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On Local Property of $|A|_k$ Summability of Factored Fourier Series

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Quite recently, Bor [J. Math. Anal. Appl. **163** (1992), 220–226] proved a result on the local property of $|\overline{N}, p_n|_k$ summability of factored Fourier series, which includes some known results. In this paper we extend his result to more general cases by taking normal matrices instead of weighted mean matrices.

1. INTRODUCTION

Let Σa_n be a given infinite series with the partial sums (s_n), and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \qquad n = 0, 1, \ldots$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [13, 14])

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

In the special case when $A = (\overline{N}, p_n)$ (resp., $k = 1$), $|A|_k$ summability is reduced to $|\overline{N}, p_n|_k$ ($|\overline{N}, p_n|$ is equivalent to $|R, P_n, 1|$) summability [5].

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Also, if we take $A = (C, \alpha)$ with $\alpha > -1$, $|A|_k$ summability is the same as $|C, \alpha, (\alpha - 1)(1 - 1/k)|_k$ in Flett's notation (see [7]).

We use the notations

$$\Delta c_n = c_n - c_{n+1} \quad \text{and} \quad \overline{\Delta} c_{nv} = c_{nv} - c_{n-1,v}, \; c_{-1,0} = 0$$

for $n, v = 0, 1, \ldots$

Let f be a periodic function with period 2π, integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of f is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} C_n(t).$$

It is well known that the convergence of the Fourier series at $t = x$ is a local property of f (i.e., depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and so the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Mohanty [12] demonstrated that the summability $|R, \log n, 1|$ of the factored Fourier series

$$\sum (\log n)^{-1} C_n(t) \tag{1}$$

at any point is a local property of f. Matsumoto [10] improved this result by replacing the series (1) by

$$\sum (\log \log n)^{-p} C_n(t), \qquad p > 1.$$

Bhatt [3] showed that the factor $(\log \log n)^{-p}$ in the above series can be replaced by the more general factor $\gamma_n \log n$ where (γ_n) is a convex sequence such that $\Sigma n^{-1}\gamma_n$ is convergent. Borwein [6] generalized Bhatt's result by proving that (λ_n) is a sequence for which

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty,$$

then the summability $|R, P_n, 1|$ of the factored Fourier series

$$\sum_{n=1}^{\infty} \lambda_n C_n(t) \tag{2}$$

at any point is a local property of f. On the other hand, Mishra [11] proved that if (γ_n) is as above, and if

$$P_n = O(np_n) \qquad \text{and} \qquad P_n \Delta p_n = O(p_n p_{n+1}),$$

the summability $|\overline{N}, p_n|$ of the series

$$\sum_{n=1}^{\infty} \gamma_n P_n (np_n)^{-1} C_n(t)$$

at any point is a local property of f. Bor [4] showed that $|\overline{N}, p_n|$ in Mishra's result can be replaced by a more general summability method $|\overline{N}, p_n|_k$. However, Baron [1, 2] established that the summability $|A|$ of the series (2) by certain triangular matrix methods is a local property of f, which includes some results mentioned above. Quite recently, Bor [5] introduced the following theorem on the local property of the summability $|\overline{N}, p_n|_k$ of the factored Fourier series, which generalizes most of the above results under more appropriate conditions than those given in them.

THEOREM A. *Let the positive sequence (p_n) and a sequence (λ_n) be such that*

$$\Delta X_n = O(1/n)$$

$$\sum_{n=1}^{\infty} n^{-1}\{|\lambda_n|^k + |\lambda_{n+1}|^k\} X_n^{k-1} < \infty$$

$$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta \lambda_n| < \infty$$

where $X_n = (np_n)^{-1} P_n$. Then the summability $|\overline{N}, p_n|_k$ $(k \geq 1)$ of the series

$$\sum_{n=1}^{\infty} \lambda_n X_n C_n(t)$$

at any point is a local property of f.

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In this paper, taking a normal matrix instead of a weighted mean matrix, we prove an analog of Theorem A. Before stating the main theorem we must first introduce some further notation.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\overline{A} = (\overline{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \qquad n, v = 0, 1, \ldots,$$

and

$$\hat{a}_{00} = \hat{a}_{00} = a_{00}, \hat{a}_{nv} = \overline{\Delta}\overline{a}_{nv}, \qquad \text{n} = 1, 2, \ldots$$

It may be noted that $\hat{A}$ and $\overline{A}$ are the well-known matrices of series-to-series and series-to-sequence transformations, respectively. Now we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \overline{a}_{nv}a_v \qquad \text{and} \qquad \overline{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{3}$$

With this notation we have the following.

THEOREM. *Suppose that* $A = (a_{nv})$ *is a positive normal matrix such that*

$$a_{n-1,v} \geq a_{nv} \qquad \text{for} \qquad n \geq v + 1 \tag{4}$$

$$\overline{a}_{n0} = 1, \qquad n = 0, 1 \ldots, \tag{5}$$

$$\sum_{v=1}^{n-1} a_{vv}\hat{a}_{n,v-1} = O(a_{nn}), \tag{6}$$

$$\Delta X_n = O(1/n), \tag{7}$$

where $X_n = (na_{nn})^{-1}$. *If a sequence* (λ_n) *holds for* $k \geq 1$ *and the following conditions,*

$$\sum_{n=1}^{\infty} n^{-1}\{|\lambda_n|^k + |\lambda_{n+1}|^k\}X_{n-1}^k < \infty \tag{8}$$

$$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta\lambda_n| < \infty, \tag{9}$$

then the summability $|A|_k$ *of the series*

$$\sum_{n=1}^{\infty} \lambda_n X_n C_n(t) \tag{10}$$

at any point is a local property of f.

Remark. The elements $\hat{a}_{nv} \geq 0$ for each v, n. In fact, it is easily seen from the positiveness of the matrix, (4) and (5), that $\hat{a}_{00} = 1$,

$$\begin{aligned} \hat{a}_{nv} &= \overline{a}_{n0} - \overline{a}_{v-1,0} + \sum_{i=0}^{v-1} (a_{n-1,i} - a_{ni}) \\ &= \sum_{i=0}^{v-1} (a_{n-1,i} - a_{ni}) \geq 0 \end{aligned}$$

for $1 \leq v \leq n$, and equal to zero otherwise.

We use the following lemma in the proof of the theorem.

LEMMA. *Suppose that the matrix A and the sequence* (λ_n) *satisfy the conditions of the theorem, and that* (s_n) *is bounded. Then the series*

$$\sum_{n=1}^{\infty} \lambda_n X_n a_n \tag{11}$$

is summable $|A|_k$, $k \geq 1$.

Proof. Let (T_n) be an A-transform of the series (11). Then we have, by (3),

$$\overline{\Delta} T_n = \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v X_v, \qquad (x_0 = 0).$$

Applying Abel's transformation to this sum we get

$$\overline{\Delta} T_n = \sum_{v=1}^{n-1} \Delta(\hat{a}_{nv} \lambda_v X_v) s_v + a_{nn} \lambda_n X_n s_n.$$

By the formula for the difference of products of sequences (see [8, p. 129]) we have

$$\begin{aligned} \Delta(\hat{a}_{nv} \lambda_v X_v) &= \lambda_v X_v \Delta \hat{a}_{nv} + \Delta(\lambda_v X_v) \hat{a}_{n,v+1} \\ &= \lambda_v X_v \overline{\Delta} a_{nv} + (X_v \Delta \lambda_v + \Delta X_v \lambda_{v+1}) \hat{a}_{n,v+1}, \end{aligned}$$

and so

$$\overline{\Delta} T_n = \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v \Delta\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta X_v s_v + \sum_{v=1}^{n-1} \overline{\Delta} a_{nv} \lambda_v X_v s_v$$

$$+ a_{nn} \lambda_n X_n s_n = T_n(1) + T_n(2) + T_n(3) + T_n(4),$$

say. For the proof of the lemma, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty} a_{nn}^{1-k} |T_n(r)|^k < \infty, \qquad \text{for } r = 1, 2, 3, 4.$$

It now follows from Hölder's inequality,

$$\left(\sum_v |y_v z_v|\right)^k = \left(\sum_v |y_v|^{k^{-1}} |z_v| |y_v|^{1-k^{-1}}\right)^k$$

$$= \sum_v |y_v| |z_v|^k \cdot \left(\sum_v |y_v|\right)^{k-1},$$

that

$$\sum_{n=2}^{m+1} a_{nn}^{1-k} |T_n(1)|^k \leq \sum_{n=2}^{m+1} a_{nn}^{1-k} \left\{\sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v |\Delta\lambda_v| |s_v|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} a_{nn}^{1-k} \left\{\sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v |\Delta\lambda_v|\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v^k |\Delta\lambda_v| \cdot \left\{a_{nn}^{-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v|\right\}^{k-1}.$$

Taking account of (4) and (5) we have, for $1 \leq v \leq n-1$,

$$\hat{a}_{n,v+1} = \sum_{r=v+1}^{n} (a_{nr} - a_{n-1,r}) = \sum_{r=0}^{v} (a_{n-1,r} - a_{nr})$$

$$\leq \sum_{r=0}^{n-1} (a_{n-1,r} - a_{nr}) = \overline{a}_{n-1,0} - \overline{a}_{n0} + a_{nn} = a_{nn},$$

which implies

$$\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v| \leq a_{nn} \sum_{v=1}^{n-1} |\Delta\lambda_v| = O(a_{nn})$$

by (9). On the other hand, it is seen from the positiveness, (4), and (5) that

$$\bar{a}_{mv} = \sum_{r=v}^{m} a_{mr} \le \sum_{r=0}^{m} a_{mr} = \bar{a}_{m0} = 1$$

for $0 \le v \le m$, and zero otherwise. Thus,

$$\begin{aligned}
\sum_{n=2}^{m+1} a_{nn}^{1-k} |T_n(1)|^k &= O(1) \sum_{n=2}^{m+1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v^k |\Delta\lambda_v| \\
&= O(1) \sum_{v=1}^{m} X_v^k |\Delta\lambda_v| \cdot \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} \\
&= O(1) \sum_{v=1}^{m} X_v^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} (\bar{a}_{n,v+1} - \bar{a}_{n-1,v+1}) \\
&= O(1) \sum_{v=1}^{m} X_v^k |\Delta\lambda_v| \bar{a}_{m+1,v+1} = O(1) \sum_{v=1}^{m} X_v^k |\Delta\lambda_v| \\
&= O(1)
\end{aligned}$$

as $m \to \infty$, by virtue of (9). Note that from (7) follows that $\Delta X_v = O(a_{vv} X_v)$. It is also seen from the Hölder inequality that, as in $T_n(1)$, by (6) we have

$$\begin{aligned}
\sum_{n=2}^{m+1} a_{nn}^{1-k} |T_n(2)|^k &\le \sum_{n=2}^{m+1} a_{nn}^{1-k} \left\{ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}| \, |\Delta X_v| \, |s_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} a_{nn}^{1-k} \left\{ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}| a_{vv} X_v \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}|^k a_{vv} X_v^k \cdot \left\{ a_{nn}^{-1} \sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}|^k a_{vv} X_v^k \\
&= O(1) \sum_{v=1}^{m} a_{vv} X_v^k |\lambda_{v+1}|^k \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} \\
&= O(1) \sum_{v=1}^{m} v^{-1} |\lambda_{v+1}|^k X_v^{k-1} = O(1)
\end{aligned}$$

as $m \to \infty$, by (8).

$$\sum_{n=2}^{m+1} a_{nn}^{1-k}|T_n(3)|^k = \sum_{n=2}^{m+1} a_{nn}^{1-k}\left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\,|\lambda_v|X_v|s_v|\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\,|\lambda_v|^k X_v^k\cdot\left\{a_{nn}^{-1}\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\right\}^{k-1}.$$

On the other hand, since, by (4) and (5),

$$\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}| = \sum_{v=1}^{n-1}(a_{n-1,v} - a_{nv}) = \bar{a}_{n-1,0} - \bar{a}_{n0} + a_{n0} - a_{n-1,0} + a_{nn}$$
$$= a_{n0} - a_{n-1,0} + a_{nn} \leq a_{nn},$$

we have

$$\sum_{n=2}^{m+1} a_{nn}^{1-k}|T_n(3)|^k = O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\,|\lambda|^k X_v^k$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|^k X_v^k\sum_{n=v+1}^{m+1}(a_{n-1,v} - a_{nv})$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|^k X_v^k a_{vv}$$
$$= O(1)\sum_{v=1}^{m} v^{-1}|\lambda_v|^k X_v^{k-1} = O(1)$$

as $m \to \infty$, by (8).

Finally,

$$\sum_{n=1}^{\infty} a_{nn}^{1-k}|T_n(4)|^k = O(1)\sum_{n=1}^{\infty}|\lambda_n|^k X_n^k a_{nn} = O(1)\sum_{n=1}^{\infty} n^{-1}|\lambda_n|^k X_n^{k-1} < \infty$$

by (8), which completes the proof of the lemma.

Proof of the Theorem. Since the convergence of the Fourier series at a point is a local property of its generating function f, the theorem follows by formula (7.1) from Chapter II of the book [15] and from the lemma.

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Applications. We now apply the Theorem to the weighted mean in which $A = (a_{nv})$ is defined as $a_{nv} = p_v P_n^{-1}$ when $0 \leq v \leq n$, where $P_n =$

$p_0 + \cdots + p_n$; therefore, it is well known that

$$\bar{a}_{nv} = P_n^{-1}(P_n - P_{v-1}) \quad \text{and} \quad \hat{a}_{n,v+1} = (P_n P_{n-1})^{-1} p_n P_v.$$

One can now easily verify that the conditions of the theorem reduce to those of Theorem A.

We may now ask whether there are some examples (other than weighted mean methods) of matrices A that satisfy the hypotheses of the theorem. For this, apply the theorem to the Cesáro method of order α with $0 < \alpha \leq 1$ in which A is given by $a_{nv} = A_{n-v}^{\alpha-1}/A_n^\alpha$. It is well known that (see [9])

$$\bar{a}_{nv} = A_{n-v}^{\alpha}/A_n^{\alpha} \quad \text{and} \quad \hat{a}_{nv} = vA_{n-v}^{\alpha-1}/(nA_n^\alpha).$$

It is now seen by taking account of $A_n^\alpha \sim n^\alpha/\Gamma(\alpha + 1)$ that conditions (4)–(7) are satisfied. Therefore the above theorem is the same as the following result.

COROLLARY. *Let $k \geq 1$ and $0 < \alpha \leq 1$. If a sequence (λ_n) holds for the following conditions,*

$$\sum_{n=1}^{\infty} n^{\alpha k-\alpha-k}(|\lambda_n|^k + |\lambda_{n+1}|^k) < \infty,$$

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty,$$

then the summability $|C, \alpha, (\alpha - 1)(1 - 1/k)|_k$ of the series (10) with $X_n = A^\alpha/n$ at any point is a local property of the generating function f.

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