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*Research Article*

# On the Leonardo Sequence via Pascal-Type Triangles

**Serpil Halıcı and Sule Curuk**

*Department of Mathematics, Pamukkale Uni., Denizli 20160, Türkiye*

Correspondence should be addressed to Serpil Halıcı; shalici@pau.edu.tr

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In this study, we discussed the Leonardo number sequence, which has been studied recently and caught more attention. We used Pascal and Hosoya-like triangles to make it easier to examine the basic properties of these numbers. With the help of the properties obtained in this study, we defined a number sequence containing the new type of Leonardo numbers created by choosing the coefficients from the bicomplex numbers. Furthermore, we gave the relationship of this newly defined sequence with the Fibonacci sequence. We also provided some important identities in the literature provided by the elements of this sequence described in this paper.

## 1. Introduction

Recently, there have been an increasing number of studies on Leonardo numbers, which are a sequence of numbers that are closely related to Fibonacci numbers and have been the subject of some articles. The first comprehensive study on these numbers belongs to Alp and Koçer. In 2012, Kocer examined the Leonardo numbers in detail and gave an important relation between Leonardo and Fibonacci numbers [1]. Furthermore, the next detailed study belongs to Catarino and Borges [2, 3]. There are only a limited number of studies on this number sequence in the literature and one of the most recent studies is by Prasad et al. [4]. In this work, Kumari generalized these numbers and used them in Octonion algebra. On the other hand, Shattuck [5] discussed the combinatorial identities of the generalized Leonardo numbers. References [6–8] can be analyzed for more information.

Leonardo's numbers are given by, for $n \in \mathbb{N}$,

$$\mathrm{Le}_{n+2} = \mathrm{Le}_{n+1} + \mathrm{Le}_n + 1, \tag{1}$$

where the initial conditions are $\mathrm{Le}_0 = \mathrm{Le}_1 = 1$. Writing the elements of the sequence provides computational convenience [2].

$$\{\mathrm{Le}_n\}_{n\geq 0} = \{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \ldots\}. \tag{2}$$

The numbers have a relationship with the Fibonacci number, due to the nature of the repeated relation in the equation (1). In 2012, Alp and Koçer examined the Leonardo numbers in detail and mentioned Fibonacci relation [1].

$$\mathrm{Le}_n = 2F_{n+1} - 1. \tag{3}$$

In 2022, Karatas handled the complex Leonardo numbers [9].

$$\mathrm{CLe}_n = \mathrm{Le}_n + \mathrm{iLe}_{n+1} = \mathrm{CLe}_{n-1} + \mathrm{CLe}_{n-2} + (1+i). \tag{4}$$

The author gave the Binet formula for these numbers in his work as follows.

$$\mathrm{CLe}_n = \frac{2\,\underline{\alpha}\,\alpha^{n+1} - 2\,\underline{\beta}\,\beta^{n+1}}{\alpha - \beta} - (1+i). \tag{5}$$

In this formula, the values of $\alpha, \beta$ are the roots of the characteristic equation containing the Fibonacci numbers. In addition, alpha and beta line values are as follows.

$$\begin{aligned} \underline{\alpha} &= 1 + i\alpha, \\ \underline{\beta} &= 1 + i\beta. \end{aligned} \tag{6}$$

And so, we write

$$\{CLe_n\}_{n\geq 0} = \{1+i, 1+3i, 3+5i, 5+9i, 9+15i, \ldots\}. \quad (7)$$

*Definition 1.* With the help of the numbers $CLe_n$, we can define the bicomplex Leonardo numbers as follows:

$$\begin{aligned} BLe_n &= CLe_n + jCLe_{n+2} \\ &= \left(Le_n + iLe_{n+1}\right) + j\left(Le_{n+2} + iLe_{n+3}\right), \end{aligned} \quad (8)$$

where

$$\begin{aligned} BLe_0 &= 1 + i + 3j + 5ij, \\ BLe_1 &= 1 + 3i + 5j + 9ij. \end{aligned} \quad (9)$$

In Table 1, we listed some of the nth Fibonacci, Lucas, Leonardo, complex Leonardo, bicomplex Fibonacci, and bicomplex Leonardo numbers to use later.

With the help of Table 1, the following equations can be written:

(i) $F_n + L_n + Le_n = 2Le_n + 1$.
(ii) $F_n + L_n - 1 = Le_n$.
(iii) $F_n + L_n + Le_n - 1 = 2Le_n$.
(iv) $F_n + L_n + Le_n + CLe_n - 1 = 3Le_n + iLe_{n+1}$.
(v) $F_n + L_n + Le_n + CLe_n + BF_n - 1 = F_n + 3Le_n + i(F_{n+4} - 1) + jCF_{n+2}$.
(vi) $F_n + L_n + Le_n + CLe_n + BLe_n - 1 = 2(Le_n + CLe_n) + jCLe_{n+2}$.

As is known, using Pascal and Pascal-like triangles is another way of obtaining various new identities. In this section, we introduce Pascal-like and Hosaya-like triangles and give new identities. Let us give new Pascal-like triangles using the numbers $Le_n$ and $CLe_n$. First, we construct a Pascal-like triangle using the numbers $CLe_n$. Next, we introduce Hosoya-like triangles using the numbers $Le_n$.

$$\begin{array}{ccccccccccc} & & & & & 1+i & & & & & \\ & & & & 1+3i & & 1+3i & & & & \\ & & & 3+5i & & 3+7i & & 3+5i & & & \\ & & 5+9i & & 7+13i & & 7+13i & & 5+9i & & \\ & 9+15i & & 13+23i & & 15+27i & & 13+23i & & 9+15i & \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \end{array} \quad (10)$$

It may be useful to use some notations here. These are

$$1 + i = \epsilon_i, 1 + 3i = \epsilon_i + 2i, 3 + 5i = 3\epsilon_i + 2i, \ldots,. \quad (11)$$

We can also examine the numbers $CLe_n$ and the elements of the Pascal-like triangle with the help of the table below.

It is observed that the first column of Table 2 consists of $CLe_n$ numbers. The following equation can be written to represent the notation $C(n,k)$ for the nth row and kth column element. For $n, k \in \mathbb{Z}^+$ and $n \geq k+1$,

$$C(n,k) = C(n-1,k-1) + C(n-1,k) + \epsilon_i. \quad (12)$$

For $n = 3, k = 2$, we get

$$\begin{aligned} C(3,2) &= (3+7i) + (3+5i) + (1+i) \\ &= 7 + 13i \\ &= 7\epsilon_i + 6i. \end{aligned} \quad (13)$$

In addition, the sum of the first column and last row elements of Table 2 gives the number $BLe_n$. Therefore, we can write the formula that gives these numbers as follows.

$$BLe_n = C(n,0) + jC(n+2,0). \quad (14)$$

For $n = 1$,

$$\begin{aligned} BLe_1 &= C(1,0) + jC(3,0) \\ &= 1 + 3i + 5j + 9ij. \end{aligned} \quad (15)$$

Using this information and the numbers $BLe_n$, we can construct the following Pascal-like triangle.

We give the elements of the Pascal-like triangle involving the numbers $BLe_n$ in the table below.

Notice that the first column of Table 3 gives the numbers $BLe_n$, and the nth row and the kth column elements give the numbers $B(n,k)$. For example, for $(n,k) = (2,1)$ and $(n,k) = (4,3)$ we get

$$\begin{aligned} B(2,1) &= B(1,0) + B(1,1) + \epsilon_i\epsilon_j \\ &= 3 + 7i + 11j + 19ij \\ &= 3\epsilon_i + 4i + j\left(11\epsilon_i + 8i\right), \\ B(4,3) &= B(3,2) + B(3,3) + \epsilon_i\epsilon_j \\ &= 13 + 23i + 37j + 61ij \\ &= 13\epsilon_i + 10i + j\left(37\epsilon_i + 24i\right), \end{aligned} \quad (16)$$

respectively.

**Theorem 2.** *For $n \geq k+1$ with $n, k \in \mathbb{Z}^+$, the following equality is satisfied.*

$$B(n,k) = B(n-1,k-1) + B(n-1,k) + \epsilon_i\epsilon_j. \quad (17)$$

Now, let us recognize Hosoya-like triangles using the numbers $Le_n$.

TABLE 1: The numbers $F_n, L_n, \text{Le}_n, \text{CLe}_n, \text{BF}_n, \text{BLe}_n$.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 |
| $L_n$ | 2 | 1 | 3 | 4 | 7 |
| $\text{Le}_n$ | 1 | 1 | 3 | 5 | 9 |
| $\text{CLe}_n$ | $1+i$ | $1+3i$ | $3+5i$ | $5+9i$ | $9+15i$ |
| $\text{BF}_n$ | $i+j+2\text{ij}$ | $1+i+2j+3\text{ij}$ | $1+2i+3j+5\text{ij}$ | $2+3i+5j+8\text{ij}$ | $3+5i+8j+13\text{ij}$ |
| $\text{BLe}_n$ | $1+i+3j+5\text{ij}$ | $1+3i+5j+9\text{ij}$ | $3+5i+9j+15\text{ij}$ | $5+9i+15j+25\text{ij}$ | $9+15i+25j+41\text{ij}$ |

TABLE 2: The numbers $\text{CLe}_n$.

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | $\epsilon_i$ | | | | | | ... |
| 1 | $\epsilon_i+2i$ | $\epsilon_i+2i$ | | | | | ... |
| 2 | $3\epsilon_i+2i$ | $3\epsilon_i+4i$ | $3\epsilon_i+2i$ | | | | ... |
| 3 | $5\epsilon_i+4i$ | $7\epsilon_i+6i$ | $7\epsilon_i+6i$ | $5\epsilon_i+4i$ | | | ... |
| 4 | $9\epsilon_i+6i$ | $13\epsilon_i+10i$ | $15\epsilon_i+12i$ | $13\epsilon_i+10i$ | $9\epsilon_i+6i$ | | ... |
| 5 | $15\epsilon_i+10i$ | $23\epsilon_i+16i$ | $29\epsilon_i+22i$ | $29\epsilon_i+22i$ | $23\epsilon_i+16i$ | $15\epsilon_i+10i$ | ... |
| ... | ... | ... | ... | ... | ... | ... | ... |

TABLE 3: The numbers $B(n,k)$.

| $n/k$ | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| 0 | $\epsilon_i+j(3\epsilon_i+2i)$ | | | | ... |
| 1 | $\epsilon_i+2i+j(5\epsilon_i+4i)$ | $\epsilon_i+2i+j(5\epsilon_i+4i)$ | | | ... |
| 2 | $3\epsilon_i+2i+3j(3\epsilon_i+2i)$ | $3\epsilon_i+4i+j(11\epsilon_i+8i)$ | $3\epsilon_i+2i+3j(3\epsilon_i+2i)$ | | ... |
| 3 | $5\epsilon_i+4i+5j(3\epsilon_i+2i)$ | $7\epsilon_i+6i+7j(3\epsilon_i+2i)$ | $7\epsilon_i+6i+7j(3\epsilon_i+2i)$ | $5\epsilon_i+4i+5j(3\epsilon_i+2i)$ | ... |
| ... | ... | ... | ... | ... | ... |

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 3 & & 1 & & 3 & & & & \\ & & & 5 & & 3 & & 3 & & 5 & & & \\ & & 9 & & 5 & & 8 & & 5 & & 9 & & \\ & 15 & & 9 & & 12 & & 12 & & 9 & & 15 & \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \end{array} \tag{18}$$

Where the initial values are $\text{Le}(0,0) = \text{Le}(1,0) = \text{Le}(1,1) = \text{Le}(2,1) = 1$ and the set $\{1, 1, 8, 19, 53, 134, 349, \ldots\}$ gives the middle sequence in this triangle.

**Theorem 3.** *For the number $\text{Le}(n, j)$, the following equalities are true.*

*(i) For $n \geq 2j,\ j \geq 2$,*

$$\text{Le}(n, j) = \text{Le}(n-1, j-1) + \text{Le}(n-2, j-2) + 2. \tag{19}$$

*(ii) For $n \geq j+2,\ j \geq 2$,*

$$\text{Le}(n, j) = \text{Le}(n-1, j) + \text{Le}(n-2, j) + 2. \tag{20}$$

*(iii) For $1 \leq k \leq n-j-1$,*

$$\begin{aligned}\text{Le}(n, j) = {} & F_{k+1}\text{Le}(n-k, j) + F_k\text{Le}(n-k-1, j) \\ & + (\text{Le}_{k+1} - 1).\end{aligned} \tag{21}$$

The following Theorem gives the relationship between Leonardo and Fibonacci numbers.

**Theorem 4.** *For $k = n - j - 1$, Leonardo numbers are*

$$\text{Le}(n, j) = F_{n-j+1}\left(2F_{j+1} + 1\right) - 2. \tag{22}$$

*Proof.* For $k = n - j - 1$, we have

$$\text{Le}(n, j) = F_{n-j}\text{Le}(j+1, j) + F_{n-j-1}\text{Le}(j, j) + \left(\text{Le}_{n-j} - 1\right). \tag{23}$$

Using the equality $\text{Le}_n = 2F_{n+1} - 1$, we can write

$$\begin{aligned}&\text{Le}(j+1, j) \\ &\quad = \text{Le}(j, j) \\ &\quad = 2F_{j+1} - 1.\end{aligned} \tag{24}$$

A simple transformation, and then we get

$$\begin{aligned}\text{Le}(n, j) &= F_{n-j}\left(2F_{j+1} - 1\right) + F_{n-j-1}\left(2F_{j+1} - 1\right) + \left(2F_{n-j+1} - 2\right), \\ \text{Le}(n, j) &= F_{n-j+1}\left(2F_{j+1} + 1\right) - 2.\end{aligned} \tag{25}$$

□

## 2. The Numbers $\text{BLe}_n$ and the Identities Involving Them

In the studies in [10–13], the authors examined the arithmetic operations in bicomplex space and their structure of functions in detail. In this section, we obtained some identities using Leonardo and bicomplex numbers.

*Definition 5.* For $n \geq 0$, the numbers $\mathrm{BLe}_n$ can be defined as follows:

$$\begin{aligned}\mathrm{BLe}_n &= \sum_{s=0}^{3} \mathrm{Le}_{n+s} e_s \\ &= \mathrm{Le}_n e_0 + \mathrm{Le}_{n+1} e_1 + \mathrm{Le}_{n+2} e_2 + \mathrm{Le}_{n+3} e_3,\end{aligned} \tag{26}$$

where $e_0 = 1, e_1 = i, e_2 = j, e_3 = \mathrm{ij}$ and

$$\begin{aligned}\mathrm{BLe}_0 &= \sum_{s=0}^{3} \mathrm{Le}_s e_s, \\ \mathrm{BLe}_1 &= \sum_{s=0}^{3} \mathrm{Le}_{s+1} e_s, \\ \mathrm{BLe}_2 &= \sum_{s=0}^{3} \mathrm{Le}_{s+2} e_s.\end{aligned} \tag{27}$$

**Corollary 6.** *For $n \geq 3$, we have*

$$\mathrm{BLe}_n = 2\mathrm{BLe}_{n-1} - \mathrm{BLe}_{n-3}. \tag{28}$$

**Theorem 7.** *For the numbers $BLe_n$, we have*

$$g(\mathrm{BLe}_n, x) = \frac{\sum_{s=0}^{3}\left(\mathrm{Le}_s + (\mathrm{Le}_{s+1} - 2\mathrm{Le}_s)x + (\mathrm{Le}_{s+2} - 2\mathrm{Le}_{s+1})x^2\right)e_s}{x^3 - 2x + 1}. \tag{29}$$

*Proof.* First, we write the equations $x^3 g(\mathrm{BLe}_n, x)$ and $2\mathrm{xg}(\mathrm{BLe}_n, x)$.

$$\begin{aligned}g(\mathrm{BLe}_n, x) &= \mathrm{BLe}_0 + \mathrm{BLe}_1 x + \mathrm{BLe}_2 x^2 + \cdots + \mathrm{BLe}_n x^n + \ldots \\ -2\mathrm{xg}(\mathrm{BLe}_n, x) &= -2\mathrm{BLe}_0 x - 2\mathrm{BLe}_1 x^2 - 2\mathrm{BLe}_2 x^3 - \ldots - 2\mathrm{BLe}_{n-1} x^n - \ldots \\ x^3 g(\mathrm{BLe}_n, x) &= \mathrm{BLe}_0 x^3 + \mathrm{BLe}_1 x^4 + \mathrm{BLe}_2 x^5 + \cdots + \mathrm{BLe}_{n-3} x^n + \ldots\end{aligned} \tag{30}$$

Later, by arranging these equations, we get

$$\begin{aligned}\left(x^3 - 2x + 1\right) g(\mathrm{BLe}_n, x) &= \mathrm{BL}e_0 + x(\mathrm{BLe}_1 - 2\mathrm{BLe}_0) + x^2(\mathrm{BLe}_2 - 2\mathrm{BLe}_1) + \ldots \\ g(\mathrm{BLe}_n, x) &= \frac{\mathrm{BLe}_0 + x(\mathrm{BLe}_1 - 2\mathrm{BLe}_0) + x^2(\mathrm{BLe}_2 - 2\mathrm{BLe}_1)}{\left(x^3 - 2x + 1\right)} \\ g(\mathrm{BLe}_n, x) &= \frac{\sum_{s=0}^{3}\left(\mathrm{Le}_s + (\mathrm{Le}_{s+1} - 2\mathrm{Le}_s)x + (\mathrm{Le}_{s+2} - 2\mathrm{Le}_{s+1})x^2\right)e_s}{x^3 - 2x + 1}.\end{aligned} \tag{31}$$

Thus, the proof is completed. □

**Theorem 8.** *For nonnegative integers n, we have*

$$\mathrm{BLe}_n = \mathrm{A}r_1^n + \mathrm{B}r_2^n + \mathrm{C}r_3^n, \tag{32}$$

*where* $r_1 = 1 + \sqrt{5}/2, r_2 = 1 - \sqrt{5}/2, r_3 = 1$,

$$A = \frac{\sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s + (-r_2 - r_3)\sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s + r_2 r_3\sum_{s=0}^{3}\mathrm{Le}_s e_s}{(r_1 - r_2)(r_1 - r_3)},$$

$$B = \frac{\sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s + (-r_1 - r_3)\sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s + r_1 r_3\sum_{s=0}^{3}\mathrm{Le}_s e_s}{(r_2 - r_1)(r_2 - r_3)},$$

$$C = \frac{\sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s + (-r_1 - r_2)\sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s + r_1 r_2\sum_{s=0}^{3}\mathrm{Le}_s e_s}{(r_3 - r_1)(r_3 - r_2)}. \tag{33}$$

*Proof.* To prove, if we use the initial values $\mathrm{BLe}_0, \mathrm{BLe}_1, \mathrm{BLe}_2$ and recurrence relation $\mathrm{BLe}_n = 2\mathrm{BLe}_{n-1} - \mathrm{BLe}_{n-3}$, then we write

$$\begin{aligned} A + B + C &= \sum_{s=0}^{3}\mathrm{Le}_s e_s, \\ \mathrm{A}r_1 + \mathrm{B}r_2 + \mathrm{C}r_3 &= \sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s, \\ \mathrm{A}r_1^2 + \mathrm{B}r_2^2 + \mathrm{C}r_3^2 &= \sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s. \end{aligned} \tag{34}$$

Here are some operations and abbreviations needed. If we also do them, then

$$A = \frac{\sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s + (-r_2 - r_3)\sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s + r_2 r_3\sum_{s=0}^{3}\mathrm{Le}_s e_s}{(r_1 - r_2)(r_1 - r_3)},$$

$$B = \frac{\sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s + (-r_1 - r_3)\sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s + r_1 r_3\sum_{s=0}^{3}\mathrm{Le}_s e_s}{(r_2 - r_1)(r_2 - r_3)},$$

$$C = \frac{\sum_{s=0}^{3}\mathrm{Le}_{s+2}e_s + (-r_1 - r_2)\sum_{s=0}^{3}\mathrm{Le}_{s+1}e_s + r_1 r_2\sum_{s=0}^{3}\mathrm{Le}_s e_s}{(r_3 - r_1)(r_3 - r_2)}. \tag{35}$$

Thus, we provided the most general formula for the number $\mathrm{BLe}_n$.

In the next theorem, we give the equation, which is significantly important for integer sequences and is known as Catalan's identity. □

**Theorem 9.** *For* $n, r \in \mathbb{Z}^+$ *and* $n \geq r$ *we have*

$$\mathrm{BLe}_{n+r}\mathrm{BLe}_{n-r} - \mathrm{BLe}_n^2 = (-1)^{n-r}\left\{\mathrm{AB}r_3^{r-n}\left(r_1^{2r} + r_2^{2r} + 2(-1)^{r+1}r_3^{-r}\right) + \mathrm{AC}r_2^{r-n}\left(r_1^{2r} + r_3^{2r} + 2(-1)^{r+1}r_2^{-r}\right) + \mathrm{BC}r_1^{r-n}\left(r_2^{2r} + r_3^{2r} + 2(-1)^{r+1}r_1^{-r}\right)\right\}. \tag{36}$$

*Proof.* We can write the following equation for the left-hand side of equation (36).

$$\begin{aligned} \mathrm{LHS} &= \left(\mathrm{A}r_1^{n+r} + \mathrm{B}r_2^{n+r} + \mathrm{C}r_3^{n+r}\right)\left(\mathrm{A}r_1^{n-r} + \mathrm{B}r_2^{n-r} + \mathrm{C}r_3^{n-r}\right) - \left(\mathrm{A}r_1^n + \mathrm{B}r_2^n + \mathrm{C}r_3^n\right)^2, \\ \mathrm{LHS} &= \mathrm{AB}(r_1 r_2)^n\left(r_1^r r_2^{-r} + r_1^{-r}r_2^r - 2\right) + \mathrm{AC}(r_1 r_3)^n\left(r_1^r r_3^{-r} + r_1^{-r}r_3^r - 2\right) + \mathrm{BC}(r_2 r_3)^n\left(r_2^r r_3^{-r} + r_2^{-r}r_3^r - 2\right). \end{aligned} \tag{37}$$

For the last equation, if we do some editing and necessary algebraic operations, then

$$\mathrm{BLe}_{n+r}\mathrm{BLe}_{n-r} - \mathrm{BLe}_n^2, \tag{38}$$

is equal to this

$$(-1)^{n-r}\left\{\mathrm{AB}r_3^{r-n}\left(r_1^{2r} + r_2^{2r} + 2(-1)^{r+1}r_3^{-r}\right) + \mathrm{AC}r_2^{r-n}\left(r_1^{2r} + r_3^{2r} + 2(-1)^{r+1}r_2^{-r}\right) + \mathrm{BC}r_1^{r-n}\left(r_2^{2r} + r_3^{2r} + 2(-1)^{r+1}r_1^{-r}\right)\right\}. \tag{39}$$

Thus, the proof is completed. □

**Corollary 10.** *For the numbers $BLe_n$, we have*

$$BLe_{n+1}BLe_{n-1} - BLe_n^2 = (-1)^{n-1}\left\{ABr_3^{1-n}\left(r_1^2 + r_2^2 + 2r_3^{-1}\right) + ACr_2^{1-n}\left(r_1^2 + r_3^2 + 2r_2^{-1}\right) + BCr_1^{1-n}\left(r_2^2 + r_3^2 + 2r_1^{-1}\right)\right\} \tag{40}$$

*Proof.* If we use the Binet formula, we write the left side of equation (40) as follows.

$$\begin{aligned} BLe_{n+1}BLe_{n-1} - BLe_n^2 &= \left(Ar_1^{n+1} + Br_2^{n+1} + Cr_3^{n+1}\right)\left(Ar_1^{n-1} + Br_2^{n-1} + Cr_3^{n-1}\right) - \left(Ar_1^n + Br_2^n + Cr_3^n\right)^2, \\ BLe_{n+1}BLe_{n-1} - BLe_n^2 &= AB\left(r_1^{n+1}r_2^{n-1} + r_2^{n+1}r_1^{n-1} - 2r_1^n r_2^n\right) \\ &\quad + AC\left(r_1^{n+1}r_3^{n-1} + r_3^{n+1}r_1^{n-1} - 2r_1^n r_3^n\right) + BC\left(r_2^{n+1}r_3^{n-1} + r_3^{n+1}r_2^{n-1} - 2r_2^n r_3^n\right), \end{aligned} \tag{41}$$

If we complete the necessary arrangements, then we obtain

$$BLe_{n+1}BLe_{n-1} - BLe_n^2 = (-1)^{n-1}\left\{ABr_3^{1-n}\left(r_1^2 + r_2^2 + 2r_3^{-1}\right) + ACr_2^{1-n}\left(r_1^2 + r_3^2 + 2r_2^{-1}\right) + BCr_1^{1-n}\left(r_2^2 + r_3^2 + 2r_1^{-1}\right)\right\} \tag{42}$$

Thus, the proof is completed. □

**Theorem 11.** *For positive numbers $m, n$, $BLe_m BLe_{n+1} - BLe_n BLe_{m+1}$ is*

$$\begin{aligned} &(-1)^{-(m+n)}\left\{ABr_3^{-(m+n)}\left(r_1^{-n}r_2^{-m+1} + r_1^{-m+1}r_2^{-n} - r_1^{-m}r_2^{-n+1} - r_1^{-n+1}r_2^{-m}\right) + ACr_2^{-(m+n)}\left(r_1^{-n}r_3^{-m+1} + r_1^{-m+1}r_3^{-n} - r_1^{-m}r_3^{-n+1} - r_1^{-n+1}r_3^{-m}\right)\right. \\ &\quad \left. + BCr_1^{-(m+n)}\left(r_2^{-n}r_3^{-m+1} + r_2^{-m+1}r_3^{-n} - r_2^{-m}r_3^{-n+1} - r_2^{-n+1}r_3^{-m}\right)\right\} \end{aligned} \tag{43}$$

*Proof.* If we want to calculate the left-hand side of equation (43), then we get

$$LHS = \left(Ar_1^m + Br_2^m + Cr_3^m\right)\left(Ar_1^{n+1} + Br_2^{n+1} + Cr_3^{n+1}\right) - \left(Ar_1^n + Br_2^n + Cr_3^n\right)\left(Ar_1^{m+1} + Br_2^{m+1} + Cr_3^{m+1}\right). \tag{44}$$

From this equality, we get

$$\begin{aligned} LHS &= AB\left(r_1^m r_2^{n+1} + r_1^{n+1}r_2^m - r_1^n r_2^{m+1} - r_1^{m+1}r_2^n\right) + AC\left(r_1^m r_3^{n+1} + r_1^{n+1}r_3^m - r_1^n r_3^{m+1} - r_1^{m+1}r_3^n\right) \\ &\quad + BC\left(r_2^m r_3^{n+1} + r_2^{n+1}r_3^m - r_2^n r_3^{m+1} - r_2^{m+1}r_3^n\right). \end{aligned} \tag{45}$$

Thus, $BLe_m BLe_{n+1} - BLe_n BLe_{m+1}$ is as follows:

$$(-1)^{-(m+n)}\Big\{ABr_3^{-(m+n)}\left(r_1^{-n}r_2^{-m+1}+r_1^{-m+1}r_2^{-n}-r_1^{-m}r_2^{-n+1}-r_1^{-n+1}r_2^{-m}\right)+ACr_2^{-(m+n)}\left(r_1^{-n}r_3^{-m+1}+r_1^{-m+1}r_3^{-n}-r_1^{-m}r_3^{-n+1}-r_1^{-n+1}r_3^{-m}\right)$$
$$+BCe_1^{-(m+n)}\left(r_2^{-n}r_3^{-m+1}+r_2^{-m+1}r_3^{-n}-r_2^{-m}r_3^{-n+1}-r_2^{-n+1}r_3^{-m}\right)\Big\} \tag{46}$$

which is the desired result. □

**Theorem 12.** *For positive numbers m and n,* $BLe_{n-1}BLe_m+BLe_nBLe_{m+1}$ *is*

$$A^2r_1^{m+n-1}\left(1+r_1^2\right)+B^2r_2^{m+n-1}\left(1+r_2^2\right)+C^2r_3^{m+n-1}\left(1+r_3^2\right)+AB\left(1+r_1r_2\right)\left(r_1^mr_2^{n-1}+r_2^mr_1^{n-1}\right)$$
$$+AC\left(1+r_1r_3\right)\left(r_3^mr_1^{n-1}+r_1^mr_3^{n-1}\right)+BC\left(1+r_2r_3\right)\left(r_3^mr_2^{n-1}+r_2^mr_3^{n-1}\right). \tag{47}$$

*Proof.* In formula $BLe_{n-1}BLe_m+BLe_nBLe_{m+1}$, if we use the Binet formula, then this value will be equal to the expression below.

$$\left(Ar_1^{n-1}+Br_2^{n-1}+Cr_3^{n-1}\right)\left(Ar_1^m+Br_2^m+Cr_3^m\right)+\left(Ar_1^n+Br_2^n+Cr_3^n\right)\left(Ar_1^{m+1}+Br_2^{m+1}+Cr_3^{m+1}\right)$$
$$=A^2r_1^{m+n-1}\left(1+r_1^2\right)+B^2r_2^{m+n-1}\left(1+r_2^2\right)+C^2r_3^{m+n-1}\left(1+r_3^2\right)$$
$$+(-1)^{m+n-1}\Big\{ABr_3^{m+n-1}\left(r_1^{-m}r_2^{-n+1}+r_2^{-m}r_1^{-n+1}+r_1^{-m+1}r_2^{-n+2}+r_2^{-m+1}r_1^{-n+2}\right)$$
$$+ACr_2^{m+n-1}\left(r_1^{-m}r_3^{-n+1}+r_3^{-m}r_1^{-n+1}+r_1^{-m+1}r_3^{-n+2}+r_3^{-m+1}r_1^{-n+2}\right)$$
$$+BCr_1^{m+n-1}\left(r_2^{-m}r_3^{-n+1}+r_3^{-m}r_2^{-n+1}+r_2^{-m+1}r_3^{-n+2}+r_3^{-m+1}r_2^{-n+2}\right)\Big\} \tag{48}$$

Then, $BLe_{n-1}BLe_m+BLe_nBLe_{m+1}$ is

$$A^2r_1^{m+n-1}\left(1+r_1^2\right)+B^2r_2^{m+n-1}\left(1+r_2^2\right)+C^2r_3^{m+n-1}\left(1+r_3^2\right)+AB\left(1+r_1r_2\right)\left(r_1^mr_2^{n-1}+r_2^mr_1^{n-1}\right)$$
$$+AC\left(1+r_1r_3\right)\left(r_3^mr_1^{n-1}+r_1^mr_3^{n-1}\right)+BC\left(1+r_2r_3\right)\left(r_3^mr_2^{n-1}+r_2^mr_3^{n-1}\right). \tag{49}$$

Thus, we prove the claim of the theorem. □

## 3. Conclusion

In this study, we discussed the Leonardo number sequence and its basic properties in detail. With the help of the properties and Pascal-like triangles, we defined a new sequence of numbers containing Leonardo numbers. We have also considered the basic identities provided by the elements of the newly defined sequence, which are often used in examining integer sequences.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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