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Research Article

Exploring EADS Modules: Properties, Direct Sums, and Applications in Matrix Rings

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We introduce the concept of an EADS module, defined such that for any decomposition $M = A \oplus B$ and any ec-complement C of A in M, the module satisfies $M = A \oplus C$. This study explores the properties of EADS modules and examines their relationships with other established properties. We particularly investigate the behavior of EADS modules concerning direct sums and direct summands. In addition, we present applications, including matrix rings over a right EADS ring.

1. Introduction

In this paper, we consider rings to be associative with identity, and R denotes such a ring. All modules discussed are unital right R-modules. Fuchs [1] defined a module M as ADS if for every decomposition $M = X \oplus Y$, any complement Z of X ensures $M = X \oplus Z$. This ADS notion has been extensively studied (see [2–6]). Every quasicontinuous module is ADS, but not vice versa. A module is termed extending (CS) or satisfying the (C_1) condition if every submodule is essential in a direct summand. Several generalizations have been examined (see [7]). Yücel and Tercan in [8] introduced ECS modules as a generalization of CS modules. A module M is ECS if every ec-closed submodule of M is a direct summand. An ec (-closed) submodule N of M is one that contains essentially a cyclic submodule, i.e., there exists $x \in N$ such that xR is essential in N.

This paper investigates a module M for which every decomposition $M = A \oplus B$ and every ec-closed C of A, we have $M = A \oplus C$. Such modules are called EADS modules. Notably, ADS modules and quasicontinuous modules are EADS, but not necessarily the other way around. A ring R is called right EADS if R_R is an EADS module.

The paper is structured as follows: Section 2 discusses fundamental properties of ec-submodules. Section 3 examines the connections between EADS conditions and various other conditions, including proving a counterexample to show that EADS and ADS modules are not identical. We also explore the behavior of EADS modules with respect to direct sums and summands, identifying conditions under which the direct sum of EADS modules is also an EADS module. Section 4 examines extension rings of EADS rings and provides applications to matrix rings over a right EADS ring.

Definitions and notations used throughout the paper are as follows: Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$, $X \leq_e M$, and $r(X)$ denote that X is a submodule of M and X is an essential submodule of M, the right annihilator of X, respectively. For R, $M_m(R)$ symbolizes the ring of m-by-m full matrices over R.

Given $X \leq M$, by a complement submodule of X in M, we mean a submodule Y of M, maximal with respect to the property $Y \cap X = 0$ (see [8]). This definition originally appears as intersection complement in Kasch's book (see [9]). Following [10], a module M is quasicontinuous if it satisfies (C_1) and (C_3) conditions: The sum of two direct summands of M with zero intersection is again a direct summand of M. For two modules M_1 and M_2, the module M_2 is M_1-ec-injective if every homomorphism $\varphi\colon K \longrightarrow M_2$, where K is an ec-closed submodule of M_1, can be extended to a homomorphism $\theta: M_1 \longrightarrow M_2$ (see [8]). Additional terminology and notation can be found in [7, 10–14].

2. Basic Results

Ec-complement submodules are fundamental in establishing EADS modules. Therefore, we start this section by noting some basic properties about them.

Lemma 1. *Let N and K be submodules of M and $\phi: N \longrightarrow K$ is an isomorphism. Any submodule of N is an ec-closed if and only if its image is an ec-closed submodule of K.*

Proof. Suppose X is an ec-closed submodule of N. This implies that $xR \leq_e X$ for some $x \in X$. Let $0 \neq Y \leq \phi(X)$. Then, $\phi^{-1}(Y) \cap xR \neq 0$. So $Y \cap \phi(x)R \neq 0$. Hence, $\phi(x)R \leq_e \phi(X)$. Since closed modules are invariant under isomorphism, $\phi(X)$ is an ec-closed submodule of K. The converse is similarly proved. □

Lemma 2. *Let M be a module, $L, N \leq M$, and N be an ec-submodule of M with $N \cap L = 0$. Then, there exists an ec-complement C of L in M such that $N \subseteq C$.*

Proof. Let $S = \{X \text{ is an ec} - \text{submodule of } M\colon\ N \leq X,\ \ X \cap L = 0\}$. Then, $N \in S$ and $S \neq 0$. The result can be obtained by Zorn's Lemma. □

Lemma 3. *Let M be an R-module, K be a submodule of M, and X be a submodule of K. Then, K is an ec-submodule of M if and only if K/X is an ec-submodule of M/X.*

Proof. Let K be an ec-submodule of M. This implies that there exists $k \in K$ such that $kR \leq_e K$. Let $X \leq N \leq K$, $0 \neq N$, and $0 \neq N/X \leq K/X$. Then, $kR \cap N \neq 0$. So there exists $r \in R$ such that $kr \in N$. Therefore, $(k + X)r = kr + X \in N/X$ and $(N/X) \cap (k + X)R \neq 0$. Hence, $(k + X)R \leq_e K/X$, so that K/X is an ec-submodule of M/X. The converse is proved similarly. □

Proposition 4. *Let $M = \oplus_{i \in I} M_i$ for some ec-submodules M_i of M. Then,*

(i) Let N be an R-module. If N is an M-ec-injective, then N is an M_i-ec-injective for every $i \in I$.

(ii) Let K be an R-module. Then, M is a K-ec-injective if and only if M_i is a K-ec-injective for every $i \in I$.

(iii) $M(I - i)$ is an M_i-ec-injective for every $i \in I$ if and only if M_i is an M_j-ec-injective for all $i \neq j \in I$.

Proof.

(i) Assume that N is an M-ec-injective. Since M_i are ec-closed submodules of M for all $i \in I$, N is an M_i-ec-injective by [[7], Proposition 2.19].

(ii) Assume that M is a K ec-injective. Let N be an ec-closed submodule of K and $\varphi: N \longrightarrow M_i$ be a homomorphism for all $i \in I$. Then, $i \circ \varphi: N \longrightarrow M$ homomorphism, where $i: M_i \longrightarrow M$ homomorphism, can be extended to $\theta: K \longrightarrow M$ by assumption. Therefore, the homomorphism $\pi_i \circ \theta$ extends to φ. Hence, M_i is a K-ec-injective for every $i \in I$. The converse is proved similarly.

(iii) It is a consequence of property (ii). □

3. EADS Modules

In this section, we first present some equivalent conditions for EADS modules, then derive structural properties, and finally investigate the direct sums and summands of EADS modules. We begin with a useful proposition for verifying the EADS property of a module.

Proposition 5. *A module is an EADS module if and only if for any decomposition $M = A \oplus B$, A and B are mutually ec-injective.*

Proof. Suppose M is an EADS and U is an ec-submodule of A with $\varphi: U \longrightarrow B$ as a homomorphism. Define $X = \{u - \varphi(u)\colon u \in U\}$. Then, $X \cap B = 0$. Let π_A denote the projection $M \longrightarrow A$. The restriction π_A to X is an isomorphism between X and $\pi_A(X)$, and since $\pi_A(X) = U$, $\pi_A(X)$ is an ec-submodule of A. By Lemma 1, X is an ec-submodule of M. Hence by Lemma 2, there exists an ec-complement C of B in M containing X. By the hypothesis, $M = C \oplus B$. Let $\pi_B: M \longrightarrow B$ be the projection with kernel C, and $\theta: A \longrightarrow B$ be the restriction of π_B to A. Then, for $0 \neq u \in U$, $\theta(u) = \pi_B(u) = \pi_B(u - \varphi(u) + \varphi(u)) = \varphi(u)$. Thus, θ extends φ, making B an A-ec-injective.

Similarly, it can be proved that A is a B-ec-injective. Conversely, let $M = A \oplus B$ and C be an ec-complement of B. Since $C \cap B = 0$, the restriction of π_A to C is an isomorphism between C and $\pi_A(C)$. By Lemma 1, $\pi_A(C)$ is an ec-submodule of A. This implies that $\pi_A(C)$ is an ec-complement of A. Let $\alpha: \pi_A(C) \longrightarrow B$ be the homomorphism defined by $\alpha(x) = \pi_B(\pi_A\mid_C)^{-1}(x)$, where $x \in \pi_A(C)$. By the hypothesis, α extends to homomorphism $\theta: A \longrightarrow B$. Define $T = \{a + \theta(a)\colon a \in A\}$. Clearly, T is a submodule of M and $M = T \oplus B$. For any $c \in C$, $\theta\pi_A(c) = \alpha\pi_A(c) = \pi_B(c)$, so that $c = \pi_A(c) + \theta\pi_A(c) \in T$. Hence, $C \leq T$. Since C is an ec-complement of B, $C \oplus B \leq_e M$ and so $C \leq_e T$. This implies that $C = T$ and $M = C \oplus B$, and thus, M is an EADS module. □

Proposition 6. *Let $M = A \oplus B$ be an R-module and $X = K \cap B$ for every K complement of A in M. If M/X is an EADS module, then M is an EADS module.*

Proof. Let $M = A \oplus B$, K be an ec-complement of A in M and $X = K \cap B$. It can be shown that $M/X = (A \oplus X)/X \oplus B/X$. By Lemma 3, K/X is an ec-submodule of M/X. Observe that K/X is complement of $(A \oplus X)/X$ in M/X. It follows from the hypothesis that $M/X = (A \oplus X)/X \oplus K/X$. Thus, $M = A \oplus K$ which implies that M is an EADS module.

Now, we locate the EADS condition with respect to several known generalizations of the quasicontinuous property. □

Proposition 7. *Let M be a module. Let us consider the following conditions:*

(i) M is quasicontinuous

(ii) M is ADS

(iii) M is EADS

Then, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). In general, the reverse implications do not hold.

Proof. (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are straightforward.

(ii) $\not\Rightarrow$ (i) Let F be a field and V a vector space over F with $\dim(V_F) = 2$. Let R be the trivial extension of F with V, i.e.,

$$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} f \in F, v \in V \right\}, \quad (1)$$

then R_R is an ADS because it is indecomposable. Since R_R is not uniform, R_R is not quasicontinuous.

(iii) $\not\Rightarrow$ (ii), let p be a prime integer and $M = \mathbb{Q} \oplus \mathbb{Z}/\mathbb{Z}p$, a $\mathbb{Z}$-module. It follows that $\mathbb{Q}$ is a $\mathbb{Z}/\mathbb{Z}p$-injective, but $\mathbb{Z}/\mathbb{Z}p$ is not $\mathbb{Q}$-injective which implies that $M_{\mathbb{Z}}$ is not an ADS module (see [[5], Example 2.4]). Since $\mathbb{Q}$ and $\mathbb{Z}/\mathbb{Z}p$ are uniform modules, they are relatively ec-injective. Hence, $M_{\mathbb{Z}}$ is an EADS module. □

Proposition 8. *Let M be a module. For the following conditions, quasicontinuous, ADS and EADS modules are equivalent:*

(i) M is a uniform module

(ii) M is a CS and an ec-module

Proof.

(i) Let M be a uniform and an ADS module. From [[15], Proposition 3.1], M is a quasicontinuous module. The converse follows from Proposition 7. Let M be a uniform and be an EADS module. Suppose $M = A \oplus B$ and C is a complement of A in M. Then, for each $c \in C$, we must have that $cR \leq_e C$. Hence, C is an ec-complement of A. By the hypothesis, $M = A \oplus C$ and so M is an ADS module. The converse follows from Proposition 7.

(ii) Let M be a CS and an ADS module. From [[15], Proposition 3.1], M is a quasicontinuous module. The converse follows from Proposition 7. Let M be a CS, an ec and an EADS module. Assume $M = A \oplus B$ and C is a complement of A in M. By [[16], Lemma 3.16], every closed submodule of M is an ec-complement submodule of M. Since M is an EADS module, $M = A \oplus C$. Hence, M is an ADS module.

The next result provides a characterization of EADS modules. □

Theorem 9. *Let M be a module. The following conditions are equivalent:*

(i) M is EADS

(ii) For any direct summand M_1 and an ec-submodule M_2, having zero intersection with M_1, the projection map π_i: $M_1 \oplus M_2 \longrightarrow M_i$ $(i = 1, 2)$ can be extended to an endomorphism (indeed a projection) of M

(iii) If $M = M_1 \oplus M_2$, then M_1 and M_2 are mutually ec-injective

(iv) For any decomposition $M = A \oplus B$, the projection π: $M \longrightarrow B$ is an isomorphism when it is restricted to any ec-complement C of A in M

Proof. (i) $\Longrightarrow$ (ii) Let M be an EADS module. By Lemma 2, there exists an ec-complement M_2' of M_1 containing M_2. Then, $M = M_1 \oplus M_2'$. Hence, the canonical projection π_1': $M_1 \oplus M_2' \longrightarrow M_1$ and π_2': $M_1 \oplus M_2' \longrightarrow M_2'$ are clearly extensions of π_1 and π_2.

(ii) $\Longrightarrow$ (i) Let $M = A \oplus B$ and C be an ec-complement of A in M. We must show that $M = A \oplus C$. By the hypothesis, the projection π: $A \oplus C \longrightarrow C$ can be extended to an endomorphism f: $M \longrightarrow M$. We claim $f(M) \subset C$. Since $A \oplus C$ is essential in M, for any $0 \neq m \in M$, there exists an essential right ideal E of R such that $0 \neq mE \subset A \oplus C$. This gives $f(m)E = \pi(mE) \subset C$. Since C is closed in M, we must have that $f(m) \in C$ which proves our claim. We also remark that $f^2 = f$, $M = \ker(f) \oplus im(f)$, and $ker f = \{m - f(m) \mid m \in M\}$. Now, we have to show that $\ker(f) = A$. For any $a \in A$, clearly $a = a - f(a) \in \mathrm{Ker}(f)$ and hence $A \subset \mathrm{Ker}(f)$. Now, let $0 \neq m - f(m) \in \mathrm{Ker}(f)$. There exists $r \in R$ such that $0 \neq (m - f(m))r \in A \oplus C$. This implies $f[m - f(m)r] = f(mr) - f(f(mr)) = f(mr) - f(mr) = 0$. Since f extends π, it follows that $0 \neq (m - f(m))r \in \mathrm{Ker}(\pi) = A$. Since A is a complement in M, then $A = \ker(f)$ which proves the result.

(i) $\Longleftrightarrow$ (iii) This is clear from Proposition 7.

(i) $\Longleftrightarrow$ (iv) Let M be an EADS module and C an ec-complement of A in M. Then, $M = A \oplus C$ and $\pi(C) = B$. Since $\mathrm{Ker}(\pi|_C) = 0$, we must have that $\pi|_C$: $C \longrightarrow B$ is an isomorphism. Conversely, let $\pi|_C$ be an isomorphism. Since $\pi|_C(C) = B$, $M = A \oplus B = A \oplus C$. Hence, M is an EADS module.

The following proposition gives that EADS modules are closed under ec-direct summand. □

Proposition 10. *Let M be a module. If M is an EADS module, then every ec-direct summand of M is an EADS module.*

Proof. Let N be an ec-direct summand of M and $N = N_1 \oplus N_2$ where $N_1, N_2 \leq N$. Then, there exists $T \leq M$ such that $M = N \oplus T$. So $M = N_1 \oplus N_2 \oplus T$. By the hypothesis, N_1 is an $N_2 \oplus T$-ec-injective. Since N is an ec-closed submodule of M, N_2 is an ec-complement submodule of N. Hence, by Proposition 4, N_1 is an N_2-ec-injective. Therefore, N is an EADS module.

The direct sum of EADS modules need not to be an EADS module. For example, let $M = (\mathbb{Z}_3 \oplus \mathbb{Z}_6)_{\mathbb{Z}}$. Then, $\mathbb{Z}_3$ and $\mathbb{Z}_6$ are EADS modules. Since $\mathbb{Z}_3$ is not $\mathbb{Z}_6$-ec-injective, we must have that M is not an EADS module. □

Proposition 11. *Let $M = \oplus_{i\in I} M_i$ be a module where M_i is a uniform module for all $i \in I$. Then, M is an EADS module if and only if M_i is an EADS module.*

Proof. Easy to check.

We now state conditions for which the direct sum of EADS modules is an EADS module. □

Proposition 12. *Let M_1 and M_2 be EADS modules such that $r(M_1) \oplus r(M_2) = R$. Then, $M_1 \oplus M_2$ is an EADS module.*

Proof. Let D be a direct summand of $M_1 \oplus M_2$. Then, there exists a submodule D' such that $M_1 \oplus M_2 = D \oplus D'$. By [[17], Proposition 3.9], $D = D_1 \oplus D_2$ and $D' = D_1' \oplus D_2'$, where D_1, D_1' and D_2, D_2' are submodules of M_1 and M_2, respectively. Let C be an ec-complement of D. So $C = M_1' \oplus M_2'$ for some ec-submodule $M_1' \leq M_1$ and $M_2' \leq M_2$. Hence, $M_1' \cap D_1 = 0$ and $M_2' \cap D_2 = 0$. So that M_1' and M_2' are ec-complement of D_1 and D_2, respectively. By the hypothesis, $M_1 = D_1 \oplus M_1'$ and $M_2 = D_2 \oplus M_2'$. Therefore, $M_1 \oplus M_2 = (D_1 \oplus M_1') \oplus (D_2 \oplus M_2') = D \oplus C$. Thus, $M_1 \oplus M_2$ is an EADS module. □

Proposition 13. *Let N be an EADS module and $N \cong M$. Then, M is an EADS module.*

Proof. Suppose $M = A \oplus B$ for some A, B are submodules of M and $f\colon M \longrightarrow N$ is an isomorphism. Since N is an EADS module, $f(A)$ is an $f(B)$-ec-injective. Let $\varphi\colon Y \longrightarrow A$ be a homomorphism, where Y is a nonzero ec-closed submodule of B. Then, $iof\colon Y \longrightarrow f(B)$ is an isomorphism. By Lemma 1, $f(Y)$ is an ec-closed submodule of $f(B)$. Since N is an EADS module, $f\varphi f^{-1}$ can be extended to $\beta\colon f(B) \longrightarrow f(A)$. Then, $f^{-1}\beta f\colon B \longrightarrow A$ is a homomorphism and for every $y \in Y$, $f^{-1}\beta f(y) = f^{-1} f \varphi f^{-1}(f(y)) = \varphi(y)$. This implies that A is a B-ec-injective. Therefore, M is an EADS module □

4. Extensions

This section explores the behavior of the EADS condition concerning various extensions of matrices in both ring and module contexts. A ring R is termed a right EADS ring if R_R satisfies the EADS module criterion. Consider, for instance, the 2×2 upper triangular matrix ring over integers, denoted as $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Here, R_R decomposes into $M_1 \oplus M_2$, where $M_1 = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$. Both M_1 and M_2 being uniform modules, as per Proposition 11, affirm R_R as an EADS module.

Now, let us commence with the subsequent ring extension.

Theorem 14. *Let R and S be two rings and M be an S-R-bimodule. Assume that $T = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ is a right EADS ring. Then,*

(i) R is a right EADS ring

(ii) If $SM = 0$, then M_R is an EADS module

Proof.

(i) Let $R_R = X \oplus Y$, K be an ec-complement submodule of X and $f\colon K \longrightarrow Y$ be an R-homomorphism. Suppose $X' = \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}$, $Y' = \begin{bmatrix} S & M \\ 0 & Y \end{bmatrix}$, and $K' = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$. It is clear that $T_T = X' \oplus Y'$. We define $\theta\colon K' \longrightarrow Y'$ via $\theta\left(\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & f(r) \end{bmatrix}$. Then, θ is a T-homomorphism and K' is an ec-complement submodule of X' in T_T. By the hypothesis, there exists a T-homomorphism $\phi\colon X' \longrightarrow Y'$ such that ϕ is an extension of θ. For any $x \in X$, $\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right) = \begin{bmatrix} s & m \\ 0 & y \end{bmatrix}$ where $s \in S$, $m \in M$, and $y \in Y$. We consider the R-homomorphism $\alpha\colon X \longrightarrow X'$ given by $\alpha(x) = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$ and $\pi\colon Y' \longrightarrow Y$ given by $\pi\left(\begin{bmatrix} s & m \\ 0 & y \end{bmatrix}\right) = y$. So $\pi\phi\alpha$ is an extension of f. Then, Y is an X-ec-injective. Therefore, R_R is an EADS module and R is an EADS ring.

(ii) Let $M_R = M_1 \oplus M_2$, N be an ec-complement submodule of M_1 and $f\colon N \longrightarrow M_2$ is an R-homomorphism. Assume $M_1' = \begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix}$, $M_2' = \begin{bmatrix} S & M_2 \\ 0 & R \end{bmatrix}$, and $N' = \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix}$. Then, $T_T = M_1' \oplus M_2'$ and N' is an ec-complement of M_1'. We define $\theta\colon N' \longrightarrow M_2'$ via $\theta\left(\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & f(n) \\ 0 & 0 \end{bmatrix}$. So θ is a T-homomorphism. By the hypothesis, there exists a T-homomorphism $M_1' \longrightarrow M_2'$ which is an extension of θ. We consider $\alpha\colon M_1 \longrightarrow M_1'$ via $\alpha(m_1) = \begin{bmatrix} 0 & m_1 \\ 0 & 0 \end{bmatrix}$ and $\pi\colon M_2' \longrightarrow M_2$ via $\pi\left(\begin{bmatrix} s & m_2 \\ 0 & r \end{bmatrix}\right) = m_2$. Then, $\alpha\phi\pi$ is an extension of f. So M_2 is an M_1-ec-injective. Hence, M_2 is an EADS module.

The following example shows that the converse of Theorem 14 is not true in general. □

Example 1. For case (i), let T be the trivial extension of $\mathbb{Z}_4$ with the $\mathbb{Z}_4$-module $2\mathbb{Z}_4$, i.e., $T = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. It is obvious that $\mathbb{Z}_4$ is an EADS module, so we have $T_T = X \oplus Y$ where $X = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. Let $N = \left\{ \begin{bmatrix} 0 & 2c \\ 0 & 2c \end{bmatrix} : c \in \mathbb{Z}_4 \right\}$. Then, N is an ec-complement of X. Since $T_T \neq X \oplus N$, T_T is not an EADS module.

For case (ii), let $T = \begin{bmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. It can be shown that $2\mathbb{Z}_4$ is an EADS module. Using the same submodule N given above, we can easily see that T_T is not an EADS module.

Lemma 15. *Let M be a right R-module and L be a submodule of M, where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$. Then,*

(i) $N \leq_e M_R$ if and only if $Ne \leq_e (Me)_S$

(ii) N is an ec-submodule of M_R if and only if Ne is an ec-submodule of $(Me)_S$

Proof.

(i) It follows from [[7], Proposition 2.77(i)].

(ii) Suppose that N is an ec-submodule of M_R. Then, $nR \leq_e N$ for some $n \in N$. This implies that $nReR \leq_e N$. By (i), $nRe \leq Ne$. So Ne is an ec-submodule of $(Me)_S$. Conversely, suppose that Ne is an ec-submodule of $(Me)_S$. Then, $neS \leq_e (Ne)_S$ for some $ne \in Ne$ so $neRe \leq_e (Ne)_S$. By (i), $neR \leq_e N$. Since $neR \leq nR \leq N$, $nR \leq_e N$. Hence, N is an ec-submodule of M_R. □

Lemma 16. *Let R be a ring where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$. Let N be a right S-module. Then, K is an ec-submodule of N if and only if $(KR)_R$ is an ec-submodule of $(NR)_R$.*

Proof. Suppose that K_S is an ec-submodule of N_S. Then, $kS \leq_e K$ for some $k \in K$ and so $keRe \leq_e KRe$. By [[7], Proposition 2.77], we obtain $\mathrm{keR} \leq_e KR$. Since $ke \in KR$, $(KR)_R$ is an ec-submodule of $(NR)_R$. Conversely, suppose that $(KR)_R$ is an ec-submodule of $(NR)_R$. Then, $krR \leq_e KR$ for some $k \in K$ and $r \in R$ so $krReR \leq_e KR$. Hence, $krReRe \leq_e KRe$. This implies that $krReeRe \leq_e KRe$. Since $K = KRe$, we must have that $krReS \leq_e K$. Hence, $krRe \in KRe = K$ which implies that K is an ec-submodule of N. □

Proposition 17. *Let M be a right R-module, where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$.*

(i) M_R is an EADS module if and only if $(Me)_S$ is an EADS module

(ii) R is a right EADS ring if and only if $(Re)_S$ is an EADS module

(iii) $(eR)_R$ is an EADS module if and only if $S = eRe$ is a right EADS ring

Proof.

(i) Let M be an EADS module, $Me = Le \oplus Ke$ and He be an ec-complement of Ke in $(Me)_S$. By [[7], Proposition 2.77(iii)], we obtain $M = L \oplus K$. From Lemma 15 (ii) and [[7], Proposition 2.77(iii)], H is an ec-complement of K in M. By the hypothesis, $M = H \oplus K$. So $Me = He \oplus Ke$. Hence, $(Me)_S$ is an EADS module. The converse is similarly proved.

(ii) Take $M = R$ in (i).

(iii) Take $M = eR$ in (i). □

Theorem 18. *Let R be any ring. Then, $M_m(R)$ is a right EADS ring, if and only if the free right R-module R^m is an EADS.*

Proof. It is clear that $M_m(R) = M_m(R)eM_m(R)$, where e is the matrix unit with 1 in the $(1, 1)$ th position and zero elsewhere. The result now follows from Proposition 17 (ii).

We conclude with the following open question: Investigate whether the EADS property remains invariant under Morita equivalence. □

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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