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Approximation by rational functions in Smirnov–Orlicz classes

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ABSTRACT

Let G be a doubly-connected domain bounded by Dini-smooth curves. In this work, the approximation properties of the Faber–Laurent rational series expansions in Smirnov–Orlicz classes $E_M(G)$ are studied.

1. Introduction, some auxilary results and main results

Let G be a doubly-connected domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curves Γ_1 and Γ_2 (the closed curve Γ_2 is in the closed curve Γ_1). Without loss of generality we assume $0 \in \operatorname{int} \Gamma_2$. Let $G_1^0 := \operatorname{int} \Gamma_1$, $G_1^\infty := \operatorname{ext} \Gamma_1$, $G_2^0 := \operatorname{int} \Gamma_2$, $G_2^\infty := \operatorname{ext} \Gamma_2$.

We denote by $\omega = \phi(z)$ the conformal mapping of G_1^∞ onto domain $D_1 := \{\omega \in \mathbb{C}\colon |\omega| > 1\}$ normalized by the conditions

$$\phi(\infty) = \infty, \qquad \lim_{z\to\infty} \frac{\phi(z)}{z} = 1$$

and let ψ be the inverse mapping of ϕ.

We denote by $\omega = \phi_1(z)$ the conformal mapping of G_2^0 onto domain $D_2 := \{\omega \in \mathbb{C}\colon |\omega| > 1\}$ normalized by the conditions

$$\phi_1(0) = \infty, \qquad \lim_{z\to 0}\bigl(z.\phi_1(z)\bigr) = 1,$$

and let ψ_1 be the inverse mapping of ϕ_1.

Let us take

$$C_{\rho_0} := \bigl\{z\colon \bigl|\phi(z)\bigr| = \rho_0 > 1\bigr\}, \qquad L_{r_0} := \bigl\{z\colon \bigl|\phi_1(z)\bigr| = r_0 > 1\bigr\}.$$

ϕ has the Laurent expansion in some neighbourhood of the point $z = \infty$ has the form

$$\phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots +$$

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and hence we have

$$\left[\phi(z)\right]^n = \gamma^n z^n + \sum_{k=0}^{n-1} \gamma_{n,k} z^k + \sum_{k<0} \gamma_{n,k} z^k.$$

The polynomial

$$\Phi_n(z) := \gamma^n z^n + \sum_{k=0}^{n-1} \gamma_{n,k} z^k$$

is called the Faber polynomial of order n for the domain G_1^0.

The function ϕ_1 has an expansion in some neighbourhood of the point origin:

$$\phi_1(z) = \frac{1}{z} + \beta_0 + \beta_1 z + \cdots + \beta_k z^k + \cdots.$$

Raising this function to the power n, we obtain

$$\left[\phi_1(z)\right]^n = F_n\left(\frac{1}{z}\right) - Q_n(z), \quad z \in G_2^0,$$

where $F_n(\frac{1}{z})$ denotes the polynomial of negative powers of z and the term $Q_n(z)$ contains non-negative powers of z; hence this is an analytic function in the domain G_2^0.

For $\Phi_n(z)$ and $F_n(\frac{1}{z})$ the following integral representations hold [21]:

1. If $z \in \operatorname{int} C_{\rho_0}$, then
$$\Phi_n(z) = \frac{1}{2\pi i} \int_{C_{\rho_0}} \frac{[\phi(\zeta)]^n}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{|\omega|=\rho_0} \frac{\psi'(\omega)\omega^n}{\psi(\omega) - z}\, d\omega. \tag{1.1}$$
2. If $z \in \operatorname{ext} C_{\rho_0}$, then
$$\Phi_n(z) = \left[\phi(z)\right]^n + \frac{1}{2\pi i} \int_{C_{\rho_0}} \frac{[\phi(\zeta)]^n}{\zeta - z}\, d\zeta. \tag{1.2}$$
3. If $z \in \operatorname{int} C_{r_0}$, then
$$F_n\left(\frac{1}{z}\right) = \left[\phi_1(z)\right]^n - \frac{1}{2\pi i} \int_{C_{r_0}} \frac{[\phi_1(\zeta)]^n}{\zeta - z}\, d\zeta. \tag{1.3}$$
4. If $z \in \operatorname{ext} C_{r_0}$ then
$$F_n\left(\frac{1}{z}\right) = -\frac{1}{2\pi i} \int_{C_{r_0}} \frac{[\phi_1(\zeta)]^n}{\zeta - z}\, d\zeta = -\frac{1}{2\pi i} \int_{|\omega|=r_0} \frac{\psi_1'(\omega)\omega^n}{\psi_1(\omega) - z}\, d\omega. \tag{1.4}$$

If a function $f(z)$ is analytic in a doubly-connected domain bounded by the curves C_{ρ_0} and Γ_{r_0}, then the following series expansion holds [17,27]:

$$f(z) = \sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} b_k F_k\left(\frac{1}{z}\right) \tag{1.5}$$

where

$$a_k = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(z)\phi'(z)}{[\phi(z)]^{k+1}}\, dz = \frac{1}{2\pi i} \int_{|\omega|=\rho_1} \frac{f[\psi(\omega)]}{\omega^{k+1}}\, d\omega \quad (1 < \rho_1 < \rho_0)$$

and

$$b_k = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(z)\phi_1'(z)}{[\phi_1(z)]^{k+1}}\, dz = \frac{1}{2\pi i} \int_{|\omega|=r_1} \frac{f[\psi_1(\omega)]}{\omega^{k+1}}\, d\omega \quad (1 < r_1 < r_0). \tag{1.6}$$

For $z \in G$ use of Cauchy theorem, gives

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\xi)}{\xi - z} \, d\xi.$$

If $z \in \operatorname{int} \Gamma_2$ and $z \in \operatorname{ext} \Gamma_1$, then

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\xi)}{\xi - z} \, d\xi = 0. \tag{1.7}$$

Let us consider

$$I_1(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \qquad I_2(z) = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\xi)}{\xi - z} \, d\xi.$$

The function $I_1(z)$ determines the functions $I_1^+(z)$ and $I_1^-(z)$ while the function $I_2(z)$ determines the functions $I_2^+(z)$ and $I_2^-(z)$. The functions $I_1^+(z)$ and $I_1^-(z)$ are analytic in $\operatorname{int} \Gamma_1$ and $\operatorname{ext} \Gamma_1$, respectively. The functions $I_2^+(z)$ and $I_2^-(z)$ are analytic in $\operatorname{int} \Gamma_2$ and $\operatorname{ext} \Gamma_2$, respectively.

Let us further assume that B is a simply-connected domain with a rectifiable Jordan boundary Γ and $B^- := \operatorname{ext} \Gamma$, further let

$$T := \{\omega \in \mathbb{C}\colon |\omega| = 1\}, \qquad D^- := \operatorname{ext} T.$$

Also, ϕ^* stand for the conformal mapping of B^- onto D^- normalized by

$$\phi^*(\infty) = \infty$$

and

$$\lim_{z \to \infty} \frac{\phi^*(z)}{z} > 0$$

and let ψ^* be the inverse of ϕ^*.

Let also χ be a continuous function on 2π. Its modulus of continuity is defined by

$$\omega(t, \chi) := \sup_{t_1, t_2 \in [0, 2\pi],\ |t_1 - t_2| < t} \left| \chi(t_1) - \chi(t_2) \right|, \quad t \geqslant 0.$$

The curve Γ is called Dini-smooth curve if it has the parametrization

$$\Gamma\colon \chi(t), \quad 0 \leqslant t \leqslant 2\pi,$$

such that $\chi'(t)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(t, \chi')}{t} \, dt < \infty$$

and

$$\chi'(t) \neq 0$$

[24, p. 48].

A convex and continuous function $M : [0, \infty) \to [0, \infty)$ is called an N-function if the conditions

$$M(0) = 0, \qquad M(x) > 0 \quad \text{for } x > 0,$$
$$\lim_{x \to 0} \frac{M(x)}{x} = 0, \qquad \lim_{x \to \infty} \frac{M(x)}{x} = \infty$$

are satisfied. The complementary N-function to M is defined by

$$N(y) := \max_{x \geqslant 0} \bigl(xy - M(x)\bigr), \quad \text{for } y \geqslant 0.$$

Let us denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M\left[\alpha\left|f(z)\right|\right]|dz| < \infty$$

for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with Orlicz norm

$$\|f\|_{L_M(\Gamma)} := \sup_{\rho(g,N)\leqslant 1} \int_\Gamma \left|f(z).g(z)\right||dz|,$$

where $g \in L_N(\Gamma)$, N is the complementary N-function to M and

$$\rho(g; N) := \int_\Gamma N\left[\left|g(z)\right|\right]|dz|.$$

The Banach space $L_M(\Gamma)$ is called an Orlicz space [26, pp. 52–68].

It is known, cf. [26, p. 50], that every function in $L_M(\Gamma)$ is integrable, i.e.

$$L_M(\Gamma) \subset L_1(\Gamma).$$

An N-function M satisfies the Δ_2-condition if

$$\lim_{x\to\infty} \sup \frac{M(2x)}{M(x)} < \infty.$$

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the N-function M and its complementary function N both satisfy Δ_2-condition [26, p. 113]. Detailed information about Orlicz spaces can be found in the books [18,26].

Let Γ_r be the image of the circle $\{\omega \in \mathbb{C}\colon\ |\omega| = r,\ 0 < r < 1\}$ under some conformal mapping of D onto B and let M be an N-function.

If an analytic function f in B satisfies

$$\int_{\Gamma_r} M\left[\left|f(z)\right|\right]|dz| < \infty,$$

uniformly in r, then it belongs to Smirnov–Orlicz class $E_M(B)$. We remark that if $M(x) := M(x, p) := x^p$, $1 < p < \infty$, then the Smirnov–Orlicz class $E_M(B)$ coincides with the usual Smirnov class $E_p(B)$.

The space $E_M(B)$ becomes a Banach space with the Orlicz norm.

Every function in the class $E_M(B)$ has [19] the non-tangential boundary values almost everywhere (a.e.) on Γ and the boundary function belongs to $L_M(\Gamma)$, and hence for $f \in E_M(B)$ the norm $E_M(B)$ can be defined as

$$\|f\|_{E_M(B)} := \|f\|_{L_M(\Gamma)}.$$

Let B be a finite domain in the complex plane by a rectifiable Jordan curve Γ and $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in B$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in B^-$$

are analytic in B and B^- respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to\infty} \frac{1}{2\pi i} \int_{\Gamma\cap\{\zeta\colon\ |\zeta - z|>\varepsilon\}} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the Cauchy singular integral of f at $z \in \Gamma$.

According to the Privalov theorem [5, p. 431], if one of the functions f^+ or f^- has the non-tangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \qquad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z)$$

and hence

$$f = f^+ - f^-$$

holds a.e. on Γ.

From the results in [20], it follows that if Γ is a Dini-smooth curve and $L_M(\Gamma)$ is a reflexive Orlicz space on Γ, the singular operator S_Γ is bounded on $L_M(\Gamma)$.

For $r > 0$ the r-th modulus of smoothness of a function $f \in L_M(T)$ is defined as

$$\omega_r(f, \delta)_M := \sup_{|h| \leqslant \delta} \left\| \Delta_h^r f \right\|_{L_M(T)}, \quad \delta > 0,\ r > 0,$$

where

$$\Delta_h^r f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\big(x + (r-k)h\big).$$

If Γ_1 and Γ_2 are Dini-smooth, then from the results in [30], it follows that

$$0 < c_1 < \left|\phi'(z)\right| < c_2 < \infty, \qquad 0 < c_3 < \left|\phi_1'(z)\right| < c_4 < \infty,$$

and

$$0 < c_5 < \left|\phi'(\omega)\right| < c_6 < \infty, \qquad 0 < c_7 < \left|\phi_1'(\omega)\right| < c_8 < \infty \tag{1.8}$$

where the constants c_1, c_2, c_3, c_4 and c_5, c_6, c_7, c_8 are independent of $z \in \bar{G}^-$ and $|\omega| \geqslant 1$, respectively.

We will say that the doubly connected domain G is bounded by the Dini-smooth curve if the domains G_1^0 and G_2^0 are bounded by the closed Dini-smooth curves.

Let Γ_i $(i = 1, 2)$ be a Dini-smooth curve and let $f_0 := f \circ \psi$ for $f \in L_M(\Gamma_1)$ and let $f_1 := f \circ \psi_1$ for $f \in L_M(\Gamma_2)$. Then by (1.8) we obtain $f_0 \in L_M(T)$ and $f_1 \in L_M(T)$. Using the non-tangential boundary values of f_0^+ and f_1^+ on T we define

$$\omega_{r,\Gamma}(f, \delta)_M := \omega_r\big(f_0^+, \delta\big)_M, \quad \delta > 0,$$
$$\tilde{\omega}_{r,\Gamma}(f, \delta)_M := \omega_r\big(f_1^+, \delta\big)_M, \quad \delta > 0,$$

for $r > 0$.

Since $f_0, f_1 \in L_M(T)$, we have $f_0^+, f_1^+ \in E_M(D)$ and $f_0^-, f_1^- \in E_M(D^-)$ such that $f_0^-(\infty) = \infty$, $f_1^-(\infty) = 0$ and

$$\left.\begin{aligned} f_0(\omega) &= f_0^+(\omega) - f_0^-(\omega), \\ f_1(\omega) &= f_1^+(\omega) - f_1^-(\omega) \end{aligned}\right\} \tag{1.9}$$

a.e. on T.

Lemma 1. *(See [4].) Let both an N-function M and its complementary function satisfy the Δ_2 condition. Then there exists a constant $c_9 > 0$, such that for every $n \in N$,*

$$\left\| g(\omega) - \sum_{k=0}^{n} d_k \omega^k \right\|_{L_M(T)} \leqslant c_9 \omega_r(g, 1/n)_M, \quad \alpha > 0$$

where d_k $(k = 0, 1, 2, \ldots)$ are the k-th Taylor coefficients of $g \in E_M(D)$ at the origin.

We set

$$R_n(f, z) := \sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=0}^{n} b_k F_k\left(\frac{1}{z}\right).$$

The rational function $R_n(f, z)$ is called the Faber–Laurent rational function of degree n of f.

Since series of Faber polynomials are a generalization of Taylor series to the case of a simply connected domain, it is natural to consider the construction of a similar generalization of Laurent series to the case of a doubly-connected domain.

In this work direct theorem of approximation theory in the Smirnov–Orlicz classes, defined in the doubly-connected domains with the Dini-smooth boundary are proved. Similar problems were studied in [17,28,29,31].

We remark that problem of approximation theory in Orlicz classes defined on the simply connected domain with boundary Γ has been investigated by Akgün and Israfilov [4] in the case that Γ is a closed Dini-smooth curve.

Similar problems for the different spaces defined on the simply connected domain of the complex plane were investigated by several authors (see for example, [1–4,6–17,22,23]). Note that the approximation of functions by polynomials and rational functions in Orlicz spaces defined on the intervals of the real line have been investigated by Ramazanov [25].

Now, in the doubly-connected domain we define Smirnov–Orlicz class. Let Γ_r be the image of the circumference $|\omega| = r$ $(r_2 < r < r_1)$ regard to a conformal mapping of the ring $0 < r_2 < |\omega| < r_1 < 1$ onto doubly-connected domain G and let M be an N-function. The class of analytic functions $f(z)$ defined on the domain G will be called Smirnov–Orlicz class $E_M(G)$ if

$$\int_{\Gamma_r} M\big[\big|f(z)\big|\big]\,|dz| < \infty,$$

uniformly in r.

Our main results are as follows.

Theorem 1. *Let G be a finite doubly-connected domain with the Dini-smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, and let $E_M(G)$ be a reflexive Smirnov–Orlicz class on G. If $r > 0$ and $f \in E_M(G)$ then for any $n = 1, 2, 3, \ldots$ there is a constant $c_{10} > 0$ such that*

$$\big\|f - R_n(.,f)\big\|_{L_M(\Gamma)} \leqslant c_{10}\big\{\tilde{\omega}_{r,\Gamma}(f,1/n)_M + \omega_{r,\Gamma}(f,1/n)_M\big\},$$

where $R_n(.,f)$ is the n-th partial sum of the Faber–Laurent series of f.

2. Proof of the new results

Proof of Theorem 1. We take the curves Γ_1, Γ_2 and $T := \{\omega \in \mathbb{C}\colon\ |\omega| = 1\}$ as the curves of integration in the formulas (1.1)–(1.4) and (1.6), respectively. (This is possible due to the conditions of Theorem 1.) Let $f \in E_M(G)$. Then $f_0, f_1 \in L_M(T)$. According to (1.9)

$$f(\zeta) = f_0^+\big(\phi(\zeta)\big) - f_0^-\big(\phi(\zeta)\big), \qquad f(\xi) = f_1^+\big(\phi_1(\xi)\big) - f_1^-\big(\phi_1(\xi)\big). \tag{2.1}$$

Let $z \in \operatorname{ext}\Gamma_1$. Then from (1.2) and (2.1) we have

$$\begin{aligned}\sum_{k=0}^{n} a_k \Phi_k(z) &= \sum_{k=0}^{n} a_k\big[\phi(z)\big]^k + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\sum_{k=0}^{n} a_k[\phi(\zeta)]^k}{\zeta - z}\,d\zeta \\ &= \sum_{k=0}^{n} a_k\big[\phi(z)\big]^k + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\sum_{k=0}^{n} a_k[\phi(\zeta)]^k - f_0^+[\phi(\zeta)]}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta - f_0^-\big[\phi(z)\big].\end{aligned} \tag{2.2}$$

For $z \in \operatorname{ext}\Gamma_2$, consideration of (1.4) and (2.1) gives

$$\begin{aligned}\sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right) &= -\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sum_{k=1}^{n} b_k[\phi_1(\xi)]^k}{\xi - z}\,d\xi \\ &= -\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sum_{k=0}^{n} b_k[\phi_1(\xi)]^k}{\xi - z}\,d\xi \\ &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_1^+[\phi_1(\xi)] - \sum_{k=0}^{n} b_k[\phi_1(\xi)]^k}{\xi - z}\,d\xi - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi - z}\,d\xi.\end{aligned} \tag{2.3}$$

For $z \in \operatorname{ext}\Gamma_1$, using (2.2), (2.3) and (1.7) we have

$$\begin{aligned}\sum_{k=0}^{n} a_k\big[\Phi_k(z)\big]^k + \sum_{k=1}^{n} a_k F_k\left(\frac{1}{z}\right) = \sum_{k=0}^{n} a_k\big[\phi(z)\big]^k - f_0^-\big[\phi(z)\big] - \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f_0^+[\phi(\zeta)] - \sum_{k=0}^{n} a_k[\phi(\zeta)]^k}{\zeta - z}\,d\zeta \\ + \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_1^+[\phi_1(\xi)] - \sum_{k=0}^{n} b_k[\phi_1(\xi)]^k}{\xi - z}\,d\xi.\end{aligned}$$

Taking the limit as $z \to z^* \in \Gamma_1$ along all non-tangential paths outside Γ_1, we obtain

$$f\left(z^*\right)-\sum_{k=0}^{n}a_k\Phi_k\left(z^*\right)-\sum_{k=1}^{n}b_kF_k\left(\frac{1}{z^*}\right)$$
$$=f_0^+\left[\phi\left(z^*\right)\right]-\sum_{k=0}^{n}a_k\left[\phi\left(z^*\right)\right]^k+\frac{1}{2}\left(f_0^+\left[\phi\left(z^*\right)\right]-\sum_{k=0}^{n}a_k\left[\phi\left(z^*\right)\right]^k\right)$$
$$+S_{\Gamma_1}\left[\left(f_0^+\circ\phi\right)-\sum_{k=0}^{n}a_k\phi^k\right]-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_1^+[\phi_1(\xi)]-\sum_{k=1}^{n}b_k[\phi_1(\xi)]^k}{\xi-z^*}\,d\xi \tag{2.4}$$

a.e. on Γ_1.

Now using (2.4), Minkowski's inequality and the boundedness of S_{Γ_1} we have

$$\left\|f-R_n(f,z)\right\|_{L_M(\Gamma_1)}\leqslant c_{11}\left\|f_0^+(\omega)-\sum_{k=0}^{n}a_k\omega^k\right\|_{L_M(T)}+c_{12}\left\|f_1^+(w)-\sum_{k=0}^{n}b_k\omega^k\right\|_{L_M(T)}. \tag{2.5}$$

That is, the Faber–Laurent coefficients a_k and b_k of the function f are the Taylor coefficients of the functions f_0^+ and f_1^+, respectively. Then by Lemma 1 and (2.5) we have

$$\left\|f-R_n(f,z)\right\|_{L_M(\Gamma_2)}\leqslant c_{13}\left\{\tilde{\omega}_{r,\Gamma}(f,1/n)_M+\omega_{r,\Gamma}(f,1/n)_M\right\}.$$

Let $z\in\operatorname{int}\Gamma_2$. Then by (1.4) and (2.1) we have

$$\sum_{k=1}^{n}b_kF_k\left(\frac{1}{z}\right)=\sum_{k=1}^{n}b_k\left[\phi_1(z)\right]^k-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sum_{k=0}^{n}b_k[\phi_1(\xi)]^k}{\xi-z}\,d\xi$$
$$=\sum_{k=1}^{n}b_k\left[\phi_1(z)\right]^k-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{(\sum_{k=1}^{n}b_k[\phi_1(\xi)]^k-f_1^+[\phi_1(\xi)])}{\xi-z}\,d\xi$$
$$-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\xi)}{\xi-z}\,d\xi-f_1^-\left[\phi_1(z)\right]. \tag{2.6}$$

For $z\in\operatorname{int}\Gamma_1$, from (1.1) and (2.1) we obtain

$$\sum_{k=1}^{n}a_k\Phi_k(z)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\sum_{k=1}^{n}a_k[\phi(\zeta)]^k}{\zeta-z}\,d\zeta$$
$$=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{(\sum_{k=1}^{n}a_k[\phi(\zeta)]^k-f_0^+[\phi(\zeta)])}{\zeta-z}\,d\zeta+\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{2.7}$$

The use of (2.6) and (2.7) for $z\in\operatorname{int}\Gamma_2$ gives

$$\sum_{k=0}^{n}a_k\Phi_k(z)+\sum_{k=1}^{n}b_kF_k\left(\frac{1}{z}\right)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{(\sum_{k=0}^{n}a_k[\phi(\zeta)]^k-f_0^+[\phi(\zeta)])}{\zeta-z}\,d\zeta$$
$$-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{(\sum_{k=1}^{n}b_k[\phi_1(\xi)]^k-f_1^+[\phi_1(\xi)])}{\xi-z}\,d\xi-f_1^-\left[\phi_1(z)\right]$$
$$+\sum_{k=1}^{n}b_k\left[\phi_1(z)\right]^k.$$

Taking the limit as $z \to z^* \in \Gamma_2$ along all non-tangential paths inside Γ_2, we reach

$$f(z^*) - \sum_{k=0}^{n} a_k \Phi_k(z^*) - \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z^*}\right) = f_1^+[\phi_1(z^*)] - \frac{1}{2}\left[\sum_{k=1}^{n} b_k[\phi_1(z^*)]^k - f_1^+[\phi_1(z^*)]\right] - S_{\Gamma_2}\left[\sum_{k=1}^{n} b_k \phi_1^k - (f_1^+ \circ \phi_1)\right] - \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\sum_{k=0}^{n} a_k[\phi(\zeta)]^k - f_0^+[\phi(\zeta)])}{\zeta - z^*}\, d\zeta \tag{2.8}$$

a.e. on Γ_2.

Consideration of (2.8), the Minkowski's inequality and the boundedness of S_{Γ_2} give rise to

$$\|f - R_n(f,z)\|_{L_M(\Gamma_2)} \leqslant c_{14}\left\| f_1^+(\omega) - \sum_{k=1}^{n} b_k \omega^k \right\|_{L_M(T)} + c_{15}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k \omega^k \right\|_{L_M(T)}. \tag{2.9}$$

Use of Lemma 1 and (2.9) leads to

$$\|f - R_n(f,z)\|_{L_M(\Gamma_2)} \leqslant c_{16}\{\tilde{\omega}_{r,\Gamma}(f,1/n)_M + \omega_{r,\Gamma}(f,1/n)_M\}.$$

Then the proof of Theorem 1 is completed. □

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