Applied Mathematics Letters 24 (2011) 126–130

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On translativity of absolute Riesz summability

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ARTICLE INFO

Article history:
Received 19 April 2010
Received in revised form 12 August 2010
Accepted 18 August 2010

Keywords:
Translativity
Absolute Riesz summability
Matrix transformations

ABSTRACT

In this work we characterize translativity of the summability $|R, p_n|_k$, $k > 1$, for any sequence (p_n) without imposing the conditions given by Orhan [C. Orhan, Translativity of absolute weighted mean summability, Czechoslovak Math. J. 48 (1998) 755–761], and so deduce some known results.

Let $\sum a_n$ be a given series with partial sums (s_n), and let (p_n) be a sequence of positive numbers such that

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty \quad \text{as } n \to \infty \qquad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (T_n) of the (R, p_n) Riesz means of the sequence (s_n), generated by the sequence coefficients (p_n). The series $\sum a_n$ is said to be summable $|R, p_n|_k$, where $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty \tag{1}$$

(see [1]) and summable $|\bar{N}, p_n|_k$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty$$

(see [2]).

Following the concept of translativity in ordinary summability, Cesco [3] introduced the concept of left translativity for the summability $|R, p_n|_k$ for the case $k = 1$ and gave sufficient conditions for $|R, p_n|$ to be left translative. Al-Madi [4] has also studied the problem of translativity for the same summability.

Analogously, we call $|R, p_n|_k$, $k \geq 1$, left translative if the summability $|R, p_n|_k$ of the series $\sum_{n=0}^{\infty} a_n$ implies the summability $|R, p_n|_k$ of the series $\sum_{n=0}^{\infty} a_{n-1}$ where $a_{-1} = 0$. $|R, p_n|_k$ is called right translative if the converse holds, and translative if it is both left and right translative.

Dealing with translativity of the summability $|R, p_n|_k$, Orhan [5] proved the following theorem which extends the known results of Al-Madi [4] and Cesco [3] to $k > 1$.

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doi:10.1016/j.aml.2010.08.031

Theorem A. *Suppose that*

$$\sum_{n=v}^{\infty} n^{k-1} \left(\frac{p_{n+1}}{P_{n+1}P_n} \right)^k = O\left(\frac{v^{k-1}}{P_{v+1}^k} \right) \tag{2}$$

and

$$\sum_{n=v}^{\infty} n^{k-1} \left(\frac{p_n}{P_{n-1}P_n} \right)^k = O\left(\frac{v^{k-1}}{P_v^k} \right) \tag{3}$$

hold, where $k \geq 1$. *Then,* $|R, p_n|_k$ *is translative if and only if*

$$\text{(a)}\ \frac{P_n}{P_{n+1}} = O\left(\frac{p_n}{p_{n+1}} \right) \quad \text{and} \quad \text{(b)}\ \frac{P_{n+1}}{P_n} = O\left(\frac{p_{n+1}}{p_n} \right). \tag{4}$$

The aim of this work is to characterize translativity of the summability $|R, p_n|_k,\ k > 1$, *for any sequence* (p_n) *without imposing the conditions* (2) *and* (3) *of* Theorem A. *Our theorems are as follows.*

Theorem 1. *Let* $1 < k < \infty$. *Then,* $|R, p_n|_k$ *is left translative if and only if* (4)(a) *holds and*

$$\left\{ \sum_{v=1}^{m-2} \frac{1}{v} \left| \frac{P_v^2}{p_v} - \frac{P_{v+1}P_{v-1}}{p_v} \right|^{k^*} \right\}^{1/k^*} \left\{ \sum_{n=m+1}^{\infty} n^{k-1} \left(\frac{p_n}{P_{n-1}P_n} \right)^k \right\}^{1/k} = O(1), \tag{5}$$

where k^* *is the conjugate index k.*

Theorem 2. *Let* $1 < k < \infty$. *Then,* $|R, p_n|_k$ *is right translative if and only if* (4)(b) *holds and*

$$\left\{ \sum_{v=1}^{m} \frac{1}{v} \left| \frac{P_v P_{v-2}}{p_v} - \frac{P_{v-1}^2}{p_v} \right|^{k^*} \right\}^{1/k^*} \left\{ \sum_{n=m}^{\infty} n^{k-1} \left(\frac{p_n}{P_{n-1}P_n} \right)^k \right\}^{1/k} = O(1), \tag{6}$$

where k^* *is the conjugate index k.*

By Theorems 1 *and* 2, *we have* Theorem A *for any sequence* (p_n) *as follows.*

Corollary 1. *Let* $1 < k < \infty$. *Then,* $|R, p_n|_k$ *is translative if and only if* (4)(a), (4)(b), (5) *and* (6) *hold.*

We require the following lemmas in the proof of the theorems.

A triangular matrix A is said to be factorable if $a_{nv} = a_n b_v$ *for* $0 \leq v \leq n$ *and zero otherwise. Then the following result of Bennett* [6] *is well known.*

Lemma 1. *Let* $1 < p \leq q < \infty$, *let a and b be sequences of non-negative numbers, and let A be a factorable matrix. Then A maps* ℓ_p *into* ℓ_q *if and only if there exists M such that, for* $m = 1, 2, \ldots,$

$$\left(\sum_{v=1}^{m} b_v^{p^*} \right)^{1/p^*} \left(\sum_{n=m}^{\infty} a_n^q \right)^{1/q} \leq M.$$

We can easily prove the following lemma by making use of Lemma 1.

Lemma 2. *Let* $1 < k < \infty$ *and let* B, C, B' *and* C' *be the matrices defined by*

$$b_{nv} = \begin{cases} P_0 n^{1/k^*} \dfrac{p_n}{P_n P_{n-1}}, & v = 0 \\ n^{1/k^*} \dfrac{p_n}{P_n P_{n-1}} \left(P_v^2 - P_{v+1}P_{v-1} \right) \dfrac{v^{-1/k^*}}{p_v}, & 1 \leq v \leq n-2 \\ 0, & v > n-2, \end{cases}$$

$$c_{nv} = \begin{cases} \left(\dfrac{n}{n-1} \right)^{1/k^*} \dfrac{p_n P_{n-1}}{P_n p_{n-1}}, & v = n-1 \\ 0, & v \neq n-1, \end{cases}$$

$$b'_{nv} = \begin{cases} n^{1/k^*} \dfrac{p_n}{P_n P_{n-1}} \left(P_v P_{v-2} - P_{v-1}^2 \right) \dfrac{v^{-1/k^*}}{p_v}, & 1 \leq v \leq n \\ 0, & v > n \end{cases}$$

and

$$c'_{nv} = \begin{cases} \left(\frac{n}{n+1}\right)^{1/k^*} \frac{p_n P_{n+1}}{P_n p_{n+1}}, & v = n+1 \\ 0, \quad v \neq n+1, \end{cases}$$

respectively. Then:

(a) *B maps ℓ_k into ℓ_k if and only if (5) is satisfied,*
(b) *C maps ℓ_k into ℓ_k if and only if (4)(a) is satisfied,*
(c) *B' maps ℓ_k into ℓ_k if and only if (6) is satisfied,*
(d) *C' maps ℓ_k into ℓ_k if and only if (4)(b) is satisfied.*

Lemma 3. *Let A be an infinite matrix. If A maps ℓ_k into ℓ_k, then there exists a constant M such that $|a_{nv}| \leq M$ for all $v, n \in \mathbb{N}$.*

Proof. A is continuous, which is immediate as ℓ_k is BK-space. Thus there exists a constant M such that

$$\|A(x)\| \leq M\|x\| \tag{7}$$

for $x \in \ell_k$. By applying (7) to $x = e_v$ for $v = 0, 1, 2, \ldots$ (e_v is the vth coordinate vector), we get

$$\left(\sum_{n=0}^{\infty} |a_{nv}|^k\right)^{1/k} \leq M,$$

which implies the result. □

We are now ready to prove our theorems.

Proof of Theorem 1. Let $(\bar{s}_n)$ denote the n-th partial sums of the series $\sum_{n=0}^{\infty} a_{n-1} (a_{-1} = 0)$. Then $\bar{s}_n = s_{n-1}, s_{-1} = 0$. Let (t_n) and (z_n) be the (R, p_n) transforms of (s_n) and $(\bar{s}_n)$, respectively. Hence we have

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v,$$

$$T_n = t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \quad \text{for } n \geq 1, \qquad T_0 = a_0 \tag{8}$$

and

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \bar{s}_v = \frac{1}{P_n} \sum_{v=0}^{n-1} p_{v+1} s_v,$$

$$Z_n = z_n - z_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-1} P_v a_v \quad \text{for } n \geq 1, \qquad Z_0 = 0. \tag{9}$$

It follows by making use of (8) that

$$\begin{aligned} Z_n &= \frac{p_n}{P_n P_{n-1}} \left[P_0 a_0 + \sum_{v=1}^{n-1} P_v \left(\frac{P_v}{p_v} T_v - \frac{P_{v-2}}{p_{v-1}} T_{v-1} \right) \right] \\ &= \frac{p_n}{P_n P_{n-1}} \left[P_0 T_0 + \sum_{v=1}^{n-2} \left(\frac{P_v^2}{p_v} - \frac{P_{v+1} P_{v-1}}{p_v} \right) T_v + \frac{P_{n-1}^2}{p_{n-1}} T_{n-1} \right]. \end{aligned}$$

Take $Z_n^* = n^{1/k^*} Z_n$, $T_n^* = n^{1/k^*} T_n$ for $n \geq 1$ and $Z_n^* = 0$, $T_0^* = T_0$. Then

$$\begin{aligned} Z_n^* &= n^{1/k^*} \frac{p_n}{P_n P_{n-1}} \left[P_0 T_0^* + \sum_{v=1}^{n-2} \left(\frac{P_v^2}{p_v} - \frac{P_{v+1} P_{v-1}}{p_v} \right) v^{-1/k^*} T_v^* + \frac{P_{n-1}^2}{p_{n-1}} (n-1)^{-1/k^*} T_{n-1}^* \right] \\ &= \sum_{v=0}^{\infty} a_{nv} T_v^*, \end{aligned}$$

where

$$a_{nv}=\begin{cases}P_0 n^{1/k^*}\dfrac{p_n}{P_nP_{n-1}}, & v=0\\ n^{1/k^*}\dfrac{p_n}{P_nP_{n-1}}\left(P_v^2-P_{v+1}P_{v-1}\right)\dfrac{v^{-1/k^*}}{p_v}, & 1\le v\le n-2\\ n^{1/k^*}(n-1)^{-1/k^*}\dfrac{p_nP_{n-1}}{P_np_{n-1}}, & v=n-1,\\ 0, & v\ge n.\end{cases}$$

Now, $|R,p_n|_k$ is left translative if and only if $\sum_{n=1}^{\infty}|Z_n^*|^k<\infty$ whenever $\sum_{n=1}^{\infty}|T_n^*|^k<\infty$, or equivalently, the matrix A maps ℓ_k into ℓ_k, i.e., $A\in(\ell_k,\ell_k)$. On the other hand, it is seen that

$$Z_n^*=\sum_{v=0}^{\infty}b_{nv}T_v^*+\sum_{v=0}^{\infty}c_{nv}T_v^*,$$

i.e., $A=B+C$. Hence, it is clear that if $B,C\in(\ell_k,\ell_k)$, then $A\in(\ell_k,\ell_k)$. Conversely, if $A\in(\ell_k,\ell_k)$, then it follows from Lemma 3 that there exists a constant M such that $|a_{n,n-1}|\le M$ for all $n\in\mathbb{N}$. By considering the definition of the matrix B, we obtain $B\in(\ell_k,\ell_k)$ by Lemma 1, which implies $C\in(\ell_k,\ell_k)$. Therefore

$$A\in(\ell_k,\ell_k)\quad\text{if and only if}\quad B,C\in(\ell_k,\ell_k).$$

This completes the proof together with Lemma 2. □

Proof of Theorem 2. It follows from (9) that

$$T_n=\frac{p_n}{P_nP_{n-1}}\left[\sum_{v=1}^{n}\left(\frac{P_vP_{v-2}}{p_v}-\frac{P_{v-1}^2}{p_{v-1}}\right)Z_v+\frac{P_{n-1}P_{n+1}}{p_{n+1}}Z_{n+1}\right]$$

and so

$$\begin{aligned}T_n^*&=\frac{p_n}{P_nP_{n-1}}\left[\sum_{v=1}^{n}\left(\frac{P_vP_{v-2}}{p_v}-\frac{P_{v-1}^2}{p_{v-1}}\right)v^{-1/k^*}Z_v^*+\frac{P_{n-1}P_{n+1}}{p_{n+1}}(n+1)^{-1/k^*}Z_{n+1}^*\right]\\&=\sum_{v=0}^{\infty}a_{nv}Z_v^*,\end{aligned}$$

where

$$a_{nv}=\begin{cases}n^{1/k^*}\dfrac{p_n}{P_nP_{n-1}}\left(\dfrac{P_vP_{v-2}}{p_v}-\dfrac{P_{v-1}^2}{p_v}\right)v^{-1/k^*}, & 1\le v\le n\\ \left(\dfrac{n}{n+1}\right)^{1/k^*}\dfrac{p_nP_{n+1}}{P_np_{n+1}}, & v=n+1\\ 0, & v\ge n+2.\end{cases}$$

The remainder is similar to the proof of Theorem 1 and so is omitted.

We now turn our attention to a result of Sarigol [7] which claims that, if $k>0$, then there exists two positive constant M and N, depending only on k, for which

$$\frac{M}{P_{v-1}^k}\le\sum_{n=v}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\le\frac{N}{P_{v-1}^k}$$

for all $v\ge 1$, where M and N are independent of (p_n). If we put $n=P_n/p_n$ in Corollary 1, then

$$\sum_{n=m}^{\infty}n^{k-1}\left(\frac{p_n}{P_{n-1}P_n}\right)^k=\sum_{n=m}^{\infty}\frac{p_n}{P_{n-1}^kP_n}.$$

So it is easy to see that conditions (5) and (6) hold. Hence we deduce the result due to Kuttner and Thorpe [8]. □

Corollary 2. *Let $1<k<\infty$. Then, $|\bar{N},p_n|_k$ is translative if and only if (4)(a) and (4)(b) hold.*

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