

Characterization of Absolute Summability Factors

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By (A, B) we denote the set of all sequences $\lambda = (\lambda_n)$ such that $\sum a_n \lambda_n$ is summable B , whenever $\sum a_n$ is summable A , where A and B are summability methods. In the present paper we characterize the sets $(|\bar{N}, p_n|, |\bar{N}, q_n|_k)$ and $(|\bar{N}, p_n|_k, |\bar{N}, q_n|)$, $1 \leq k < \infty$, using functional analytic techniques, and also extend some known results.

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1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums s_n and let (p_n) be a sequence of positive numbers such that $P_n = p_0 + p_1 + \dots + p_n \rightarrow \infty$ as $n \rightarrow \infty$ ($P_{-i} = p_{-i} = 0, i \geq 1$). The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v,$$

defines the sequence (t_n) of the (\bar{N}, p_n) means of the sequence (s_n) generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k, 1 \leq k < \infty$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1}$$

In the special case when $k = 1$, $|\overline{N}, p_n|_k$ summability is the same as $|\overline{N}, p_n|$ summability. Throughout the paper (p_n) and (q_n) will be denote positive sequences with $P_n \rightarrow \infty$ and $Q_n = q_0 + q_1 + \cdots + q_n \rightarrow \infty$ as $n \rightarrow \infty$.

If a sequence λ has the property that $\sum a_n \lambda_n$ is summable B whenever $\sum a_n$ is summable A , where A and B are two methods of summability, then we say that λ is a summability factor of type (A, B) and we write $\lambda \in (A, B)$. It is known that the summability $|\overline{N}, p_n|$ and the summability $|\overline{N}, q_n|_k$ are, in general, independent of each other. It is, therefore, natural to ask for suitable summability factors of the types $(|\overline{N}, p_n|, |\overline{N}, q_n|_k)$ and $(|\overline{N}, p_n|_k, |\overline{N}, q_n|)$, $1 \leq k < \infty$.

2. THE MAIN RESULTS

We shall prove the following.

THEOREM 2.1. *Let $1 \leq k < \infty$. Then $\lambda \in (|\overline{N}, p_n|, |\overline{N}, q_n|_k)$ if and only if*

$$\begin{aligned} \text{(a)} \quad & \lambda_n = O(1) \\ \text{(b)} \quad & \Delta\lambda_n = O(p_n/P_n) \\ \text{(c)} \quad & \lambda_n = O\{(p_n/P_n)(Q_n/q_n)^{1/k}\} \end{aligned} \tag{2}$$

as $n \rightarrow \infty$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

THEOREM 2.2. *Let $1 < k < \infty$. Then $\lambda \in (|\overline{N}, p_n|_k, |\overline{N}, q_n|)$ if and only if*

$$\begin{aligned} \text{(a)} \quad & \sum_{v=1}^{\infty} (p_v/P_v) |(P_v/p_v) \Delta\lambda_v + \lambda_{v+1}|^{k^*} < \infty \\ \text{(b)} \quad & \sum_{v=1}^{\infty} (p_v/P_v) \left\{ \frac{q_v P_v}{p_v Q_v} |\lambda_v| \right\}^{k^*} < \infty, \end{aligned} \tag{3}$$

where k^* is the conjugate index of k ; i.e., $k^* = k/(k - 1)$.

3. NEEDED LEMMAS

We need the following lemmas for the proof of our theorems.

LEMMA 3.1 [2]. *Let $k \geq 1$. Then there are two positive constants A and B , depending only on k ; we have for all $v \geq 1$,*

$$\frac{A}{P^{k_{v-1}}} \leq \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leq \frac{B}{P^{k_{v-1}}},$$

where A and B are independent of (p_n) .

LEMMA 3.2 [6]. Let $1 \leq k < \infty$. Then $T = (a_{nv}) \in (l_1, l_k)$ if and only if

$$\sup_v \sum_{n=1}^{\infty} |a_{nv}|^k < \infty,$$

where (l_1, l_k) denotes the set of all infinite matrices T which map l_1 into $l_k = \{a = (a_v) : \sum |a_v|^k < \infty, k \geq 1\}$.

LEMMA 3.3 [8]. Let $1 \leq k < \infty$. Then $[\bar{N}_p]_k = \{x = (x_i) : \sum x_i \text{ is summable } [\bar{N}, p_n]_k\}$ is a Banach space with respect to the norm

$$\|x\| = \left\{ |x_0|^k + \sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{i=1}^n P_{i-1} x_i \right|^k \right\}^{1/k}.$$

LEMMA 3.4 [7]. Let $1 \leq k < \infty, 1 \leq r < \infty$, and suppose that x, y, u , and v are related as

$$y_n = \sum_{v=0}^{\infty} a_{nv} x_v \quad (n = 0, 1, \dots), \quad v_v = \sum_{n=0}^{\infty} a_{nv} u_v \quad (v = 0, 1, \dots).$$

Then $y \in l_r$ whenever $x \in l_k$ if and only if $v \in l_{k^*}$ whenever $u \in l_{r^*}$, where k^* and r^* are the conjugate indices of k and r , respectively.

LEMMA 3.5 [7]. $1 < k < \infty$ and $y_n = \sum_{v=0}^n a_{nv} x_v$ and $y \in l_1$ whenever $x \in l_k$. Then $(a_{nn}) \in l_{k^*}$.

LEMMA 3.6. Let $k \geq 1$. If $\sum a_n \lambda_n$ is summable $[\bar{N}, q_n]_k$ whenever $\sum a_n$ is summable $[\bar{N}, p_n]$ then $\lambda_n = O(1)$.

Proof. Let (t_n) and (T_n) be the sequences of (\bar{N}, p_n) and (\bar{N}, q_n) means of the series $\sum a_n$ and $\sum a_n \lambda_n$, respectively. Then we have

$$\begin{aligned} t_n &= \frac{1}{P_n} \sum_{v=0}^n p_v s_v = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v, \\ x_n &= t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v, \quad n \geq 1, x_0 = a_0, \end{aligned} \tag{4}$$

and similarly for $n \geq 1$,

$$y_n = T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} a_v \lambda_v, \quad y_0 = a_0 \lambda_0. \quad (5)$$

We are given that

$$\sum_{n=1}^{\infty} (Q_n/q_n)^{k-1} |y_n|^k < \infty \quad (6)$$

whenever

$$\sum_{n=1}^{\infty} |x_n| < \infty. \quad (7)$$

Now the space of sequences satisfying (7) if normed by

$$\|x\| = \sum_{n=0}^{\infty} |x_n|$$

is a Banach space. We are considering the space of these sequences (y_n) such that they satisfy statement (6). This is again a Banach space, by Lemma 3.3, with respect to the norm

$$\|y\| = \left\{ \sum_{n=0}^{\infty} (Q_n/q_n)^{k-1} |y_n|^k \right\}^{1/k}.$$

We are given that (5) transforms the space of sequences satisfying (7) into the space satisfying (6); applying the Banach–Steinhaus theorem in the usual way we find that there is some constant C , such that for all sequences satisfying (7) we have

$$\|y\| \leq C \|x\|. \quad (8)$$

Taking $v \geq 1$, we are applying (8) with

$$a_v = 1, \quad a_r = 0, \quad (r \neq v).$$

Hence (4) and (5) give, respectively,

$$x_n = \begin{cases} 0, & n < v, \\ \frac{P_{v-1} p_n}{P_n P_{n-1}}, & n \geq v, \end{cases}$$

and

$$y_n = \begin{cases} 0, & n < v, \\ \frac{Q_{v-1}q_n\lambda_v}{Q_nQ_{n-1}}, & n \geq v, \end{cases}$$

and so

$$\|x\| = 1, \quad \|y\| = Q_{v-1}|\lambda_v| \left\{ \sum_{n=v}^{\infty} \frac{q_n}{Q_nQ_{n-1}^k} \right\}^{1/k}.$$

Therefore it follows from (8) and Lemma 3.1 that $\lambda_n = O(1)$.

Proof of Theorem 2.1. We use the notation of Lemma 3.6. Putting the inverse of (4) in (5), we have for $n \geq 1$

$$\begin{aligned} y_n &= \frac{q_n}{Q_nQ_{n-1}} \sum_{v=1}^n Q_{v-1}\lambda_v \left(\frac{P_v}{p_v}x_v - \frac{P_{v-2}}{p_{v-1}}x_{v-1} \right) \\ &= \frac{q_n}{Q_nQ_{n-1}} \sum_{v=1}^{n-1} \left(\frac{Q_vP_v \Delta\lambda_v}{p_v} - \frac{q_vP_v\lambda_v}{p_v} + Q_v\lambda_{v+1} \right) x_v + \frac{q_nP_n\lambda_n}{p_nQ_n} x_n. \end{aligned}$$

Take $Y_n = (Q_n/q_n)^{1-(1/k)} y_n$ for $n \geq 0$. Then for $n \geq 1$

$$Y_n = \sum_{v=1}^n a_{nv} x_v,$$

where

$$a_{nv} = \begin{cases} (q_n/Q_n)^{1/k}(1/Q_{n-1}) \left(\frac{Q_vP_v \Delta\lambda_v}{p_v} - \frac{q_vP_v\lambda_v}{p_v} + Q_v\lambda_{v+1} \right), & 1 \leq v \leq n-1, \\ (q_n/Q_n)^{1/k} (P_n/p_n)\lambda_n, & v = n. \end{cases}$$

Now, $\sum a_n\lambda_n$ is summable $[\bar{N}, q_n]_k$ whenever $\sum a_n$ is summable $[\bar{N}, p_n]$ if and only if $Y \in l_k$ whenever $x \in l_1$, or, equivalently,

$$\sup_v \sum_{n=1}^{\infty} |a_{nv}|^k < \infty, \tag{9}$$

by Lemma 3.2. But (9) is the same as

$$(q_v/Q_v)(P_v|\lambda_v/p_v)^k + \left| \frac{Q_v P_v \Delta \lambda_v}{p_v} - \frac{q_v P_v \lambda_v}{p_v} + Q_v \lambda_{v+1} \right|^k \sum_{n=v+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}^k} \quad (10)$$

$$= O(1), \quad \text{as } v \rightarrow \infty.$$

This holds if and only if

$$(q_v/Q_v)(P_v|\lambda_v/p_v)^k = O(1) \quad (11)$$

and

$$\left(\frac{Q_v P_v \Delta \lambda_v}{p_v} - \frac{q_v P_v \lambda_v}{p_v} + Q_v \lambda_{v+1} \right) \left\{ \sum_{n=v+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}^k} \right\}^{1/k} = O(1). \quad (12)$$

Because of Lemma 3.1, (12) is satisfied if and only if

$$P_v \Delta \lambda_v / p_v - \frac{q_v P_v \lambda_v}{Q_v p_v} + \lambda_{v+1} = O(1). \quad (13)$$

Since $q_v P_v \lambda_v / Q_v p_v = O\{(q_v/Q_v)^{1-1/k}\} = O(1)$, by (11), statement (13) is equivalent to

$$P_v \Delta \lambda_v / p_v + \lambda_{v+1} = O(1). \quad (14)$$

Thus it follows from (11) and (14) that hypotheses (a), (b), and (c) are sufficient. However, it is from Lemma 3.6, (11), and (14) that hypotheses of Theorem 2.1 are necessary, completing the proof.

Proof of Theorem 2.2. According to the notation of Lemma 3.6, we know that

$$y_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left\{ (P_v/p_v) \Delta(Q_{v-1} \lambda_v) + Q_v \lambda_{v+1} \right\} x_v + \frac{q_n P_n \lambda_n}{Q_n P_n} x_n, \quad n \geq 1.$$

If we put $X_n = (P_n/p_n)^{1-1/k} x_n$ for $n \geq 1$, then we have

$$y_n = \sum_{v=1}^n a_{nv} X_v,$$

where

$$a_{nv} = \begin{cases} \frac{q_n}{Q_n Q_{n-1}} \{(P_v/p_v) \Delta(Q_{v-1} \lambda_v) + Q_v \lambda_{v+1}\} (P_v/p_v)^{(1/k)-1}, & 1 \leq v \leq n-1, \\ \frac{q_n P_n \lambda_n}{Q_n P_n} (P_n/p_n)^{(1/k)-1}, & v = n. \end{cases}$$

Therefore $\sum a_n \lambda_n$ is summable $[\bar{N}, q_n]$ whenever $\sum a_n$ is summable $[\bar{N}, p_n]$ if and only if $y \in I_1$ whenever $X \in I_k$. By Lemma 3.4, the necessary and sufficient conditions for the same are

$$\sum_{n=v}^{\infty} a_{nv} z_n \text{ converges for every } z_n = O(1), \quad v = 1, 2, \dots,$$

and

$$\sum_{v=1}^{\infty} \left| \sum_{n=v}^{\infty} a_{nv} z_n \right|^{k^*} < \infty \quad \text{whenever } z_n = O(1).$$

Now,

$$\begin{aligned} \sum_{n=v}^{\infty} a_{nv} z_n &= \frac{q_v}{Q_v} (P_v/p_v)^{1/k} \lambda_v z_v + \{(P_v/p_v) \Delta(Q_{v-1} \lambda_v) \\ &\quad + Q_v \lambda_{v+1}\} (P_v/p_v)^{(1/k)-1} d_v, \end{aligned}$$

where

$$d_v = \sum_{n=v+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} z_n,$$

which converges, since $z_n = O(1)$. Hence, the necessary and sufficient condition for the conclusion of the theorem is

$$\begin{aligned} \sum_{v=1}^{\infty} \{ &(q_v/Q_v)(P_v/p_v)^{1/k} \lambda_v z_v + \{(P_v/p_v) \Delta(Q_{v-1} \lambda_v) \\ &+ Q_v \lambda_{v+1}\} (P_v/p_v)^{(1/k)-1} d_v \}^{k^*} < \infty \end{aligned} \tag{15}$$

whenever $z_v = O(1)$. Take $z_v = 1$ for all v . Then $d_v = 1/Q_v$. Thus it follows from (15) that

$$\sum_{v=1}^{\infty} (p_v/P_v)|(P_v/p_v) \Delta\lambda_v + \lambda_{v+1}|^{k^*} < \infty. \tag{16}$$

In addition to this, we have, by Lemma 3.5,

$$\sum_{v=1}^{\infty} (p_v/P_v) \left\{ \frac{q_v P_v}{Q_v p_v} |\lambda_v| \right\}^{k^*} < \infty. \tag{17}$$

Therefore, the hypotheses of the theorem are necessary. The sufficiency is easily seen from (16), (17), Lemma 3.1, and the inequality

$$(a + b)^{k^*} \leq 2^{k^*} (a^* + b^*), \quad a, b \geq 0,$$

since (15) holds whenever $z_v = O(1)$. Thus the proof is completed.

4. SPECIAL CASES

Theorem 2.1 and Theorem 2.2 include some known results as special cases and completed some of them in the necessary and sufficient form. We list examples.

COROLLARY 4.1. *Necessary and sufficient conditions for $\lambda \in (\overline{N}, p_n, \overline{N}, q_n)$ are*

- (a) $\lambda_n = O(1)$,
 - (b) $\Delta\lambda_n = O(p_n/P_n)$,
 - (c) $\lambda_n = O(p_n Q_n/q_n P_n)$.
- (18)

Khan-Alauddin [4] established the sufficiency of the conclusion of Corollary 4.1 under stronger conditions (18b), (18c), and $p_n Q_n = O(q_n P_n)$.

COROLLARY 4.2. *Necessary and sufficient conditions for $(p_n Q_n/q_n P_n) \in (\overline{N}, p_n, \overline{N}, q_n)$ are*

$$p_n Q_n = O(P_n q_n) \text{ and } \Delta(Q_n/q_n) + (P_n/P_{n+1})(Q_{n+1}/q_{n+1})(\Delta p_n/p_n) = O(1).$$

Kishore and Hotta [5] proved the only sufficiency of this result under stronger conditions.

COROLLARY 4.3. *Necessary and sufficient condition in order that every \overline{N}, p_n summable series is summable \overline{N}, q_n is $q_n P_n = O(p_n Q_n)$. This result due to Sunouchi [10] and Bosanquet [3].*

COROLLARY 4.4. *Necessary and sufficient condition in order that every $|\overline{N}, p_n|$ summable series is summable $|\overline{N}, q_n|_k$, $k \geq 1$, is $q_n P_n^k = O(Q_n p_n^k)$. This is the main result of [9].*

It should be noted that Corollaries 4.1–4.4 are trivial special cases of Theorem 2.1. In fact we obtain Corollary 4.1 from Theorem 2.1 with $k = 1$. We get Corollary 4.2 from Theorem 2.1 with $k = 1$ and $\lambda_n = p_n Q_n / P_n q_n$. Finally, if we take $k = \lambda_n = 1$ and $k = 1$ in Theorem 2.1, then we get Corollary 4.3 and Corollary 4.4, respectively.

We now note that if we apply Theorem 2.2 with $\lambda_n = 1$, condition (3a) is reduced to

$$\sum_{v=1}^{\infty} (p_v / P_v) < \infty,$$

which is impossible by the Abel-Dini theorem. So we have the following that solves the open problem raised by the first author in [9].

COROLLARY 4.5. *Let $1 < k < \infty$. Then there exists a $|\overline{N}, p_n|_k$ summable series which is not $|\overline{N}, q_n|$ summable.*

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