MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
https://doi.org/10.36753/mathenot.1471156
12 (4) 196-206 (2024) - Research Article
ISSN: 2147-6268

# Characterizations of Some New Classes of Four-Dimensional Matrices on the Double Series Spaces of First Order Cesàro Means

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**Abstract**
The main purpose in this study is to investigate some topological and algebraic properties of the absolutely double series spaces $|C_{1,1}|_k$ defined by combining the first order Cesàro means with the concept of absolute summability for $k \geq 1$. Beside this, we determine the $\alpha-$dual of the space $|C_{1,1}|_1$ and the $\beta(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \geq 1$. Finally, we characterize some new four-dimensional matrix classes $(|C_{1,1}|_k, \upsilon)$, $(|C_{1,1}|_1, \upsilon)$, $(|C_{1,1}|_1, \mathcal{L}_k)$, $(|C_{1,1}|_k, \mathcal{L}_u)$, $(\mathcal{L}_u, |C_{1,1}|_k)$ and $(\mathcal{L}_k, |C_{1,1}|_1)$, where $\upsilon \in \{\mathcal{M}_u, \mathcal{C}_{bp}\}$ for $1 \leq k < \infty$. Hence, some important results concerned on Cesàro matrix summation methods have been extended to double sequences.

***Keywords:*** *Double sequences, Dual spaces, Four dimensional Cesàro matrix, Four dimensional matrix transformations, Pringsheim convergence*

***AMS Subject Classification (2020):*** *46B45; 40C05; 46A45; 40F05*

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## 1. Introduction

Recently, studies on the generalization of single sequence spaces to double sequence spaces have increased. Important studies on some double sequence spaces are included in [1–12]. Using Cesàro and weighted means for single series, Hazar, Hazar and Sarıgöl [13–15] have defined new series spaces. Later, Sarıgöl has extended some results to doubly infinite series by two dimensional weighted means [16]. Further, Başar and Sever have introduced the Banach space $\mathcal{L}_k$ of double sequences corresponding to the well-known classical sequence space $\ell_k$ of single sequences [17].Also, for the special case $k = 1$, the space $\mathcal{L}_k$ is reduced to the space $\mathcal{L}_u$, which was introduced by Zeltser [18].

A double sequence $x = (x_{rs})$ is a double infinite array of elements $x_{rs}$ for all $r, s \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, ...\}$. We denote the set of all complex-valued double sequences by $\Omega$ which is a vector space with coordinatewise addition

***Received :*** *19-04-2024,* ***Accepted :*** *31-07-2024,* ***Available online :*** *31-07-2024*

*(**Cite as** "O. Bodur, C. H. Güleç, Characterizations of Some New Classes of Four-Dimensional Matrices on the Double Series Spaces of First Order Cesàro Means, Math. Sci. Appl. E-Notes, 12(4) (2024), 196-206")*

and scalar multiplication of double sequences. Any vector subspace of $\Omega$ is called as a double sequence space.

A double sequence $x = (x_{rs})$ of complex numbers is called bounded if $\|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$ which is a Banach space with the norm $\|.\|_\infty$ . Consider the double sequence $x = (x_{mn}) \in \Omega$. If for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ and $L \in \mathbb{C}$ such that $|x_{mn} - L| < \epsilon$ for all $m, n > n_0$, then we say that the double sequence $x = (x_{mn})$ is convergent in the Pringsheim's sense to the limit point $L$, where $\mathbb{C}$ denotes the complex field. Then, we write $p - \lim_{m,n\to\infty} x_{mn} = L$ and $L \in \mathbb{C}$ is called the Pringsheim limit of $x$. The space of all convergent double sequences in the Pringsheim's sense is denoted by $\mathcal{C}_p$. Unlike single sequences, $p-$convergent double sequences need not be bounded. Namely, the set $\mathcal{C}_p - \mathcal{M}_u$ is not empty. So, we consider the set $\mathcal{C}_{bp}$ of double sequences which are both convergent in the Pringsheim's sense and bounded, i.e, $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. Hardy [19] proved that a sequence in the space $\mathcal{C}_p$ is said to be regularly convergent if it is a single convergent sequence with respect to each index and the space of all such double sequences is denoted by $\mathcal{C}_r$.

Here and after, we assume that $\upsilon$ denotes any of the symbols $p$, $bp$ or $r$, and $k'$ denotes the conjugate of $k$, that is, $\frac{1}{k} + \frac{1}{k'} = 1$ for $1 < k < \infty$, and $\frac{1}{k'} = 0$ for $k = 1$.

Let $x = (x_{mn})$ be a double sequence and define the sequence $s = (s_{mn})$ as

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$$

for all $m, n \in \mathbb{N}$. For brevity, here and in what follows we use the abbreviation $\sum_{i,j=1}^{m,n} x_{ij}$ for the summation $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$. Then, the pair of $(x, s)$ is called as a double series and is denoted by $\sum_{i,j=1}^{\infty} x_{ij}$, or briefly by $\sum_{i,j} x_{ij}$. Let $\lambda$ be a space of double sequence, converging with respect to some linear convergence rule $\upsilon - \lim : \lambda \to \mathbb{C}$. The sum of a double series $\sum_{i,j} x_{ij}$ according to this rule is defined by $\upsilon - \sum_{i,j} x_{ij} = \upsilon - \lim_{m,n\to\infty} s_{mn}$.

Let us consider double sequence spaces $\lambda$ and $\mu$, and four dimensional infinite matrix $A = (a_{mnij})$ . Then we say that $A$ defines a matrix mapping from $\lambda$ into $\mu$ if for every double sequence $x = (x_{ij}) \in \lambda$, $Ax = \{(Ax)_{mn}\}_{i,j\in\mathbb{N}}$ , the $A$- transform of $x$, is in $\mu$, where

$$(Ax)_{mn} = \upsilon - \sum_{i,j} a_{mnij} x_{ij} \tag{1.1}$$

provided that the double series exists for each $m, n \in \mathbb{N}$. By $(\lambda, \mu)$ , we denote the set of such all four dimensional matrices transforming the space $\lambda$ into the space $\mu$. Thus, $A = (a_{mnij}) \in (\lambda, \mu)$ if and only if the double series on the right side of (1.1) converges in the sense of $\upsilon$ for each $m, n \in \mathbb{N}$ and $Ax \in \mu$ for all $x \in \lambda$.

The $\alpha - dual$ $\lambda^\alpha$, $\beta(\upsilon) - dual$ $\lambda^{\beta(\upsilon)}$ in regard to the $\upsilon-$convergence for $\upsilon \in \{p, bp, r\}$ , and the $\gamma - dual$ $\lambda^\gamma$ of a double sequence space $\lambda$ are respectively described as

$$\lambda^\alpha := \left\{ \varepsilon = (\varepsilon_{ij}) \in \Omega : \sum_{i,j} |\varepsilon_{ij} x_{ij}| < \infty \text{ for all } (x_{ij}) \in \lambda \right\},$$

$$\lambda^{\beta(\upsilon)} := \left\{ \varepsilon = (\varepsilon_{ij}) \in \Omega : \upsilon - \sum_{i,j} \varepsilon_{ij} x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\},$$

and

$$\lambda^\gamma := \left\{ \varepsilon = (\varepsilon_{ij}) \in \Omega : \sup_{m,n\in\mathbb{N}} \left| \sum_{i,j=1}^{m,n} \varepsilon_{ij} x_{ij} \right| < \infty \text{ for all } (x_{ij}) \in \lambda \right\}.$$

The $\upsilon-$summability domain $\lambda_A^{(\upsilon)}$ of a four dimensional complex infinite matrix $A = (a_{mnij})$ in a space $\lambda$ of double sequences is introduced by

$$\lambda_A^{(\upsilon)} = \left\{ x = (x_{ij}) \in \Omega : Ax = \left( \upsilon - \sum_{i,j} a_{mnij} x_{ij} \right)_{m,n\in\mathbb{N}} \text{ exists and is in } \lambda \right\}.$$

The four dimensional Cesàro matrix $C = (c_{mnij})$ of order one is defined by

$$c_{mnij} = \begin{cases} \frac{1}{mn}, & 1 \le i \le m,\ 1 \le j \le n \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n, i, j \in \mathbb{N}$.

Let $\sum_{i,j} x_{ij}$ be a doubly infinite series with partial sums $(s_{mn})$. The Cesàro mean $T_{mn}$ of order one of a double sequence $s = (s_{mn})$ is defined by

$$T_{mn} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij}, \ \ (m, n \in \mathbb{N}).$$

We say that $s = (s_{mn})$ is $(C, 1, 1)$ summable or double Cesàro summable to some number $\ell$ if

$$p - \lim T_{mn} = \ell.$$

From the notation of Rhoades [20], a double series $\sum_{i,j} x_{ij}$ is called absolutely double Cesàro summable $|C, 1, 1|_k$, $k \geq 1$, if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left|\bar{\Delta} T_{mn}\right|^k < \infty,$$

where, for $m, n \geq 2$,

$$\bar{\Delta} T_{m1} = T_{m1} - T_{m-1,1} \,,$$

$$\bar{\Delta} T_{1n} = T_{1n} - T_{1,n-1},$$

$$\bar{\Delta} T_{mn} = T_{mn} - T_{m-1,n} - T_{m,n-1} + T_{m-1,n-1.}$$

Further, it is easily seen that

$$T_{mn} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \left(m - i + 1\right)\left(n - j + 1\right).$$

So, we have for $m, n = 1$,

$$\bar{\Delta} T_{11} = x_{11}, \tag{1.2}$$

and, for $m, n \geq 2$,

$$\bar{\Delta} T_{m1} = \frac{1}{m\left(m-1\right)} \sum_{i=2}^{m} x_{i1} \left(i - 1\right), \tag{1.3}$$

$$\bar{\Delta} T_{1n} = \frac{1}{n\left(n-1\right)} \sum_{j=2}^{n} x_{1j} \left(j - 1\right), \tag{1.4}$$

and

$$\bar{\Delta} T_{mn} = \sum_{i=2}^{m} \sum_{j=2}^{n} \frac{x_{ij}\left(i-1\right)\left(j-1\right)}{\left(m-1\right)\left(n-1\right) mn}. \tag{1.5}$$

Now, referring Sarıgöl [16], we show the double series space $|C_{1,1}|_k$ by the set of all series summable by absolutely double Cesàro summability method of order one $|C, 1, 1|_k$, that is,

$$|C_{1,1}|_k = \left\{ x = (x_{ij}) \in \Omega : \sum_{i,j} x_{ij} \text{ is summable } |C, 1, 1|_k \right\}.$$

More recently, Mursaleen and Başar [12] have introduced the spaces $\tilde{\mathcal{M}}_u, \tilde{\mathcal{C}}_p, \tilde{\mathcal{C}}_{bp}, \tilde{\mathcal{C}}_r$ and $\tilde{\mathcal{L}}_u$ of double sequences whose Cesàro transforms of order one are in the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively. Also, they examine some properties of those sequence spaces, determine certain dual spaces and give some matrix characterizations. In this paper, we investigate some topological and algebraic properties of the absolutely double series spaces $|C_{1,1}|_k$ for $k \geq 1$ taking account of the first order double Cesàro means with the concept of absolute summability. Beside this, we determine the alpha-dual of the space $|C_{1,1}|_1$ and the $\beta\,(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \geq 1$. Finally, we characterize some new four-dimensional matrix classes $\left(|C_{1,1}|_k, \upsilon\right)$, $\left(|C_{1,1}|_1, \upsilon\right)$, $\left(|C_{1,1}|_1, \mathcal{L}_k\right)$, $\left(|C_{1,1}|_k, \mathcal{L}_u\right)$, $\left(\mathcal{L}_u, |C_{1,1}|_k\right)$ and $\left(\mathcal{L}_k, |C_{1,1}|_1\right)$, where $\upsilon \in \{\mathcal{M}_u, \mathcal{C}_{bp}\}$ for $1 \leq k < \infty$.

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## 2. Double series spaces of first order Cesàro means

In this section, we give some new results on the absolutely double Cesàro spaces $|C_{1,1}|_k$ for $k \geq 1$. Also, we determine the $\alpha-$ dual of the space $|C_{1,1}|_1$ , $\beta(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $1 \leq k < \infty$.

**Theorem 2.1.** *The set $|C_{1,1}|_k$ becomes a linear space with the coordinatewise addition and scalar multiplication, and $|C_{1,1}|_k$ is a Banach space with the norm*

$$\|x\|_{|C_{1,1}|_k} = \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left|\bar{\Delta} T_{mn}\right|^k \right)^{1/k}, \tag{2.1}$$

*which is linearly norm isomorphic to the space $\mathcal{L}_k$ for $1 \leq k < \infty$.*

*Proof.* Since the initial assertion is routine verification and so we omit it.

To prove the fact that $|C_{1,1}|_k$ is norm isomorphic to the space $\mathcal{L}_k$, we should show the existence of a linear and norm preserving bijection between the spaces $|C_{1,1}|_k$ and $\mathcal{L}_k$ for $1 \leq k < \infty$. Consider the transformation $B$ defined by

$$B : |C_{1,1}|_k \to \mathcal{L}_k$$

$$x \to y = B(x)$$

where $B(x) = (y_{mn})$ is defined by

$$B_{mn}(x) = y_{mn} = (mn)^{1-1/k} \bar{\Delta} T_{mn} \tag{2.2}$$

for $m, n \geq 1$ and $\bar{\Delta} T_{mn}$ is as in $(1.2 - 1.5)$. The linearity of $B$ is clear. Also, $x = \theta$ whenever $B(x) = \theta$, which says us that $B$ is injective.

Let $y = (y_{mn}) \in \mathcal{L}_k$ and define the sequence $x = (x_{mn})$ via $y$ by

$$\begin{aligned} x_{mn} = \quad & \frac{1}{(n-1)(m-1)} \Big[ m^{1/k} (m-1) \Big( y_{mn} n^{1/k} (n-1) - y_{m,n-1} (n-1)^{1/k} (n-2) \Big) \\ & - (m-1)^{1/k} (m-2) \Big( y_{m-1,n} n^{1/k} (n-1) - y_{m-1,n-1} (n-1)^{1/k} (n-2) \Big) \Big], \end{aligned} \tag{2.3}$$

$$x_{m1} = \frac{1}{m-1} \Big[ m^{1/k} (m-1) y_{m1} - (m-1)^{1/k} (m-2) y_{m-1,1} \Big], \tag{2.4}$$

$$x_{1n} = \frac{1}{n-1} \Big[ n^{1/k} (n-1) y_{1n} - (n-1)^{1/k} (n-2) y_{1,n-1} \Big], \tag{2.5}$$

for $m, n \geq 2$, and

$$x_{11} = y_{11}. \tag{2.6}$$

In that case, it seen that

$$\|x\|_{|C_{1,1}|_k} = \|B(x)\|_{\mathcal{L}_k} = \left( \sum_{m,n} |B_{mn}(x)|^k \right)^{1/k} = \|y\|_{\mathcal{L}_k} < \infty$$

for $1 \leq k < \infty$. So, this yields that $B$ is surjective and norm preserving. Thus, $B$ is a linear and norm preserving bijection which says the spaces $|C_{1,1}|_k$ and $\mathcal{L}_k$ are norm isomorphic for $1 \leq k < \infty$, as desired.

Now, we may show that $|C_{1,1}|_k$ is a Banach space with norm defined by $(2.1)$. To prove this, we can consider "Let $(X, \rho)$ and $(Y, \sigma)$ be semi-normed spaces and $F : (X, \rho) \to (Y, \sigma)$ be an isometric isomorphism. Then $(X, \rho)$ is complete if and only if $(Y, \sigma)$ is complete. In particular, $(X, \rho)$ is a Banach space if and only if $(Y, \sigma)$ is a Banach space." which can be found section (b) of Corollary 6.3.41 in [21]. Since the transformation $B$ defined from $|C_{1,1}|_k$ into $\mathcal{L}_k$ by $(2.2)$ is an isometric isomorphism and the double sequence space $\mathcal{L}_k$ is a Banach space from Theorem 2.1 in [17], we deduce that the space $|C_{1,1}|_k$ is a Banach space. This is the result that we desired. □

Now we have the following significant lemma, which will be used in the following theorems in order to calculate the $\alpha-$, $\beta(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \geq 1$.

**Lemma 2.1.** *[22] Let $A = (a_{mnij})$ be any four dimensional infinite matrix. At that case, the following statements are satisfied:*

***(a)*** *Let $0 < k \leq 1$. Then, $A \in (\mathcal{L}_k, \mathcal{M}_u)$ iff*

$$\xi_1 = \sup_{m,n,i,j \in \mathbb{N}} |a_{mnij}| < \infty. \tag{2.7}$$

***(b)*** *Let $1 < k < \infty$. Then, $A \in (\mathcal{L}_k, \mathcal{M}_u)$ iff*

$$\xi_2 = \sup_{m,n \in \mathbb{N}} \sum_{i,j} |a_{mnij}|^{k'} < \infty. \tag{2.8}$$

***(c)*** *Let $0 < k \leq 1$ and $1 \leq k_1 < \infty$. Then, $A \in (\mathcal{L}_k, \mathcal{L}_{k_1})$ iff*

$$\sup_{i,j \in \mathbb{N}} \sum_{m,n} |a_{mnij}|^{k_1} < \infty.$$

***(d)*** *Let $0 < k \leq 1$.Then, $A \in (\mathcal{L}_k, \mathcal{C}_{bp})$ iff the condition (2.7) holds and there exists a $(\lambda_{ij}) \in \Omega$ such that*

$$bp - \lim_{m,n \to \infty} a_{mnij} = \lambda_{ij}. \tag{2.9}$$

***(e)*** *Let $1 < k < \infty$. Then, $A \in (\mathcal{L}_k, \mathcal{C}_{bp})$ iff (2.8) and (2.9) are satisfied.*

**Lemma 2.2.** *[23] Let $1 < k < \infty$ and $A = (a_{mnrs})$ be a four dimensional infinite matrix of complex numbers. Define $W_k(A)$ and $w_k(A)$ by*

$$W_k(A) = \sum_{r,s=1}^{\infty} \left( \sum_{m,n=1}^{\infty} |a_{mnrs}| \right)^k,$$

$$w_k(A) = \sup_{M \times N} \sum_{r,s=1}^{\infty} \left| \sum_{(m,n) \in M \times N} a_{mnrs} \right|^k,$$

*where the supremum is taken through all finite subsets $M$ and $N$ of $\mathbb{N}$. Then, the following statements are equivalent:*

$$i)\ W_{k'}(A) < \infty\ , \qquad ii)\ A \in (\mathcal{L}_k, \mathcal{L}_u)$$

$$iii)\ A^t \in (\mathcal{L}_\infty, \mathcal{L}_{k'}) < \infty\ , \qquad ii) w_{k'}(A) < \infty.$$

*To shorten the following theorems and their proofs let us define the sets $\psi_p$ with $p \in \{1,2,3,4\}$ as follows:*

$$\psi_1 = \left\{ b = (b_{mn}) \in \Omega : \sup_{i,j \in \mathbb{N}} \sum_{m,n} |g_{mnij}| < \infty \right\}, \tag{2.10}$$

$$\psi_2 = \left\{ b = (b_{mn}) \in \Omega : \sup_{r,s,i,j \in \mathbb{N}} \left| \sum_{m=i}^{r} \sum_{n=j}^{s} b_{mn} f_{mnij}^{(1)} \right| < \infty \right\}, \tag{2.11}$$

$$\psi_3 = \left\{ b = (b_{mn}) \in \Omega : bp - \lim_{r,s \to \infty} \sum_{m=i}^{r} \sum_{n=j}^{s} b_{mn} f_{mnij}^{(k)} \text{ exists} \right\}, \tag{2.12}$$

$$\psi_4 = \left\{ b = (b_{mn}) \in \Omega : \sup_{r,s \in \mathbb{N}} \sum_{i,j} \left| \sum_{m=i}^{r} \sum_{n=j}^{s} b_{mn} f_{mnij}^{(k)} \right|^{k'} < \infty \right\}, \tag{2.13}$$

*where the 4-dimensional matrices $G = (g_{mnij})$ and $F^{(k)} = \left( f_{mnij}^{(k)} \right)$ are defined by*

$$g_{mnij} = \begin{cases} \dfrac{b_{mn}}{(n-1)(m-1)} (-1)^{m+n-(i+j)} (i-1)(j-1)\, ij, \ m-1 \leq i \leq m \text{ and } n-1 \leq j \leq n \\ \dfrac{b_{m1}}{m-1} (-1)^{m-i} (i-1)\, i, \ m-1 \leq i \leq m \text{ and } n = 1 \\ \dfrac{b_{1n}}{n-1} (-1)^{n-j} (j-1)\, j, \ n-1 \leq j \leq n \text{ and } m = 1 \\ b_{11}, \ n = m = 1 \end{cases} \tag{2.14}$$

*and*

$$f_{mnij}^{(k)}=\begin{cases}\dfrac{(-1)^{m+n-(i+j)}}{(n-1)(m-1)}(i-1)(j-1)(ij)^{1/k}, & m-1\leq i\leq m \text{ and } n-1\leq j\leq n\\ \dfrac{(-1)^{m-i}}{m-1}(i-1)(i)^{1/k}, & m-1\leq i\leq m \text{ and } n=1\\ \dfrac{(-1)^{n-j}}{n-1}(j-1)(j)^{1/k}, & n-1\leq j\leq n \text{ and } m=1\\ 1, & n=m=1\end{cases}, \tag{2.15}$$

*respectively.*

Now we give theorems determining the $\alpha-$dual of the space $|C_{1,1}|_1$ and $\beta-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$.

**Theorem 2.2.** *Let the set $\psi_1$ and the 4-dimensional matrix $G=(g_{mnij})$ be defined as in (2.10) and (2.14), respectively. Then, $\left(|C_{1,1}|_1\right)^{\alpha}=\psi_1$.*

*Proof.* Let $b=(b_{mn})\in\Omega$ , $x=(x_{mn})\in|C_{1,1}|_1$ and $y=(y_{ij})\in\mathcal{L}_u$. Taking account of relations in $(2.3-2.6)$ for $m,n\geq 1$, we obtain the following equalities: for $m,n\geq 2$

$$b_{mn}x_{mn}=\frac{b_{mn}}{(n-1)(m-1)}\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}(-1)^{m+n-(i+j)}(i-1)(j-1)\,ijy_{ij}=(Gy)_{mn},$$

for $n=1$ and $m\geq 2$

$$b_{m1}x_{m1}=\frac{b_{m1}}{m-1}\sum_{i=m-1}^{m}(-1)^{m-i}(i-1)\,iy_{i1}=(Gy)_{m1},$$

for $m=1$ and $n\geq 2$

$$b_{1n}x_{1n}=\frac{b_{1n}}{n-1}\sum_{j=n-1}^{n}(-1)^{n-j}(j-1)\,jy_{1j}=(Gy)_{1n}$$

and for $n=m=1$

$$b_{11}x_{11}=b_{11}y_{11}=(Gy)_{11},$$

where the four-dimensional matrix $G=(g_{mnij})$ defined by (2.14). In this fact, we see that $bx=(b_{mn}x_{mn})\in\mathcal{L}_u$ whenever $x\in|C_{1,1}|_1$ iff $Gy\in\mathcal{L}_u$ whenever $y\in\mathcal{L}_u$. This leads that $b=(b_{mn})\in\left(|C_{1,1}|_1\right)^{\alpha}$ iff $G\in(\mathcal{L}_u,\mathcal{L}_u)$. Then, we deduce by using (c) of Lemma 2.1 with $k_1=k=1$ that

$$\sup_{i,j\in\mathbb{N}}\sum_{m,n}|g_{mnij}|<\infty.$$

Hence, we have $\left(|C_{1,1}|_1\right)^{\alpha}=\psi_1$, as desired. This step concludes the proof. □

**Theorem 2.3.** *Let the sets $\psi_2$, $\psi_3$, $\psi_4$ and the 4-dimensional matrix $F^{(k)}=\left(f_{mnij}^{(k)}\right)$ be defined as in $(2.11-2.13)$ and (2.15), respectively. Then, $\left(|C_{1,1}|_1\right)^{\beta(bp)}=\psi_2\cap\psi_3$ for $k=1$ and $\left(|C_{1,1}|_k\right)^{\beta(bp)}=\psi_3\cap\psi_4$ for $1<k<\infty$.*

*Proof.* Let $b=(b_{mn})\in\Omega$ and $x=(x_{mn})\in|C_{1,1}|_k$ be given. Then, we write from Theorem 2.1 that there exists a double sequence $y=(y_{ij})\in\mathcal{L}_k$. Therefore, by using the equations $(2.3-2.6)$ we obtain that

$$z_{rs}=\sum_{m=1}^{r}\sum_{n=1}^{s}b_{mn}x_{mn}=\sum_{i=1}^{r}\sum_{j=1}^{s}\left(\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f_{mnij}^{(k)}\right)y_{ij}=(Dy)_{rs}$$

for every $r,s\in\mathbb{N}$. Thus, we see that $bx=(b_{mn}x_{mn})\in\mathcal{CS}_{bp}$ whenever $x=(x_{mn})\in|C_{1,1}|_k$ iff $z=(z_{rs})\in\mathcal{C}_{bp}$ whenever $y=(y_{ij})\in\mathcal{L}_k$. This leads to the fact that $b=(b_{mn})\in\left(|C_{1,1}|_k\right)^{\beta(bp)}$ iff $D\in(\mathcal{L}_k,\mathcal{C}_{bp})$, where the four-dimensional matrix $D=(d_{rsij})$ is defined by

$$d_{rsij}=\begin{cases}\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f_{mnij}^{(k)}, 1\leq i\leq r \text{ and } 1\leq j\leq s\\ 0, \text{ otherwise}\end{cases}$$

for every $r,s,i,j\in\mathbb{N}$. Hence, we deduce $\left(|C_{1,1}|_1\right)^{\beta(bp)}=\psi_2\cap\psi_3$ and $\left(|C_{1,1}|_k\right)^{\beta(bp)}=\psi_3\cap\psi_4$ for $1<k<\infty$ from parts (d) and (e) of Lemma 2.1, respectively. □

**Theorem 2.4.** *Let the sets* $\psi_2$, $\psi_4$ *and the* 4*-dimensional matrix* $F^{(k)} = \left(f^{(k)}_{mnij}\right)$ *be defined as in* $(2.11)$, $(2.13)$ *and* $(2.15)$, *respectively. Then,* $\left(|C_{1,1}|_1\right)^{\gamma} = \psi_2$ *and* $\left(|C_{1,1}|_k\right)^{\gamma} = \psi_4$ *for* $1 < k < \infty$.

*Proof.* This theorem can be proved by analogy with the proof Theorem 2.3 using Parts (a) and (b) of Lemma 2.1 in place of parts (d) and (e) of Lemma 2.1, respectively. So we leave the details to readers. □

## 3. Characterizations of some classes of four-dimensional matrices

In the present section, we characterize some matrix mappings from double series spaces $|C_{1,1}|_1$ and $|C_{1,1}|_k$ to the double sequence spaces $\mathcal{M}_u$, $\mathcal{C}_{bp}$, $\mathcal{L}_u$ and $\mathcal{L}_k$ for $1 \leq k < \infty$. Although the theorem characterizing matrix mappings from double series spaces $|C_{1,1}|_1$ and $|C_{1,1}|_k$ to the double sequence space $\mathcal{M}_u$ is given with proof, other theorems characterizing other mappings are given without proof since the proof techniques are similar.

**Theorem 3.1.** *Suppose that* $A = (a_{mnij})$ *be an arbitrary* $4-$*dimensional infinite matrix and the* 4*-dimensional matrix* $F^{(k)} = \left(f^{(k)}_{mnij}\right)$ *be defined as in* $(2.15)$ *for* $1 \leq k < \infty$. *In that case, the following statements hold:*

***(a)*** $A \in \left(|C_{1,1}|_1, \mathcal{M}_u\right)$ *if and only if*

$$A_{mn} \in \left(|C_{1,1}|_1\right)^{\beta(bp)} \tag{3.1}$$

*and*

$$\sup_{m,n,u,v\in\mathbb{N}} \left|\sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f^{(1)}_{ijuv}\right| < \infty. \tag{3.2}$$

***(b)*** *Let* $1 < k < \infty$. *Then,* $A \in \left(|C_{1,1}|_k, \mathcal{M}_u\right)$ *if and only if*

$$A_{mn} \in \left(|C_{1,1}|_k\right)^{\beta(bp)} \tag{3.3}$$

*and*

$$\sup_{m,n\in\mathbb{N}} \sum_{u,v} \left|\sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f^{(k)}_{ijuv}\right|^{k'} < \infty. \tag{3.4}$$

*Proof.* The part (a) can be proved by using Lemma 2.1 (a) in a similar way to that used in the proof of the part (b) of Theorem, so, we give the proof only for $1 < k < \infty$ to avoid the repetition of similar statements.

**(b)** Let $1 < k < \infty$ and $x = (x_{ij}) \in |C_{1,1}|_k$. Then, there exists a double sequence $y = (y_{mn}) \in \mathcal{L}_k$. By using the equalities $(2.3 - 2.6)$, for $(s,t)$th rectangular partial sum of the series $\sum_{i,j} a_{mnij}x_{ij}$, we have

$$\begin{aligned} (Ax)^{[s,t]}_{mn} &= \sum_{i,j=1}^{s,t} a_{mnij}x_{ij} \\ &= \sum_{i,j}^{s,t} a_{mnij} \sum_{u,v}^{i,j} f_{ijuv}y_{uv} \\ &= \sum_{u,v=1}^{s,t}\left(\sum_{i=u}^{s}\sum_{j=v}^{t} f_{ijuv}a_{mnij}\right) y_{uv} \\ &= \sum_{u,v=1}^{s,t} h^{mn}_{stuv}y_{uv} \end{aligned} \tag{3.5}$$

for every $m,n,s,t \in \mathbb{N}$, where the $4-$ dimensional matrix $H_{mn} = (h^{mn}_{stuv})$ is defined by

$$h^{mn}_{stuv} = \begin{cases} \sum_{i=u}^{s}\sum_{j=v}^{t} f_{ijuv}a_{mnij}, & 1 \leq u \leq s \text{ and } 1 \leq v \leq t \\ 0, & \text{otherwise} \end{cases}$$

for every $s,t,u,v \in \mathbb{N}$. Then, the equality $(3.5)$ can be written as

$$(Ax)^{[s,t]}_{mn} = (H_{mn}y)_{[s,t]}\,. \tag{3.6}$$

Therefore, it follows from (3.6) that the *bp*-convergence of $(Ax)_{mn}^{[s,t]}$ and the statement $H_{mn} \in (\mathcal{L}_k, \mathcal{C}_{bp})$ are equivalent for all $x \in |C_{1,1}|_k$ and $m, n \in \mathbb{N}$. Hence, the condition (3.3) is satisfied for each fixed $m, n \in \mathbb{N}$, that is, $A_{mn} \in \left(|C_{1,1}|_k\right)^{\beta(bp)}$ for each fixed $m, n \in \mathbb{N}$ and $1 < k < \infty$.

If we take *bp*-limit in the terms of the matrix $H_{mn} = (h_{stuv}^{mn})$ while $s, t \to \infty$, we obtain that

$$bp - \lim_{s,t\to\infty} h_{stuv}^{mn} = \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}. \tag{3.7}$$

With the relation (3.7), we can define the 4-dimensional matrix $H = (h_{mnuv})$ as

$$h_{mnuv} = \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}$$

for all $m, n, u, v \in \mathbb{N}$. In this situation, we deduce from the equations (3.6) and (3.7) that

$$bp - \lim_{s,t\to\infty} (Ax)_{mn}^{[s,t]} = bp - \lim (Hy)_{mn}. \tag{3.8}$$

Thus, one can write that $A = (a_{mnij}) \in \left(|C_{1,1}|_k, \mathcal{M}_u\right)$ if and only if $H \in (\mathcal{L}_k, \mathcal{M}_u)$, by having in mind the relation (3.8).

Therefore, using Lemma 2.1 (b), we obtain that

$$\sup_{m,n\in\mathbb{N}} \sum_{u,v} \left| \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}^{(k)} \right|^{k'} < \infty,$$

which satisfies the condition (3.4).

So, we conclude that $A = (a_{mnij}) \in \left(|C_{1,1}|_k, \mathcal{M}_u\right)$ if and only if the conditions (3.3) and (3.4) are satisfied. This completes the proof. □

**Theorem 3.2.** *Suppose that $A = (a_{mnij})$ be an arbitrary 4−dimensional infinite matrix and the 4-dimensional matrix $F^{(k)} = \left(f_{mnij}^{(k)}\right)$ be defined as in (2.15) for $1 \le k < \infty$. In that case, the following statements hold:*

***(a)*** *$A \in \left(|C_{1,1}|_1, \mathcal{C}_{bp}\right)$ if and only if (3.1), (3.2) hold and there exists $\left(\alpha_{uv}^{(1)}\right) \in \Omega$ such that*

$$bp - \lim_{m,n\to\infty} \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}^{(1)} = \alpha_{uv}^{(1)}.$$

***(b)*** *Let $1 < k < \infty$. Then, $A \in \left(|C_{1,1}|_k, \mathcal{C}_{bp}\right)$ if and only if (3.3), (3.4) hold and there exists $(\alpha_{uv}) \in \Omega$ such that*

$$bp - \lim_{m,n\to\infty} \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}^{(k)} = \alpha_{uv}.$$

*Proof.* This theorem can be proved by using Lemma 2.1 (d) and (e) in a similar way to that used in the proof of Theorem 3.1. □

**Theorem 3.3.** *Suppose that $A = (a_{mnij})$ be an arbitrary 4−dimensional infinite matrix and the 4-dimensional matrix $F^{(k)} = \left(f_{mnij}^{(k)}\right)$ be defined as in (2.15) for $1 \le k < \infty$. In that case, the following statements hold:*

***(a)*** *Let $1 \le k < \infty$. Then, $A \in \left(|C_{1,1}|_1, \mathcal{L}_k\right)$ if and only if (3.1) holds and*

$$\sup_{r,s\in\mathbb{N}} \sum_{m,n} \left| \sum_{i=r}^{\infty}\sum_{j=s}^{\infty} a_{mnij} f_{ijrs}^{(1)} \right|^{k} < \infty.$$

***(b)*** *Let $1 < k < \infty$. Then, $A = (a_{mnij}) \in \left(|C_{1,1}|_k, \mathcal{L}_u\right)$ if and only if* (3.3) *holds and*

$$\sum_{r,s=1}^{\infty}\left(\sum_{m,n}^{\infty}\left|\sum_{i=r}^{\infty}\sum_{j=s}^{\infty}a_{mnij}f_{ijrs}^{(k)}\right|\right)^{k'} < \infty.$$

*Proof.* This theorem can be proved by using Lemma 2.1 (c) and Lemma 2.2 in a similar way to that used in the proof of Theorem 3.1. □

**Lemma 3.1.** *[22] Let $\lambda$ and $\mu$ be two double sequence spaces in $\Omega$, $A = (a_{mnij})$ an arbitrary 4-dimensional infinite matrix and $\Phi = (\phi_{mnuv})$ be triangle 4-dimensional infinite matrix. Then, $A \in (\lambda, \mu_\Phi)$ if and only if $\Phi A \in (\lambda, \mu)$.*

*Now, we can give the final results of our work by considering the Lemma 2.1, 2.2 and 3.1.*

**Corollary 3.1.** *Let $A = (a_{mnij})$ and $\Phi = (\phi_{mnuv})$ four dimensional matrices be given by the relation*

$$\phi_{mnuv} = \sum_{i,j=1}^{m,n} b_{mnij}a_{ijuv},$$

*where $B = (b_{mnij})$ is defined as*

$$b_{mnij} = \begin{cases} 1, \ m = n = 1 \\ \dfrac{(i-1)}{m^{1/k}(m-1)}, \ 2 \le i \le m \text{ and } n = 1 \\ \dfrac{(j-1)}{n^{1/k}(n-1)}, \ 2 \le j \le n, \text{ and } m = 1 \\ \dfrac{(i-1)(j-1)}{(m-1)(n-1)(mn)^{1/k}}, \ 2 \le i \le m \text{ and } 2 \le j \le n \\ 0, \ \text{otherwise} \end{cases}$$

*and, by considering the relation* (2.1)*. Then, the necessary and sufficient conditions for the classes $\left(\mathcal{L}_u, |C_{1,1}|_k\right)$ and $\left(\mathcal{L}_q, |C_{1,1}|_1\right)$ can be found as follows:*

***(a)*** *$A = (a_{mnij}) \in \left(\mathcal{L}_u, |C_{1,1}|_k\right)$ if and only if*

$$\sup_{u,v\in\mathbb{N}}\sum_{m,n}|\phi_{mnuv}|^k < \infty$$

*holds for $1 \le k < \infty$.*

***(b)*** *$A = (a_{mnij}) \in \left(\mathcal{L}_q, |C_{1,1}|_1\right)$ if and only if*

$$\sum_{u,v=1}^{\infty}\left(\sum_{m,n=1}^{\infty}|\phi_{mnuv}|\right)^{q'} < \infty$$

*holds for $1 < q < \infty$ and $k = 1$.*

## 4. Conclusion

In this study, we investigate some topological and algebraic properties of the absolutely double series spaces $|C_{1,1}|_k$ defined by combining the first order Cesàro means with the concept of absolute summability for $k \ge 1$. Beside this, we determine the $\alpha-$dual of the space $|C_{1,1}|_1$ and the $\beta\,(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \ge 1$. Finally, we characterize some new four-dimensional matrix classes on the absolutely double series spaces $|C_{1,1}|_k$. Hence, some important results concerned on Cesàro matrix summation methods have been extended to double sequences.

## Article Information

**Acknowledgements:** The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

**Author's contributions:** All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Conflict of Interest Disclosure:** No potential conflict of interest was declared by the authors.



**Supporting/Supporting Organizations:** No grants were received from any public, private or non-profit organizations for this research.

**Ethical Approval and Participant Consent:** It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

**Plagiarism Statement:** This article was scanned by the plagiarism program.

## References

[1] Bromwich, T. J.:*An Introduction to the Theory of Infinite Series.* Macmillan, New York, NY, USA, (1965).

[2] Demiriz, S., Duyar, O.:*Domain of difference matrix of order one in some spaces of double sequences.* Gulf Journal of Mathematics. **3**(3), 85–100 (2015).

[3] Demiriz, S., Duyar, O.:*Domain of the generalized double Cesàro matrix in some paranormed spaces of double sequences.* Tbilisi Mathematical Journal. **10**(2), 43-56 (2017).

[4] Demiriz, S., Erdem, S.:*Domain of Euler-totient matrix operator in the space $\mathcal{L}_p$.* Korean Journal of Mathematics. **28**(2), 361-378 (2020).

[5] Demiriz, S., Erdem, S.:*Domain of binomial matrix in some spaces of double sequences.* Punjab University Journal of Mathematics. **52**(11), 65-79 (2020).

[6] Erdem, S., Demiriz, S.:*A study on strongly almost convergent and strongly almost null binomial double sequence spaces.* Fundamental Journal of Mathematics and Applications. **4**(4), 271-279 (2021).

[7] Erdem, S., Demiriz, S.:*A new RH-regular matrix derived by Jordan's function and its domains on some double sequence spaces.* Journal of Function Spaces. **2021**, Article ID 5594751, 9 pages, (2021).

[8] Erdem, S., Demiriz, S.:*$q-$Cesàro double sequence space $\tilde{\mathcal{L}}_q$ derived by $q-$analog.*Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica. **22**, 111-126 (2023).

[9] Móricz, F.:*Extensions of the spaces $c$ and $c_0$ from single to double sequences.* Acta Mathematica Hungarica. **57**(1-2), 129–136 (1991).

[10] Móricz, F., Rhoades, B. E.:*Almost convergence of double sequences and strong regularity of summability matrices.* Mathematical Proceedings of the Cambridge Philosophical Society. **104**(2), 283–294 (1988).

[11] Mursaleen, M.:*Almost strongly regular matrices and a core theorem for double sequences.* Journal of Mathematical Analysis and Applications. **293**(2), 523–531 (2004).

[12] Mursaleen, M., Başar, F.:*Domain of Cesàro mean of order one in some spaces of double sequences.* Studia Scientiarum Mathematicarum Hungarica. **51**, 335-356 (2014).

[13] Hazar Güleç, G. C.:*Compact matrix operators on absolute Cesàro spaces.* Numerical Functional Analysis and Optimization. **41**(1), 1-15 (2020).

[14] Hazar Güleç, G. C.:*Characterization of some classes of compact and matrix operators on the sequence spaces of Cesàro means.* Operators and Matrices. **13**(3), 809-822 (2019).

[15] Hazar Güleç, G. C., Sarigöl, M. A.:*Matrix mappings and norms on the absolute Cesàro and weighted spaces.* Quaestiones Mathematicae. **43**(1), 117-130 (2020).

[16] Sarıgöl, M. A.:*On equivalence of absolute double weighted mean methods.* Quaestiones Mathematicae. **44**(6), 755-764 (2021).

[17] Başar, F., Sever, Y.:*The space $\mathcal{L}_k$ of double sequences.* Mathematical Journal of Okayama University. **51**, 149–157 (2009).

[18] Zeltser, M.:*On conservative matrix methods for double sequence spaces.* Acta Mathematica Hungarica. **95**(3), 225-242 (2002).

[19] Hardy, G. H.:*On the convergence of certain multiple series.* Proceedings of the Cambridge Philosophical Society. **19**, 86–95 (1917).

[20] Rhoades, B. E.:*Absolute comparison theorems for double weighted mean and double Cesàro means.* Mathematica Slovaca. **48**, 285-291 (1998).

[21] Boos, J.:*Classical and Modern Methods in Summability.* Oxford University Press, New York, (2000).

[22] Yeşilkayagil, M., Başar, F.:*On the domain of Riesz mean in the space $\mathcal{L}_k^*$.* Filomat. **31**(4), 925-940 (2017).

[23] Sarıgöl, M. A.:*Four dimensional matrix mappings on double summable spaces.* Filomat. **37**(4), 1277-1290 (2023).

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