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On Oresme Numbers and Their Geometric Interpretations

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ABSTRACT:

In this study, we examined the Oresme sequences defined by Nicole Oresme. We examined the geometric interpretation of Oresme sequences with rational coefficients which are defined by A.F. Horadam as $O_{n+2} = O_{n+1} - \frac{1}{4}O_n$ with initial conditions $O_0 = 0$ and $O_1 = \frac{1}{2}$. We defined the the nth vector of the Oresme sequence. We calculated the area and volume. We gave the general solution for four squares equation involving Oresme vectors. We calculated the Heron Formula of Oresme sequences. We defined the angle value between these sequences. We also obtained a relationship between the Oresme sequence and the generalized Fibonacci sequence in vector space. We calculated the area and volume of these sequence. We obtained important definitions and theorems for these sequences.

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INTRODUCTION

Integer sequences have been studied in many fields of science. Among them, the most widely used and studied sequences is Horadam sequence. These sequences are a generalization of many sequences and have an important place in the literature. For $n \geq 0$, the initial conditions of the sequence Horadam, shown as $\{w_n(a,b;p,q)\}$, are $a = w_0$ and $b = w_1$. This sequence is given by the following recurrence relation (Horadam, 1965).

$$w_n = pw_{n-1} - qw_{n-2}. \tag{1}$$

Many new sequences were studied by changing the initial conditions of the Horadam sequence. Some of them are Fibonacci, Pell, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell-Lucas and Oresme sequences.

Among these sequences, Oresme (Oresme, 1961) was the first to introduce a sequence defined differently as initial conditions. In 1974, Horadam discussed this sequence and gave its recursive relation and related identities (Horadam, 1974). For $n \geq 0$ and $O_0 = 0$, $O_1 = \frac{1}{2}$ the recurrence relation is given by

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n. \tag{2}$$

Many authors have done some studies on Oresme numbers. Cook (Cook, 2004) is one of those who do these studies. Cook obtained some fundamental equations related to these numbers such as

$$O_{n+1}O_{n-1} - {O_n}^2 = \left(\frac{1}{4}\right)^n, \tag{3}$$

$$O_{n+r}O_{n-r} - {O_n}^2 = \left(\frac{1}{4}\right)^{n-r+1} F_{r-1}^2, \tag{4}$$

$$\frac{1}{2}O_{2n-1} = {O_n}^2 - \frac{1}{4}O_{n-1}, \tag{5}$$

$${O_{n+1}}^2 - \frac{1}{16}{O_{n-1}}^2 = \frac{1}{2}O_{2n+1} + \frac{1}{8}O_{2n-1}. \tag{6}$$

In (Cook, 2004), the author, for $k > 2$, $n \geq 2$, defined a different generalization which as called k-Oresme:

$${O_{n+2}}^{(k)} = {O_{n+1}}^{(k)} - \frac{1}{k^2}{O_n}^{(k)} \tag{7}$$

where the initial conditions are ${O_0}^{(k)} = 0$ and ${O_1}^{(k)} = \frac{1}{k}$. It can be clearly seen that $w_n\left(0,\frac{1}{k};1,\frac{1}{k^2}\right) = {O_n}^{(k)}$ and ${O_n}^{(2)} = O_n$. In (Halici, 2022), Halici et al. defined the *nth* k- Oresme polynomial by the following reccurence relation. With initial condition ${O_0}^{(k)}(x) = 0$, ${O_1}^{(k)}(x) = \frac{1}{kx}$, this sequnce is

$${O_{n+2}}^{(k)}(x) = {O_{n+1}}^{(k)}(x) - \frac{1}{k^2x^2}{O_n}^{(k)}(x), \tag{8}$$

where $x \in \mathbb{R}$ and $n \in \mathbb{N}$. For $z \in \mathbb{R}$, by the aid of the equation (8), its generating function can be written as

$$f(z) = \sum_{i\geq 0} {O_i}^{(k)}(x)z^i = \frac{\frac{z}{kx}}{1 - z + \frac{z^2}{k^2x^2}}. \tag{9}$$

In (Halici and Gur, 2023), the authors examined some sum formulas and derivatives of these polynomials.

$$\sum_{i=0}^{n} {O_i}^{(k)}(x) = k^2x^2\left(\frac{1}{kx} - {O_{n+2}}^{(k)}(x)\right). \tag{10}$$

For $p = 1$, $q = \frac{1}{4}$ Horadam gave the following equation (Horadam, 1974).

$$O_n = \frac{1}{2}U_{n-1}. \tag{11}$$

The most commonly used sequence among the sequences examined is the Fibonacci sequence.

$$F_{n+1} = F_n + F_{n-1}. \tag{12}$$

This sequence is

$\{\ldots, F_{-n}, \ldots, -1, 1, 0, 1, 1, 2, 3, 5, \ldots, F_n, \ \ldots\}.$

This sequence is used in many areas of mathematics. Some of these are the number theory, cryptology, and geometry. Fibonacci vector geometry is a study of the properties of vectors that accept sequences of integers produced in linear recurrence relations as their coordinates. In general, the vectors consist of triplets of integers taken from an integer sequence. Since geometrically, this vectors can be represented as points in $\mathbb{Z}^3$, vector sequences are connected with polygons. That is, polygons are on planes and various geometric objects associated with these polygons can be defined as

$$\overrightarrow{F_n} = [F_{n-1} \quad F_n \quad F_{n+1}]^T. \tag{13}$$

In (Salter, 2005), the author defined the vector version of closed formula for all integers F_n as follows:

$$\overrightarrow{F_n} = \frac{1}{\alpha-\beta}\left(\alpha^n\vec{a} - \beta^n\vec{b}\right) \tag{14}$$

where $\vec{a} = [1 \quad \alpha \quad \alpha^2]^T$ and $\vec{b} = [1 \quad \beta \quad \beta^2]^T$. For all integers *n*, the Fibonacci *r*- vector F_n is defined by

$$\overrightarrow{F_n} = [F_n \quad F_{n+1} \quad F_{n+2} \ldots F_{n+r-2} \quad F_{n+r-1}]^T \tag{15}$$

where F_n is *nth* Fibonacci number (Salter, 2005). The matrix related to these vectors was defined by Cetinberk et al. This matrix is a 3×3 anti-symmetric matrix (Cetinberk et al, 2020).

$$\mathbb{F}_n = \begin{bmatrix} 0 & -F_{n+2} & F_{n+1} \\ F_{n+2} & 0 & -F_n \\ -F_{n+1} & F_n & 0 \end{bmatrix}. \tag{16}$$

By the vector $\overrightarrow{F_n}$, a triangle for the Fibonacci sequence can be created (Atanassov, 2002). The area of this triangle is

$$\Delta_n = \frac{1}{2}\sqrt{(F_nF_{n-1})^2 + (F_nF_{n+1})^2 + (F_{n-1}F_{n+1})^2}\ . \tag{17}$$

The other different representation for the area of this triangle is

$$\Delta_n = \frac{1}{2}(F_{2n-1} + F_nF_{n-1}). \tag{18}$$

Using these representations, the general solution for Fibonacci vector sequences can be obtained. The general solution is the sum of the four squares of the points (Atanassov, 2002):

$$(F_nF_{n-1})^2 + (F_nF_{n+1})^2 + (F_{n-1}F_{n+1})^2 = \left({F_{n-1}}^2 + {F_n}^2 + F_nF_{n-1}\right)^2. \tag{19}$$

The equations with Fibonacci triangles are shown on the plane and their geometric properties are examined (Atanassov, 2002). The vector F_n is studied by many authors and they also gave some new results (see Munarini, 1997;Hilton and Pedersen, 1994).

In (Kızılateş, 2021), the author investigated new families of Horadam numbers associated with finite operators.

In our current study, we define Oresme vectors and give some geometric properties of Oresme triangles.

MATERIALS AND METHODS

In this study, some geometric interpretations of Oresme sequence were examined. Geometric properties of Oresme triangles were given and some geometric equations were obtained. Also, a relationship between the Oresme sequence and the generalized Fibonacci sequence was obtained.

Definition 1. We define the nth Oresme vector as

$$\boldsymbol{O_n} = (-O_{n-1},\ 4O_n,\ 4O_{n+1}). \tag{13}$$

The Oresme vector is denoted by the letter $\vec{O}$. By using the recurrence relation and initial conditions of the Oresme sequence, we can write

$A = (-O_{n-1},\ 0,\ 0), B = (0,\ 4O_n,\ 0), C = (0, 0,\ 4O_{n+1}), n \geq 1.$

The points *A*, *B*, *C* are

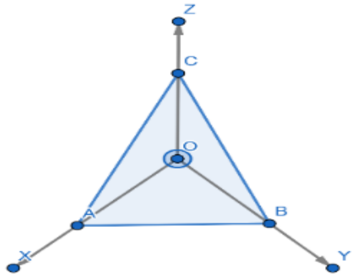

Figure 1. The Points *A*, *B*, *C*

Theorem 1. For the points A, B, C the following equality is satisfied.

$$\Delta_n = 2\sqrt{(O_n O_{n-1})^2 + 16(O_n O_{n+1})^2 + (O_{n-1} O_{n+1})^2}. \tag{14}$$

Proof. The vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are

$\overrightarrow{AC} = (O_{n-1}, 0\ ,\ 4O_{n+1}),\ \overrightarrow{BC} = (0, -\ 4O_n,\ 4O_{n+1}).$

Then, we calculate the area $|\overrightarrow{AC} \times \overrightarrow{BC}\ |$.

$$|\overrightarrow{AC} \times \overrightarrow{BC}\ | = \begin{vmatrix} O_{n-1} & 0 & 4O_{n+1} \\ 0 & -4O_n & 4O_{n+1} \end{vmatrix} = (16O_n O_{n+1}, -4O_{n-1}O_{n+1}, -4O_n O_{n-1}).$$

So, we get the following equation.

$$\Delta_n = \frac{1}{2}\sqrt{16(O_n O_{n-1})^2 + 256(O_n O_{n+1})^2 + 16(O_{n-1} O_{n+1})^2}.$$

Thus, the proof is completed.

The calculation of the first three values Δ_n is as follows. For $n = 1, 2, 3$ we have

$$\Delta_1 = 2\sqrt{(O_1 O_0)^2 + 16(O_1 O_2)^2 + (O_0 O_2)^2} = 2,$$

$$\Delta_2 = 2\sqrt{(O_3 O_1)^2 + 16(O_2 O_3)^2 + (O_1 O_3)^2} = \frac{13}{8},$$

$$\Delta_3 = 2\sqrt{(O_3 O_2)^2 + 16(O_3 O_4)^2 + (O_2 O_4)^2} = \frac{1}{2}$$

respectively.

In the next Corollary, using the areas Δ_n , the relationship between consecutive areas is given.

Corollary 1. The ratio of areas Δ_n is as follows.

$$\lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n} = \alpha^2. \tag{15}$$

Proof. First, let us calculate the limit for the ratios of the areas Δ_n and Δ_{n+1}.

$$\lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n} = \lim_{n\to\infty} \sqrt{\frac{(O_{n+1}O_n)^2 + (O_{n+1}O_{n+2})^2 + (O_n O_{n+2})^2}{(O_n O_{n-1})^2 + (O_n O_{n+1})^2 + (O_{n-1} O_{n+1})^2}},$$

$$= \lim_{n\to\infty} \sqrt{\frac{(O_{n+1}O_n)^2+\left(1+\left(\frac{O_nO_{n+2}}{O_nO_{n+1}}\right)^2+\left(\frac{O_{n+1}O_{n+2}}{O_nO_{n+1}}\right)^2\right)}{(O_nO_{n-1})^2+\left(1+\left(\frac{O_{n+1}O_{n-1}}{O_nO_{n-1}}\right)^2+\left(\frac{O_nO_{n+1}}{O_nO_{n-1}}\right)^2\right)}},$$

$$= \lim_{n\to\infty} \sqrt{\frac{{O_{n+1}}^2}{{O_{n-1}}^2}\frac{\left(1+\alpha^2+\left(\frac{O_{n+2}}{O_n}\right)^2\right)}{\left(1+\alpha^2+\left(\frac{O_{n+1}}{O_{n-1}}\right)^2\right)}}.$$

Using by the recurrence relation, we get

$$lim_{n\to\infty}\frac{\Delta_{n+1}}{\Delta_n} = \lim_{n\to\infty} \sqrt{\frac{{O_{n+1}}^2}{{O_{n-1}}^2}\frac{\left(1+\alpha^2+\left(\frac{O_{n+1}}{O_n}-\frac{1}{4}\right)^2\right)}{\left(1+\alpha^2+\left(\frac{O_n}{O_{n-1}}-\frac{1}{4}\right)^2\right)}}.$$

The limit of consecutive terms of Oresme numbers is $\lim_{n\to\infty}\frac{O_{n+1}}{O_n}=\frac{1}{2}=\alpha$ and so, we define

$$lim_{n\to\infty}\frac{\Delta_{n+1}}{\Delta_n} = \lim_{n\to\infty} \sqrt{\frac{{O_{n+1}}^2}{{O_{n-1}}^2}\frac{\left(1+\alpha^2+\left(\alpha-\frac{1}{4}\right)^2\right)}{\left(1+\alpha^2+\left(\alpha-\frac{1}{4}\right)^2\right)}} = \alpha^2 .$$

Thus, the proof is completed.

In the theorem below, we have calculated the four square equation using Oresme identities.

Theorem 2. The four square equation is

$$(O_nO_{n-1})^2+16(O_nO_{n+1})^2+(O_{n-1}O_{n+1})^2=\left(4O_n{}^2+4O_{n+1}{}^2-4O_nO_{n+1}\right)^2. \quad (16)$$

Proof. Let us take w^2 as $w^2=(-4O_nO_{n-1})^2+(16O_nO_{n+1})^2+(-4O_{n-1}O_{n+1})^2$.

Then, we write

$$w^2 = 16(O_nO_{n-1})^2+256(O_nO_{n+1})^2+16(O_{n-1}O_{n+1})^2,$$
$$= 16\left(O_n(4O_n-4O_{n+1})\right)^2+256(O_nO_{n+1})^2+16\left(O_{n+1}(4O_n-4O_{n+1})\right)^2,$$
$$= 16\left(4O_n{}^2-4O_nO_{n+1}\right)^2+256(O_nO_{n+1})^2+16\left(4O_nO_{n+1}-4O_{n+1}{}^2\right)^2,$$
$$= 256O_n{}^4-512O_n{}^3O_{n+1}+768(O_nO_{n+1})^2-512O_{n+1}{}^3O_n+256O_{n+1}{}^4,$$
$$= 16\left(16O_n{}^4+16O_{n+1}{}^4-32O_{n+1}{}^3O_n-32O_n{}^3O_{n+1}+48(O_nO_{n+1})^2\right).$$

If the needed operations are completed, then we get

$$w^2 = 16\left(4O_n{}^2+4O_{n+1}{}^2-4O_nO_{n+1}\right)^2.$$

That is $w=4\left(O_n{}^2+O_{n+1}{}^2-O_nO_{n+1}\right)$.

Theorem 3. The area formula of the shape formed by the points *A*, *B*, *C* is

$$\Delta_n = 32O_{n+1}{}^2+8O_nO_{n-1}. \quad (17)$$

Proof. Using the equality (16), we have

$$\Delta_n = 2\left(16\left(O_n{}^2+O_{n+1}{}^2-O_nO_{n+1}\right)\right),$$
$$\Delta_n = 32O_n{}^2+32O_{n+1}{}^2-O_nO_{n+1},$$

By the aid the recurrence relation related of the Oresme numbers, we get

$$\Delta_n = 32\left[O_n{}^2+\left(O_n-\frac{1}{4}O_{n-1}\right)^2-O_n\left(O_n-\frac{1}{4}O_{n-1}\right)\right],$$
$$\Delta_n = 32\left[O_n{}^2+O_n{}^2-\frac{1}{2}O_nO_{n-1}+\frac{1}{16}O_{n-1}{}^2-O_n{}^2+\frac{1}{4}O_nO_{n-1}\right].$$

Using the equations (5) and (6), we write

$$\Delta_n = 32\left[\frac{3}{8}O_{2n-1}-\frac{1}{2}O_{2n+1}+O_n{}^2-\frac{1}{2}O_{2n-1}+O_{n+1}{}^2-\frac{1}{4}O_nO_{n-1}\right],$$

$\Delta_n = 32\left[\left(O_n - \frac{1}{4}O_{n-1}\right)^2 + \frac{1}{2}O_nO_{n-1}\right]$,

$\Delta_n = 32{O_{n+1}}^2 + 16O_nO_{n-1}$.

Thus, the proof is completed.

Using the information obtained, the general solution of the four square equation was given. Moreover, for the equation U_n, if the generalized Fibonacci numbers are written instead of Oresme numbers in equation (16), a similar solution to this equation can be found.

Theorem 4. The four squares equation of generalized Fibonacci numbers is

$$(U_{n-1}U_{n-2})^2 + 16(U_nU_{n-1})^2 + (U_{n-2}U_n)^2 = \left(4{U_{n-1}}^2 + 4{U_n}^2 - 4U_{n-1}U_n\right)^2. \quad (18)$$

Proof. The proof of the theorem can be easily seen.

For $m = 3, 4, \ldots$, the general $m-$squares equation can be represented by

$${x_1}^2 + {x_2}^2 + {x_3}^2 + \cdots + {x_{m-1}}^2 = {x_m}^2$$

The two squares equation is as follows.

Theorem 5. The two squares equation is

$$\left(16{O_{n+1}}^2 + 16{O_n}^2\right)^2 - 256O_{n+1}O_n\left(2{O_{n+1}}^2 + 2{O_n}^2 - O_{n+1}O_n\right) = \left(16{O_n}^2 - 4O_nO_{n-1}\right)^2. \quad (19)$$

In the next section, we give some geometric properties of Oresme triangles.

RESULTS AND DISCUSSION

In this section, some geometric results related to the ABC triangle are given.

Definition 2. The general solution of generalized Fibonacci triangles is

$$\frac{-2x}{U_{n-2}} + \frac{y}{2U_{n-1}} + \frac{z}{2U_n} = 1. \quad (20)$$

Where, $A = \left(-\frac{1}{2}U_{n-2}, 0, 0\right)$, $B = (0, 2U_{n-1}, 0)$ and $C = (0, 0, 2U_n)$.

Specifically, if we use the equation (7) in the last equation above, then the general solution is as follows.

$$\frac{-x}{O_{n-1}} + \frac{y}{4O_n} + \frac{z}{4O_{n+1}} = 1. \quad (21)$$

Where, $A = (-O_{n-1}, 0, 0)$, $B = (0, 4O_n, 0)$ and $C = (0, 0, 4O_{n+1})$.

Now, let's take the line PN as the normal line descending from the point U_n to the plane ABC. The length of this normal line is given in the next theorem.

Theorem 6. For the points U_n, we have

$$|PN| = \frac{U_{n-2}U_{n-1}U_n}{{U_{n-1}}^2 + {U_n}^2 - U_nU_{n-1}}. \quad (22)$$

Proof. Let $(x_0, y_0, z_0) = \left(-\frac{1}{2}U_{n-2},\ 2U_{n-1},\ 2U_n\right)$, where $a = \frac{1}{-\frac{1}{2}U_{n-2}}$, $b = \frac{1}{2U_{n-1}}$, $c = \frac{1}{2U_n}$ and $d = -1$.

$$|PN| = \frac{|1+1+1-1|}{\sqrt{\left(\frac{-2}{U_{n-2}}\right)^2 + \left(\frac{1}{2U_{n-1}}\right)^2 + \left(\frac{1}{2U_n}\right)^2}} = \frac{2}{\sqrt{\frac{4}{{U_{n-2}}^2} + \frac{1}{4{U_{n-1}}^2} + \frac{1}{4{U_n}^2}}},$$

$$|PN| = \frac{8U_nU_{n-1}U_{n-2}}{2\sqrt{16(U_nU_{n-1})^2 + (U_nU_{n-2})^2 + (U_{n-1}U_{n-2})^2}}.$$

If we use the equation (18), then we get the following equation.

$$|PN| = \frac{U_{n-2}U_{n-1}U_n}{{U_{n-1}}^2 + {U_n}^2 - U_nU_{n-1}}.$$

Thus, the proof is completed.

In the next theorem, we give similar a result for the Oresme number sequence.

Theorem 7. For the points O_n, we have

$$|PN| = \frac{2O_{n-1}O_nO_{n+1}}{{O_n}^2+{O_{n+1}}^2-O_nO_{n+1}}. \quad (23)$$

Proof. Let us take the desired point as $(x_0, y_0, z_0) = (-O_{n-1},\ 4O_n,\ 4O_{n+1})$,

where $a = \frac{-1}{O_{n-1}}$, $b = \frac{1}{4O_n}$, $c = \frac{1}{4O_{n+1}}$ and $d = -1$.

$$|PN| = \frac{|1+1+1-1|}{\sqrt{\left(\frac{-1}{O_{n-1}}\right)^2+\left(\frac{1}{4O_n}\right)^2+\left(\frac{1}{4O_{n+1}}\right)^2}} = \frac{2}{\sqrt{\frac{1}{{O_{n-1}}^2}+\frac{1}{16{O_n}^2}+\frac{1}{16{O_{n+1}}^2}}},$$

$$|PN| = \frac{32O_{n-1}O_nO_{n+1}}{\sqrt{256(O_nO_{n+1})^2+16(O_nO_{n-1})^2+16(O_{n+1}O_{n-1})^2}}.$$

If we substitute equation (19), then we get the following equation.

$$|PN| = \frac{2O_{n-1}O_nO_{n+1}}{{O_n}^2+{O_{n+1}}^2-O_nO_{n+1}}.$$

In the continuation of this section, we defined the angle between the *nth* Oresme and generalized Fibonacci triangles.

Theorem 8. The angle between the vectors O_n and U_n is

$$\cos\theta = 1. \quad (24)$$

Proof. For the nth generalized Fibonacci triangle we can write as

$$D_U = \frac{1}{4}{U_{n-2}}^2 + 4{U_{n-1}}^2 + 4{U_{n-2}}^2.$$

Similarly, for the Oresme triangles we write $D_O = {O_{n-1}}^2 + 16{O_n}^2 + 16{O_{n+1}}^2$. Then, we get.

$$\cos\theta = \frac{O_n\, U_n}{|D_O||\, D_U|},$$

$$\cos\theta = \frac{\frac{1}{2}O_{n-1}U_{n-2} + 8O_nU_{n-1} + 8O_{n+1}U_n}{\sqrt{\left(\frac{1}{4}{U_{n-2}}^2 + 4{U_{n-1}}^2 + 4{U_{n-2}}^2\right)\left({O_{n-1}}^2 + 16{O_n}^2 + 16{O_{n+1}}^2\right)}}.$$

If we write the relation between the generalized Fibonacci sequence and the Oresme sequence and the Oresme sequence, then we get $\cos\theta = 1$. This case is shown in the following figure.

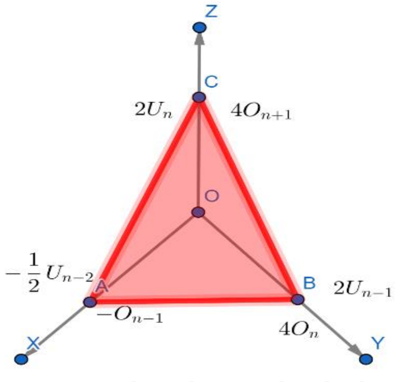

Figure 2. Oresme Triangle and Generalized Fibonacci Triangle

Corollary 2. For the areas of two consecutive Oresme triangles, we have

$$\Delta_{n+1} - \Delta_n = -{O_n}^2\left(1 + \frac{7}{n}\right). \quad (25)$$

Proof. It can be easily seen by substituting the Binet formula for the Oresme numbers.

Let us define triangle O to use in the corollary below. One of the sides of the O triangle is the hypotenuse of a right triangle formed by two of the reference axes. Using the Pisagor theorem and Oresme identities, the following equations can be given. Where, $a = -O_{n-1}$, $b = 4O_n$ and $c = 4O_{n+1}$.

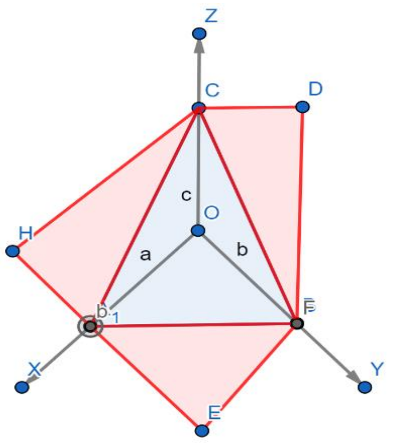

Figure 3. Right Triangle

Corollary 3. For the values $|AB| = k = \sqrt{a^2+b^2}$, $|BC| = l = \sqrt{b^2+c^2}$ ve $|AC| = m = \sqrt{a^2+c^2}$, the following results are satisfied.

i) $k = \sqrt{20{O_n}^2 - 2O_{2n-1}}$. (26)

ii) $l = \sqrt{20{O_n}^2 + 8O_{2n+1}}$. (27)

iii) $m = \sqrt{8{O_n}^2 - 4O_{2n-1} + 8O_{n-1}O_{n+1}}$. (28)

The following equation is known as Heron's Formula for the areas Δ_n. Now let us give this formula.

Corollary 4. The following equation is satisfied..

$$\Delta_n = \sqrt{s\left(s-\sqrt{20{O_n}^2 - 2O_{2n-1}}\right)\left(s-\sqrt{20{O_n}^2 + 8O_{2n+1}}\right)\left(s-\sqrt{8{O_n}^2 - 4O_{2n-1} + 8O_{n-1}O_{n+1}}\right)},$$
(29)

where $s = \frac{u+v+w}{2}$.

Let's consider the tetrahedron formed when we connect the corners of triangle O to the starting the point $Q(0,0,0)$ and examine its volume.

Theorem 9. For the Oresme vectors, we have

$V_n = Vol(T_n) = \frac{8}{3}O_{n+1}O_nO_{n-1}$. (30)

Proof. Let us calculate the equation V_n.

$$V_n = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -O_{n-1} & 0 & 0 & 1 \\ 0 & 4O_n & 0 & 1 \\ 0 & 0 & 4O_{n+1} & 1 \end{vmatrix},$$

$$V_n = \frac{8}{3}O_{n+1}O_nO_{n-1}.$$

Thus, the proof is completed.

Corollary 4. The limit and difference of the ratio of consecutive terms are as follows.

i) $\lim_{n\to\infty} \frac{V_{n+1}}{V_n} = \alpha^3$. (31)

ii) $V_{n+1} - V_n = \frac{1}{6}O_{n+1}O_n(6O_{n+1} - 5O_n)$. (32)

Now, let us calculate the volume of the generalized Fibonacci sequence from the equality of the Oresme sequence and the generalized Fibonacci sequence.

Theorem 10. For the generalized Fibonacci vectors, we have

$$V_n = Vol(T_n) = \frac{1}{3} U_n U_{n-1} U_{n-2}. \quad (33)$$

Proof. The proof can be seen easily.

$$V_n = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -\frac{1}{2}U_{n-2} & 0 & 0 & 1 \\ 0 & 2U_{n-1} & 0 & 1 \\ 0 & 0 & 2U_n & 1 \end{vmatrix},$$

$$V_n = Vol(T_n) = \frac{1}{3} U_n U_{n-1} U_{n-2}.$$

Corollary 5. For the generalized Fibonacci numbers, the following equations are satisfied.

i) $\lim_{n\to\infty} \frac{V_{n+1}}{V_n} = \alpha^3.$ (34)

ii) $V_{n+1} - V_n = \frac{1}{3} U_{n-1} U_n (U_n - U_{n-1}).$ (35)

Definition 3. In general system of linear equations, for the Oresme sequence, when the initial values are $x_{1=}\ a$ and $x_2 = b$, the following vector can be defined.

$$\overrightarrow{O_n} = a, b, O_1 a + O_2 b\ , O_2 a + O_3 b, O_3 a + O_4 b, \dots, O_{n-2} a + O_{n-1} b, \dots \quad (36)$$

Theorem 11. For the generalized Oresme vectors, we have

$$l_i = \frac{a_i + \alpha b_i}{\sum_{j=1}^{3} \left(a_j + \alpha b_j\right)^2}, \quad i = 1,2,3, \dots \quad (37)$$

Proof. For generalized Oresme vectors, the number L consisting of the limit values of consecutive terms is $L = (l_1, l_2, l_3) = (a_1 + \alpha b_1, a_2 + \alpha b_2, a_3 + \alpha b_3)$. Thus,

$GO_n = (a_1, a_2, a_3) O_{n-2} + (b_1, b_2, b_3) O_{n-1} = (k_1, k_2, k_3)$**.**

can be written. If we use the knowledge of the equation of a line passing through two points then, then we write

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \quad (38)$$

From the formula (38), we write

$$\frac{x-0}{k_1-0} = \frac{y-0}{k_2-0} = \frac{z-0}{k_3-0}, \quad \frac{x}{a_1 O_{n-2} + b_1 O_{n-1}} = \frac{y}{a_2 O_{n-2} + b_2 O_{n-1}} = \frac{z}{a_3 O_{n-2} + b_3 O_{n-1}}, \quad \frac{x}{a_1 + b_1 \frac{O_{n-1}}{O_{n-2}}} = \frac{y}{a_2 + b_2 \frac{O_{n-1}}{O_{n-2}}} = \frac{z}{a_3 + b_3 \frac{O_{n-1}}{O_{n-2}}}.$$

Then, we obtain that

$$\overrightarrow{QPN} = \frac{x}{a_1 + b_1 \frac{O_{n-1}}{O_{n-2}}} = \frac{y}{a_2 + b_2 \frac{O_{n-1}}{O_{n-2}}} = \frac{z}{a_3 + b_3 \frac{O_{n-1}}{O_{n-2}}}.$$

In the case $n \to \infty$, $\frac{O_{n-1}}{O_{n-2}} \to \frac{1}{2} = \alpha$ is obtained. Then, the desired formula is

$$(l_1, l_2, l_3) = \frac{(a_1 O_{n-2} + b_1 O_{n-1}, a_2 O_{n-2} + b_2 O_{n-1}, a_3 O_{n-2} + b_3 O_{n-1})}{\sqrt{(a_1 O_{n-2} + b_1 O_{n-1})^2 + (a_2 O_{n-2} + b_2 O_{n-1})^2 + (a_3 O_{n-2} + b_3 O_{n-1})^2}} \quad (39)$$

CONCLUSION

In this study, we give a geometric interpretation for Oresme sequences. For this purpose, we defined the nth vector related with the Oresme sequence and its the formulas area, volume. Moreover, we obtain the general solution of the four-squares equation involving Oresme vectors. We also obtain an important relationship between the Oresme sequence and the generalized Fibonacci sequence.

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Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

REFERENCES

Atanassov K., 2002. New visual perspectives on Fibonacci numbers. World Scientific.

Cetinberk K., Yuce, S., 2020. On Fibonacci Vectors. Hagia Sophia Journal of Geometry, 2(2), 12-25.

Cook C. K., 2004. Some sums related to sums of Oresme numbers. In Applications of Fibonacci

Halici S., Gur Z., 2023. On Some Derivatives of *k*- Oresme Polynomials. Bulletin of The International Mathematical Virtual Institute, 13(1), 41-50.

Halici S., Gur Z., Sayin E., 2022. *k*- Oresme Polynomials and Their Derivatives, Third International Conference on Mathematics and Its Applications in Science and Engineering, Bucharest, Romania, 4-7 July.

Hilton P., Pedersen J., 1994. A note on a geometrical property of Fibonacci numbers, The Fibanacci Quarterly, 32, 386-388.

Horadam A. F., 1965. Basic properties of a certain generalized sequence of numbers, The Fibonacci Quarterly 3(3), 161–176.

Horadam A. F., 1974. Oresme Numbers, The Fibonacci Quarterly 12(3), 267– 271.

Kızılates C., 2021. New families of Horadam numbers associated with finite operators and their applications. Mathematical Methods in the Applied Sciences, 44(18), 14371-14381.

Munarini E., 1997. A combinatorial interpretation of the generalized Fibonacci numbers. Advances in Applied Mathematics, 19(3), 306-318.

Numbers and Their Applications , 87-99.

Numbers: Volume 9: Proceedings of The Tenth International Research Conference on Fibonacci

Oresme N., 1961. Quaestiones super geometriam Euclidis, ed. by HLL Busard, 2 Vols.

Salter E., 2005. Fibonacci Vectors. Graduate Theses and Dissertations, University of South Florida, USA.