sqrt{105}-15\right)a_0\left(1\pm\tanh\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)}{2}$$

$$u_{15,16}=\frac{\left(15-\sqrt{105}\right)a_0\left(1\pm\coth\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)^2}{4}$$

$$a_0+\frac{\left(\sqrt{105}-15\right)a_0\left(1\pm\coth\left(30a_0^2\sqrt{a_0\left(\sqrt{105}-15\right)}\left(\frac{\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{\beta}-\frac{x}{60a_0^2}\right)\right)\right)}{2}$$

3.1 Results and discussion

It is understood that auxiliary equation method is an efficient, applicable and effective method to solve FNLPDE with beta derivative. Also in this part, both 3D and 2D graphical illustrations for the solutions of Eq. 6 are given. As it can be seen from the Figs. 1, 2, 3 and 4, dark soliton, singular soliton, and traveling wave solutions are obtained. The term soliton is used in optics to refer to any optical field that does not change during propagation due to a delicate balance between nonlinear and linear effects present in the medium. Figures 5, 6 and 7 show the effect of β to the solutions with different values.

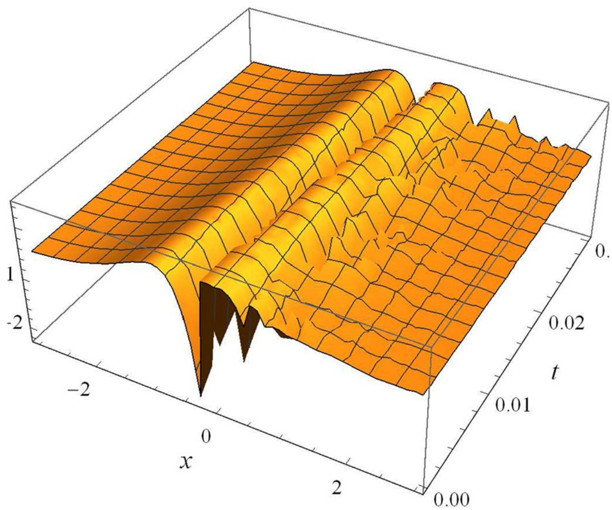

Fig. 1 The 3D surface of the solution $u_1(x,t)$ at $a_0=-0.55, \beta=0.65$ gives a dark soliton solution

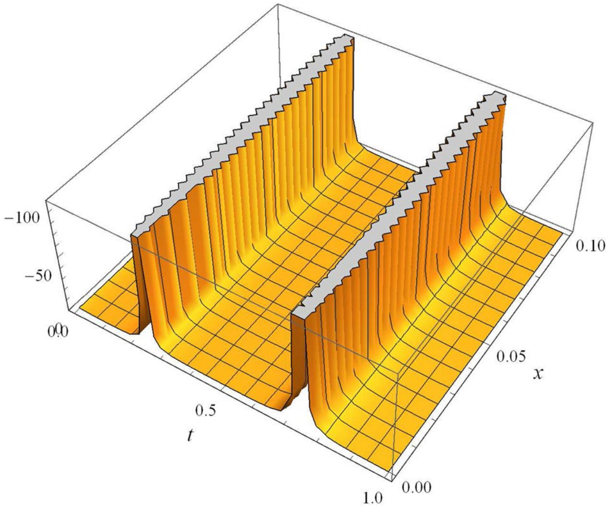

Fig. 2 The 3D surface of the solution $u_5(x, t)$ at $a_0 = 0.36, \beta = 0.25$ gives a singular soliton solution

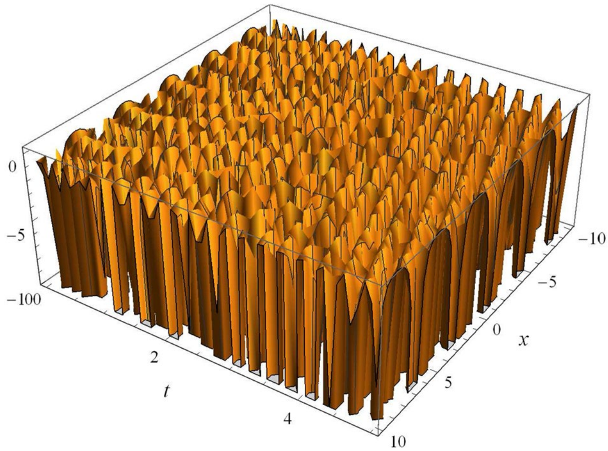

Fig. 3 The 3D surface of the solution $u_7(x, t)$ at $a_0 = 0.6, beta = 0.7$

4 Conclusion

In this work, an auxiliary equation method is used to acquire the new exact solutions of time fractional Caudrey–Dodd–Gibbon equation arising in nonlinear optics. So singular soliton, dark soliton and traveling wave solutions are obtained. These solutions are firstly seen in the literature and may give a new perspective to the scientists studying on fluid dynamics, chemical kinetics, plasma physics, quantum field theory, crystal dislocations, and nonlinear optics, plasma physics and laser optics. This research reveals that the auxiliary equation method provides a reliable, effective and useful way for solving other FNLPDEs in mathematical physics. On the other hand, using the beta fractional

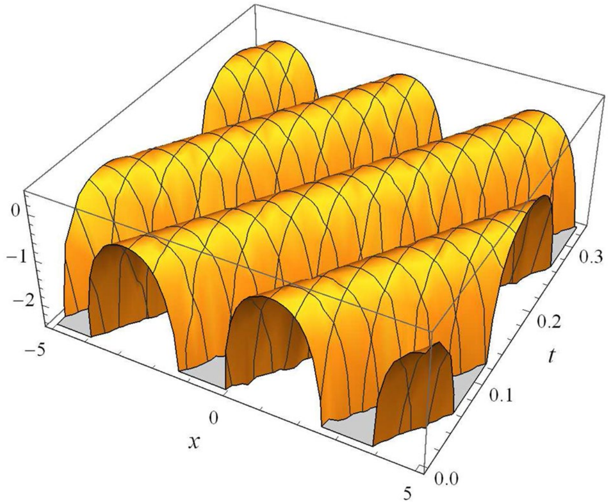

Fig. 4 The 3D surface of the solution $u_9(x,t)$ at $a_0 = 0.55, \beta = 0.65$

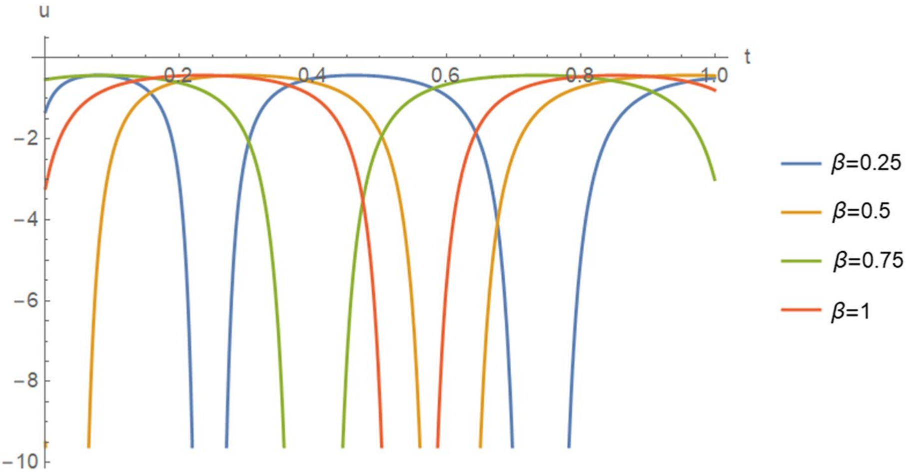

Fig. 5 The 2D graphical simulation of the solution $u_1(x,t)$ at $a_0 = -0.55, x = 0$

derivative gives us some conveniences such as using the chain rule which isn't satisfied by Caputo and Reimann–Liouville derivative definitions. Finally, the calculations were made with the computer software called Wolfram Mathematica. By the given graphical illustrations the scientists who study on optics can interpret the physical behavior of the solutions. This article may give a new insight to the scientists on fractional optical modeling. Many different models of optical theory involving beta fractional derivative can be discussed by using auxiliary equation method to express the physical nature and behavior of the solutions.

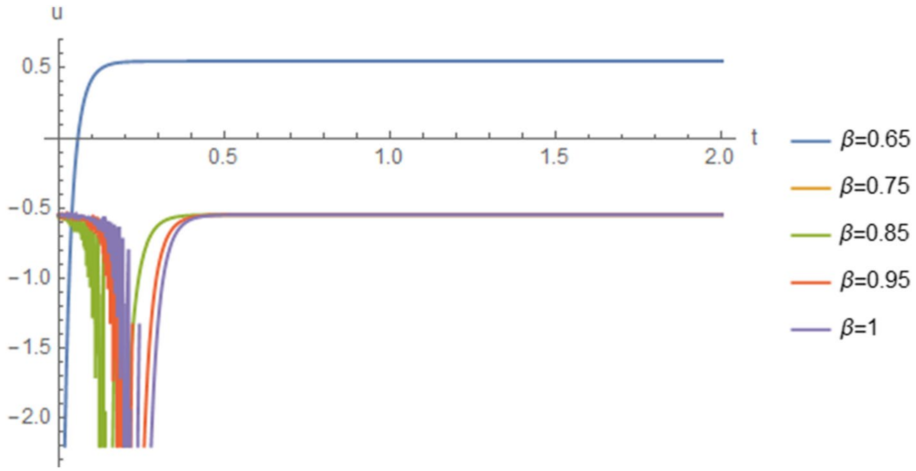

Fig. 6 The 2D graphical simulation of the solution $u_5(x,t)$ at $a_0 = 0.36, x = 0$

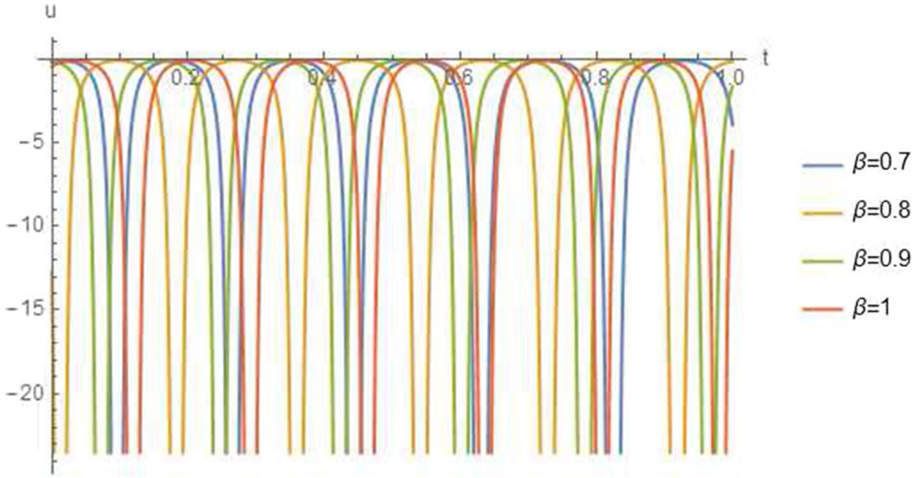

Fig. 7 The 2D graphical representation of the solution $u_7(x,t)$ at $a_0 = 0.6, x = 0$

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

Abdelwahed, H.G., Alsarhan, A.F., El-Shewy, E.K., Abdelrahman, M A.: Novel explosive and super fractional nonlinear Schrödinger structures. J. Math. 2023 (2023)

Abdelwahed, H.G., El-Shewy, E.K., Alghanim, S., Abdelrahman, M.A.: On the physical fractional modulations on Langmuir plasma structures. Fractal Fract. **6**(8), 430 (2022)

Abdou, M.A.: A generalized auxiliary equation method and its applications. Nonlinear Dyn. **52**(1), 95–102 (2008)

Akar, M., Özkan, E.M.: On exact solutions of the $(2 + 1)$-dimensional time conformable Maccari system. Int. J. Mod. Phys. B **37**, 2350219 (2023)

Akbulut, A., Kaplan, M.: Auxiliary equation method for time-fractional differential equations with conformable derivative. Comput. Math. Appl. **75**, 876–882 (2018)

Atangana, A.: A novel model for the lassa hemorrhagic fever: deathly disease for pregnant women. Neural Comput. Appl. **26**, 1895–1903 (2015)

Atangana, A., Doungmo Goufo, E.F.: Extension of matched asymptotic method to fractional boundary layers problems. Math. Probl. Eng. **2014**, 1–7 (2014)

Atangana, A., Baleanu, D., Alsaedi, A.: Analysis of time-fractional Hunter-Saxton equation: a model of neumatic liquid crystal. Open Phys. **14**(1), 145–149 (2016)

Chen, C., Jiang, Y.L.: Simplest equation method for some time-fractional partial differential equations with conformable derivative. Comput. Math. Appl. **75**(8), 2978–2988 (2018)

Daftardar-Gejji, V., Jafari, H.: Adomian decomposition: a tool for solving a system of fractional differential equations. J. Math. Anal. Appl. **301**(2), 508–518 (2005)

Duan, J.S., Rach, R., Baleanu, D., Wazwaz, A.M.: A review of the Adomian decomposition method and its applications to fractional differential equations. Commun. Fract. Calc. **3**(2), 73–99 (2012)

Dubey, S., Chakraverty, S.: Application of modified extended tanh method in solving fractional order coupled wave equations. Math. Comput. Simul. **198**, 509–520 (2022)

Durur, H., Tasbozan, O., Kurt, A.: New analytical solutions of conformable time fractional bad and good modified Boussinesq equations. Appl. Math. Nonlinear Sci. **5**, 447–454 (2020)

Ekici, M., Ünal, M.: Application of the exponential rational function method to some fractional soliton equations. In: Emerging applications of differential equations and game theory. IGI Global. pp. 13-32 (2020)

Elsherbeny, A.M., Bekir, A., Arnous, A.H., Sadaf, M., Akram, G.: Solitons to the time-fractional Radhakrishnan–Kundu–Lakshmanan equation with β and M-truncated fractional derivatives: a comparative analysis. Opt. Quantum Electron. **55**(12), 1112 (2023)

Fan, Z.Y., Ali, K.K., Maneea, M., Inc, M., Yao, S.W.: Solution of time fractional Fitzhugh–Nagumo equation using semi analytical techniques. Results Phys. **51**, 106679 (2023)

Fathima, D., Alahmadi, R.A., Khan, A., Akhter, A., Ganie, A.H.: An efficient analytical approach to investigate fractional Caudrey–Dodd–Gibbon equations with non-singular Kernel derivatives. Symmetry **15**(4), 850 (2023)

Göktaş, M.A., Yılmaz, K., Tasbozan, O.: Analytical solutions of (1+1)- dimensional distributed long wave equation with auxiliary equation method. J. Univers. Math. **5**, 88–94 (2022)

Huang, S.J., Yang, N.J.: Auto-Bäcklund transform and exact solutions to local conformable time-fractional viscous Burgers system. Europhys. Lett. **125**(1), 15003 (2019)

Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. comput. appl. math. **264**, 65–70 (2014)

Li, L., Li, D.: Exact solutions and numerical study of time fractional Burgers' equations. Appl. Math. Lett. **100**, 106011 (2020)

Ma, H.C., Yu, Y.D., Ge, D.J.: The auxiliary equation method for solving the Zakharov Kuznetsov (ZK) equation. Comput. Math. Appl. **58**, 2523–2527 (2009)
Majeed, A., Rafiq, M.N., Kamran, M., Abbas, M., Inc, M.: Analytical solutions of the fifth-order time fractional nonlinear evolution equations by the unified method. Mod. Phys. Lett. B **36**(02), 2150546 (2022)
Malfliet, W.: Solitary wave solutions of nonlinear wave equations. Am. J. Phys. **60**(7), 650–654 (1992)
Neamaty, A., Agheli, B., Darzi, R.: Exact travelling wave solutions for some nonlinear time fractional fifth-order Caudrey–Dodd–Gibbon equation by (G'/G) -expansion method. SeMA J. **73**, 121–129 (2016)
Odibat, Z., Momani, S.: The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics. Comput. Math. Appl. **58**(11–12), 2199–2208 (2009)
Ozkan, E.M.: New exact solutions of some important nonlinear fractional partial differential Equations with beta derivative. Fractal Fract. **6**(3), 173 (2022)
Özkan, E.M., Akar, M.: Analytical solutions of (2+1)-dimensional time conformable Schrödinger equation using improved sub-equation method. Optik **267**, 169660 (2022)
Özkan, E.M., Özkan, A.: The soliton solutions for some nonlinear fractional differential equations with Beta-derivative. Axioms **10**(3), 203 (2021)
Ozkan, E.M., Yildirim, O., Ozkan, A.: On the exact solutions of optical perturbed fractional Schrödinger equation. Phys. Scr. **98**(11), 115104 (2023)
Singh, J., Gupta, A., Baleanu, D.: On the analysis of an analytical approach for fractional Caudrey–Dodd–Gibbon equations. Alex. Eng. J. **61**(7), 5073–5082 (2022)
Sirendaoreji, Jiong, S.: Auxiliary equation method for solving nonlinear partial differential equations. Phys. Lett. A **309**, 387–396 (2003)
Tasbozan, O., Çenesiz, Y., Kurt, A.: New solutions for conformable fractional Boussinesq and combined KdV-mKdV equations using Jacobi elliptic function expansion method. Eur. Phys. J. Plus **131**, 1–14 (2016)
Tasbozan, O., Çenesiz, Y., Kurt, A.: New solutions for conformable fractional Boussinesq and combined KdV-mKdV equations using Jacobi elliptic function expansion method. Eur. Phys. J. Plus **131**(7), 1–14 (2016)
Tchier, F., Inc, M., Korpinar, Z.S., Baleanu, D.: Solutions of the time fractional reaction-diffusion equations with residual power series method. Adv. Mech. Eng. **8**(10), 1687814016670867 (2016)
Veeresha, P., Prakasha, D.G.: A reliable analytical technique for fractional Caudrey–Dodd–Gibbon equation with Mittag–Leffler kernel. Nonlinear Eng. **9**(1), 319–328 (2020)
Wang, Q.: Homotopy perturbation method for fractional KdV equation. App. Math. Comput. **190**(2), 1795–1802 (2007)
Wang, K.J., Wang, G.D., Zhu, H.W.: A new perspective on the study of the fractal coupled Boussinesq–Burger equation in shallow water. Fractals **29**(05), 2150122 (2021)
Wang, K.J., Shi, F., Liu, J.H., Si, J.: Application of the extended F-expansion method for solving the fractional Gardner equation with conformable fractional derivative. Fractals **30**(07), 2250139 (2022)
Wazwaz, A.M.: Analytic study of the fifth order integrable nonlinear evolution equations by using the tanh method. Appl. Math. Comput. **174**, 289–299 (2006)
Yılmaz, S., Tasbozan, O.: Analytıcal solutions of conformable Boussinesq–Double–Sinh–Gordon and first Boussinesq–Liouville equations with the help of auxiliary equation method. Ikonion J. Math. **2**, 40–54 (2020)
Zafar, A., Raheel, M., Mahnashi, A.M., Bekir, A., Ali, M.R., Hendy, A.S.: Exploring the new soliton solutions to the nonlinear M-fractional evolution equations in shallow water by three analytical techniques. Results Phys. **54**, 107092 (2023)
Zheng, B.: (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. Commun. Theor. Phys. **58**(5), 623 (2012)

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